A SIMPLE COMPLEX ANALYSIS AND AN ADVANCED CALCULUS PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

ANTON R. SCHEP

It is hard not to have Ray Redheffer's title of [19] as a reaction to another article on the Fundamental Theorem of Algebra. In fact at least 26 notes have appeared in this Monthly (see the References) about this theorem. In this note we present nevertheles two proofs of the Fundamental Theorem of Algebra, which do not seem to have been observed before and which we think are worth recording. The first one uses Cauchy's Integral Theorem and is, in the author's opinion, as simple as the most popular complex analysis proof based on Liouville's theorem (see [20] for this and three other proofs using complex analysis). The second one considers the integral obtained by parameterizing the contour integral from the first proof and uses only results from advanced calculus. This proof is similar to the proof of [23], where the same ideas were used to prove the non-emptiness of the spectrum of an element in a complex Banach algebra. There the companion matrix of a polynomial was used then to derive the Fundamental Theorem of Algebra.

Theorem (Fundamental Theorem of Algebra)**.** *Every polynomial of degree $n \geq 1$ with complex coefficients has a zero in $\mathbb{C}$.*

Proof. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ and assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$.
First Proof: By Cauchy's integral theorem we have

$$\oint_{|z|=r} \frac{1}{zp(z)}\, dz = \frac{2\pi i}{p(0)} \neq 0,$$

where the circle is traversed counter clockwise. On the other hand $|p(z)| = |z|^n|(1+ \frac{a_1}{z} + \cdots + \frac{a_0}{z^n})| \to \infty$ as $|z| \to \infty$ implies that

$$\left|\oint_{|z|=r} \frac{1}{zp(z)}\, dz\right| \leq 2\pi r \cdot \max_{|z|=r} \frac{1}{|zp(z)|} = 2\pi \cdot \max_{|z|=r} \frac{1}{|p(z)|} \to 0,$$

as $r \to \infty$, which is a contradiction.
Second Proof: Define $g : [0,\infty) \times [0, 2\pi] \to \mathbb{C}$ by $g(r, \theta) = \frac{1}{p(re^{i\theta})}$. Then the function g is continuous on $[0,\infty) \times [0, 2\pi]$ and has continuous partials on $(0,\infty) \times (0, 2\pi)$, given by

$$\frac{\partial g}{\partial r}(r,\theta) = \frac{-e^{i\theta}p'(re^{i\theta})}{p^2(re^{i\theta})}$$
$$\frac{\partial g}{\partial \theta}(r,\theta) = \frac{-rie^{i\theta}p'(re^{i\theta})}{p^2(re^{i\theta})}.$$

Define now $F : [0,\infty) \to \mathbb{C}$ by $F(r) = \int_0^{2\pi} g(r,\theta)\, d\theta$. Then F is continuous on $[0,\infty)$ by the uniform continuity of g on $[0,M] \times [0,2\pi]$ for all $M > 0$ and by Leibniz's rule for differentiation under the integral sign we have for all $r > 0$

$$F'(r) = \int_0^{2\pi} \frac{\partial g}{\partial r}(r,\theta)\, d\theta = \int_0^{2\pi} \frac{-e^{i\theta}}{p^2(re^{i\theta})}\, d\theta,$$

so that

$$riF'(r) = \int_0^{2\pi} \frac{-rie^{i\theta}}{p^2(re^{i\theta})}\, d\theta = \int_0^{2\pi} \frac{\partial g}{\partial \theta}(r,\theta)\, d\theta = g(r,2\pi) - g(r,0) = 0,$$

by the fundamental theorem of calculus. Hence $F'(r) = 0$ for all $r > 0$. This implies that F is constant on $[0,\infty)$ with $F(r) = F(0) = \frac{2\pi}{p(0)} \neq 0$. On the other hand $|p(z)| \to \infty$ as $|z| \to \infty$ implies that $g(r,\theta) \to 0$ as $r \to \infty$ uniformly in θ. Therefore $F(r) \to 0$ as $r \to \infty$, which is a contradiction. $\square$

References

[1] N. C. Ankeny, One More Proof of the Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **54** (1947) 464.

[2] B. H. Arnold, A Topological Proof of the Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **56** (1949) 465 – 466.

[3] B. H. Arnold and I. Niven, A Correction A Topological Proof of the Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **58** (1951) 104.

[4] J. L. Brenner and R. C. Lyndon, Proof of the Fundamental Theorem of Algebra, this Monthly **88** (1981) 253 – 256.

[5] J. C. d. O. Santos, Another Proof of the Fundamental theorem of Algebra, this Monthly **112** (2005) 76.

[6] H. Derksen, The fundamental theorem of algebra and linear algebra, this Monthly **110** (2003) 620 – 623.

[7] J. E. Eaton, The Fundamental Theorem of Algebra (in Classroom Notes), this Monthly **67** (1960) 578 – 579.

[8] C. Fefferman, An Easy Proof of the Fundamental Theorem of Algebra (in Classroom Notes), this Monthly **74** (1967) 854 – 855.

[9] F. Forelli, An Advanced Calculus Proof of the Fundamental Theorem of Algebra (in Classroom Notes), this Monthly **88** (1981) 347 – 348.

[10] R. P. B. Jr, A Proof of the Fundamental Theorem of Algebra (in Questions, Discussions, and Notes), this Monthly **42** (1935) 501 – 502.

[11] ———, Yet Another Proof of the Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **71** (1964) 180.

[12] T. E. Korner, On the Fundamental Theorem of Algebra, this Monthly **113** (2006) 347.

[13] G. Leibman, A Nonstandard Proof of the Fundamental Theorem of Algebra, this Monthly **112** (2005) 705.

[14] M. Leum and M. F. Smiley, A Matric Proof of the Fundamental Theorem of Algebra for Real Quaternions (in Mathematical Notes), this Monthly **60** (1953) 99 – 100.

[15] P. Loya, Green's theorem and the fundamental theorem of algebra, this Monthly **10** (2003) 944 – 946.

[16] R. E. Moritz, On Certain Proofs of the Fundamental Theorem of Algebra, this Monthly **10** (1903) 159 – 161.

[17] I. Niven, Extension of a Topological Proof of the Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **57** (1950) 246 – 248.

[18] R. Redheffer, The Fundamental Theorem of Algebra (in Mathematical Notes), this Monthly **64** (1957) 582 – 585.

[19] R. M. Redheffer, What! Another Note Just on the Fundamental Theorem of Algebra? (in Classroom Notes), this Monthly **71** (1964) 180 – 185.

[20] R. Remmert, *Theory of complex functions*, Graduate Texts in Mathematics, vol. 122, Springer-Verlag, New York, 1991, Translated from the second German edition by Robert B. Burckel, Readings in Mathematics.
[21] P. C. Rosenbloom, An Elementary Constructive Proof of the Fundamental Theorem of Algebra, this MONTHLY **52** (1945) 562 – 570.
[22] A. Sen, Fundamental theorem of albegra - yet another proof, this MONTHLY **107** (2000) 842.
[23] D. Singh, The spectrum in a Banach algebra, this MONTHLY **113** (2006) 756 – 758.
[24] S. Stein, The Fundamental Theorem of Algebra (in Mathematical Notes), this MONTHLY **61** (1954) 109.
[25] F. Terkelsen, The Fundamental Theorem of Algebra (in Classroom Notes), this MONTHLY **83** (1976) 647.
[26] S. Wolfenstein, Proof of the Fundamental Theorem of Algebra (in Classroom Notes), this MONTHLY **74** (1967) 853 – 854.
[27] H. Zassenhaus, On the Fundamental Theorem of Algebra, this MONTHLY **74** (1967) 485 – 497.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208, USA
Email address: schep@math.sc.edu