Homework 14.

(1) (Schwarz’s lemma) Let $f$ be a holomorphic function on $B(0,1)$ with $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$.
   a. Define $f_1(z) = \frac{f(z)}{z}$ for $z \neq 0$ in $B(0,1)$. Prove that $z = 0$ is a removable singularity of $f_1$.
   b. Prove that $|f_1(z)| \leq \frac{1}{r}$ on $B(0,r)$ for all $0 < r < 1$. (Hint: use the maximum modulus principle.)
   c. Conclude that $|f(z)| \leq |z|$ for all $z \in B(0,1)$. Moreover if equality holds for some $z_0 \neq 0$, then there exists $c$ with $|c| = 1$ such that $f(z) = cz$ for all $z \in B(0,1)$.

(2) Compute
$$\int_0^\infty \frac{x^2}{x^4 + x^2 + 1} \, dx.$$

(3) Compute
$$\int_{-\infty}^\infty \frac{\cos \pi x}{x^2 - 2x + 2} \, dx$$
by integrating $f(z) = \frac{e^{\pi iz}}{z^2 - 2z + 2}$ over a semi-circular path.

(4) (Quals ’06) Let $f$ be a holomorphic function on $|z| < 1$. Assume $f(\frac{1}{n}) \in \mathbb{R}$ for $n \geq 2$. Prove $f(x) \in \mathbb{R}$ for all $-1 < x < 1$. (Hint: Use Problem 4 from HW 8.)