THE MEASURE OF NON-COMPACTNESS OF A DISJOINTNESS PRESERVING OPERATOR

Anton R. Schep

University of South Carolina

Abstract. Let E and F be Banach lattices and assume E^* has no atoms. Let $T : E \to F$ be a norm bounded disjointness preserving operator from E into F. Then $\beta(T) = \alpha(T) = \|T\|_e = \|T\|$.

1. Introduction

Let E be a Banach space and D a norm bounded subset of E. Then the *Kuratowski measure of non-compactness* of D is defined as

$$\alpha(D) = \inf\{\lambda : D \subset \bigcup_{j=1}^{m} D_j, \operatorname{diam}(D_j) \leq \lambda\}$$

and the *Hausdorff measure of non-compactness* of D is defined as

$$\beta(D) = \inf\{r : D \subset \bigcup_{j=1}^{m} B(x_j, r), x_j \in E\},$$

where $B(x_j, r)$ denotes the ball in E with center x_j and radius r. If E and F are Banach spaces and $T : E \longrightarrow F$ is a bounded linear operator , then one defines for T the corresponding measures of non-compactness

$$\alpha(T) = \inf\{k : \alpha(T(D)) \leq k\alpha(D) \text{ for all bound ed } D \subset E\}$$

and

$$\beta(T) = \inf\{k : \beta(T(D) \leq k\beta(D) \text{ for all bounded } D \subset E\} = \beta(T(B_E)),$$

where B_E denotes the unit ball in E. We recall some of the basic properties of $\alpha(T)$, respectively$\beta(T)$:

(1) $\frac{1}{2}\alpha(T) \leq \beta(T) \leq 2\alpha(T)$,
(2) $\alpha(T^*) \leq \beta(T)$ and $\alpha(T) \leq \beta(T^*)$ (see [5]),
(3) $\alpha(T(B_E)) = \alpha(T^*(B_{F^*}))$ (see [1]),
(4) $\max\{\alpha(T), \beta(T)\} \leq \|T\|_e$, where $\|T\|_e$ denotes the essential norm of T.

In this paper we are interested in $\alpha(T)$ and $\beta(T)$ for a special class of operators on Banach lattices. For general information on Banach lattices we refer to the monographs [4], [7] and [10]. For specific results on measures of non-compactness of operators on Banach lattices we refer to [6], [8] and [9]. From now on E and F will denote Banach lattices. A linear operator T from E into F is called *disjointness preserving* if $x \wedge y = 0$ implies $|Tx| \wedge |Ty| = 0$. It was shown in [6, theorem 3.10], that if E^* is non-atomic and $T : E \to F$ is a norm bounded disjointness preserving operator, then $\beta(T) \geq \frac{1}{2}\|T\|$. It was indicated in [6] that no example was known for which $\beta(T) < \|T\|$. Moreover for special classes of spaces (e.g. $F = L_p, 1 \leq p < \infty$) it was indicated in [6] that one always has $\beta(T) = \|T\|$. It will be shown in this paper that in fact under the above hypotheses one always has $\beta(T) = \|T\|$. Our approach follows [6], with one major difference: we employ the Kuratowski measure of non-compactness α, whereas [6] only used the Hausdorff measure of non-compactness β. It is this difference which allows us to obtain the improved result.

2. The main result

We denote by E^* the dual space of E and by E_n^* the space of order continuous linear functionals on E. For $0 \leq \phi \in E^*$ we denote by p_ϕ the seminorm $p_\phi(f) = \phi(|f|)$. The following lemma is an easy consequence of the result [3, Theorem 4] that a probability measure μ on a complete Boolean algebra has a continuous spectral resolution. For the benefit of the reader we provide a direct short proof.

lemma 2.1. *Let E be a Dedekind complete non-atomic Banach lattice and let $0 \leq u \in E$ and $0 \leq \phi \in E_n^*$ with $\phi(u) = 1$. Then for all $t \in [0,1]$ there exists a band projection P_t such that $\phi(P_t u) = t$ and such that $t \leq s$ implies $P_t \leq P_s$.*

Proof. Let $P_0 = 0$ and P_1 be the band projection on $\{u\}^{dd}$. By Zorn's lemma we can find a maximal chain $\{P_\tau\}$ of band projections such that $0 \leq P_\tau \leq P_1$. Then we note that for each $0 < t < 1$ there exists $\tau_0 \in \{\tau\}$ such that $\phi(P_{\tau_0} u) = t$, since E is non-atomic and ϕ is order continuous. Define now $P_t = \sup\{P_\tau : \phi(P_\tau u) = t\}$. The order continuity of ϕ implies now $\phi(P_t u) = t$ and obviously $t \leq s$ implies $P_t \leq P_s$. $\square$

In the following lemma we denote by S^n the n-sphere in $\mathbf{R}^{n+1}$, i.e . $S^n = \{(x_1, \ldots, x_{n+1}) : (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$ with $x_1^2 + \cdots + x_{n+1}^2 = 1\}$.

lemma 2.2. *Let E, u and ϕ be as in Lemma 2.1. Then for all $n \in \mathbf{N}$ there exists a p_ϕ-continuous map $F_n : S^n \to \{v \in E : |v| = u\}$ such that $F_n(-x_1, \ldots, -x_{n+1}) = -F_n(x_1, \ldots, x_{n+1})$ for all $(x_1, \ldots, x_{n+1})$ inS^n.*

Proof. Let P_t be a collection of band projections as in Lemma 2.1. We shall construct F_n inductively. To define F_1 we will parametrize S^1 as $\{e^{2\pi i t} : 0 \leq t < 1\}$. Define then

$$F_1(e^{2\pi i t}) = \begin{cases} 2P_{2t}u - u & \text{for } 0 \leq t \leq \frac{1}{2} \\ -2P_{2t-1}u + u & \text{for } \frac{1}{2} < t < 1 \end{cases}$$

Note that $|2P_{2t}u - u| = |P_{2t}u + P_{2t}u - P_1 u| = |P_{2t}u + (P_1 - P_{2t})u| = u$, since $P_{2t} \perp P_1 - P_{2t}$ so that $|F_1(e^{2\pi i t})| = u$ for all t. Also observe that if $0 \leq t < \frac{1}{2}$, then $F_1(-e^{2\pi i t}) = F_1(e^{2\pi i (t + \frac{1}{2})}) = -2P_{2t}u + u = -F_1(e^{2\pi i t})$. To show that F_1 is p_ϕ-continuous, we only have to show that $P_t u$ is a p_ϕ-continuous function of t, which is obvious from the fact that $p_\phi(P_t u - P_s u) = |t-s|$ for all $t, s \in [0,1]$. Hence F_1 satisfies all the requirements. Assume now that $F_{n-1} : S^{n-1} \longrightarrow \{v \in E : |v| = u\}$ has been constructed. Then define F_n as follows: $F_n(x_1, \ldots, x_{n+1}) =$

$$\begin{cases} u & \text{if } x_{n+1} = 1 \\ (P_1 - P_{x_{n+1}})F_{n-1}\left(\frac{x_1}{(1-x_{n+1}^2)^{\frac{1}{2}}}, \ldots, \frac{x_n}{(1-x_{n+1}^2)^{\frac{1}{2}}}\right) + P_{x_{n+1}}u & \\ & \text{if } 0 \leq x_{n+1} < 1 \\ -F_n(-x_1, \ldots, -x_{n+1}) & \text{if } x_{n+1} < 0 \end{cases}$$

It is easy to see that for all $(x_1, ..., x_{n+1}) \in S^n$ we have $|F_n(x_1, ..., x_{n+1})| = u$ and $F_n(-x_1, ..., -x_{n+1}) = -F_n(x_1, ..., x_{n+1})$, since $F_n(x_1, ..., x_n, 0) = F_{n-1}(x_1, ..., x_n)$. To show that F_n is p_ϕ-continuous at all $(x_1, ..., x_{n+1}) \in S^n$ one has to consider 3 cases: $x_{n+1} = 0, 0 < x_{n+1} < 1$ and $x_{n+1} = 1$. First we consider the case $x_{n+1} = 0$. Then $F_n(x_1, \ldots, x_{n+1}) = F_{n-1}(x_1, \ldots, x_n)$. The continuity of F_n at $(x_1, \ldots, x_{n+1})$ follows now from the continuity of F_{n-1} and the fact that $P_{x_{n+1,k}}u \downarrow 0$ as $k \to \infty$ for any sequence $x_{n+1,k} \downarrow^k 0$. In case $0 < x_{n+1} < 1$, then we denote by

$$X = \left(\frac{x_1}{(1-x_{n+1}^2)^{\frac{1}{2}}}, \ldots, \frac{x_n}{(1-x_{n+1}^2)^{\frac{1}{2}}}\right)$$

the corresponding point in S^{n-1}. Let now $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ be points in S^n with $0 < x_{n+1}, y_{n+1} < 1$. Then we have

$$\begin{aligned} & p_\phi(F_n(x_1, \ldots, x_{n+1}) - F_n(y_1, \ldots, y_{n+1})) \leq \\ & \phi(|(P_1 - P_{x_{n+1}})F_{n-1}(X) - (P_1 - P_{y_{n+1}})F_{n-1}(Y)|) + \phi(P_{x_{n+1}}u - P_{y_{n+1}}u) \\ & \leq \phi(|F_{n-1}(X) - F_{n-1}(Y)|) + 2\phi(|P_{x_{n+1}}u - P_{y_{n+1}}u|), \end{aligned}$$

which implies that F_n is p_ϕ-continuous at $(x_1, \ldots, x_{n+1})$. We leave it to the reader to verify that F_n is continuous at $(0, \ldots, 1)$. $\square$

We now define a measure of non-compactness associated to p_ϕ. If $D \subset E$ is norm bounded, then define:

$$\alpha_\phi(D) = \inf \{\lambda : D \subset \bigcup_{j=1}^m D_j, p_\phi\text{-diam}(D_j) \leq \lambda\}.$$

It is easy to see that $\alpha_\phi(D) \leq \|\phi\| \alpha(D)$.

Lemma 2.3.. *Let E, u and ϕ be as above. Then $\alpha_\phi([-u, u]) = 2$.*

Proof. Since the p_ϕ-diameter of $[-u, u]$ is 2, we have $\alpha_\phi([-u, u]) \leq 2$. Assume now that $[-u, u] \subset \cup_{j=1}^n D_j$. Decompose E as $N_\phi \oplus N_\phi^d$, where N_ϕ denotes $\{x \in E : \phi(|x|) = 0\}$. We can then assume that the principal ideal E_u generated by u is contained in N_ϕ^d and then replace D_j by $D_j \cap E_u$. Then we denote by $\tilde{D}_j$ the p_ϕ-closure of D_j in the completion of (E_u, p_ϕ). Let F_{n-1} be the map constructed in the previous lemma. Then $\cup_{j=1}^n F_{n-1}^{-1}(\tilde{D}_j)$ is a covering of S^{n-1} with n closed sets.

By the Lusternik–Schnirelman–Borsuk theorem ([2]) there exists an index j_0 and $(x_1, \ldots, x_n) \in S^{n-1}$ so that $\pm(x_1, \ldots, x_n) \in F_{n-1}^{-1}(\tilde{D}_{j_0})$, i.e. $\pm F_{n-1}(x_1, \ldots, x_n) \in \tilde{D}_{j_0}$. Hence

$$p_\phi\text{-diam } (D_{j_0}) = p_\phi\text{-diam } (\tilde{D}_{j_0}) \geq 2p_\phi(F_{n-1}(x_1, \ldots, x_n)) = 2,$$

and the proof of the lemma is complete. $\square$

Remark. The above lemma says essentially that $\alpha([-\chi_X, \chi_X]) = 2$ in the space $L_1(X, \mu)$, where μ is a non-atomic probability measure. The next proposition shows how to compute $\alpha([-u, u])$ in a large class of Banach lattices, in particular the following proposition holds for $E = L_p(X, \mu)$, where $1 \leq p \leq \infty$.

Proposition 2.4. *Let E be a Dedekind complete non-atomic Banach lattice and assume $\|u\| = sup\ \{< \phi, u >: 0 \leq \phi \in E_n^*, \|\phi\| = 1\}$ for all $0 \leq u \in E$. Then $\alpha([-u, u]) = 2\|u\|$.*

Proof. Let $\epsilon > 0$ and $0 \leq u \in E$ with $u \neq 0$. Then by assumption there exists $0 \leq \phi \in E_n^*, \|\phi\| = 1$ with $\phi(u) > (1 - \epsilon)\|u\|$. It follows now from the above lemma, using a scaling of ϕ, that $\alpha_\phi([-u, u]) = 2\phi(u)$. Hence $\alpha([-u, u]) \geq \alpha_\phi([-u, u]) > 2(1 - \epsilon)\|u\|$ for all $\epsilon > 0$. Hence $\alpha([-u, u]) = 2\|u\|$. $\square$

Recall now that a positive linear operator T from a Banach lattice E into a Banach lattice F is called a *Maharam operator* (or *interval preserving*) if $T[0, u] = [0, Tu]$ for all $0 \leq u \in E$.

Proposition 2.5. *Let E and F be Banach lattices with F Dedekind complete, non-atomic and such that $\|f\| = sup\{< |f|, \phi >: 0 \leq \phi \in F_n^*, \|\phi\| \leq 1\}$ for all $f \in F$. If $0 \leq T : E \to F$ is a Maharam operator, then $\alpha(T(B_E)) = 2\|T\|$.*

Proof. Let $\epsilon > 0$. Then there exists $0 \leq u \in E$ such that $\|u\| = 1$ and $\|Tu\| \geq \|T\| - \epsilon$. Then $[-Tu, Tu] = T[-u, u] \subseteq T(B_E)$ implies that $\alpha(T(B_E)) \geq \alpha([-Tu, Tu]) = 2\|Tu\| \geq 2(\|T\| - \epsilon)$, and hence $\alpha(T(B_E)) = 2\|T\|$. $\square$

We now derive, along the same lines as in [6] ,the main result of this pape r.

Theorem 2.6. *Let E and F be Banach lattices such that E^* is non -atomic. If $T : E \to F$ is a norm bounded disjointness preserving operator, then $\alpha(T) = \beta(T) = \|T\|_e = \|T\|$.*

Proof. As noted in [6], $|T^*|$ is an order continuous Maharam operator and there exists $\pi \in Z(F^*)$, the center of F^*, such that $T^* = |T^*| \circ \pi$ and $|\pi| = I$. Now E^* satisfies the hypotheses of the previous proposition, so $\alpha(|T^*|(B_{F^*})) = 2\||T^*|\|$. Since π is an isometry, we conclude that $\alpha(T^*(B_{F^*})) = 2\|T^*\|$. From [1] we know that $\alpha(T^*(B_{F^*})) = \alpha(T(B_E))$, so that we conclude that $\alpha(T(B_E)) = 2\|T\|$. Now the inequalities $\alpha(T(B_E)) \leq 2\alpha(T)$ and $\alpha(T(B_E)) \leq 2\beta(T(B_E)) = 2\beta(T)$ imply that $\beta(T) = \alpha(T) = \|T\|$. The theorem follows now, since we always have $\beta(T) \leq \|T\|_e \leq \|T\|$. $\square$

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Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208