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ANTON R. SCHEP

1. The Darboux definition of the Riemann Integral

Let $f : [a, b] \to \mathbb{R}$ be a it bounded function.

Definition 1.1. A partition $\mathcal{P}$ of $[a, b]$ is collection of points $\{x_0, \cdots, x_n\}$ which satisfiy $a = x_0 < x_1 < \cdots < x_n = b$.

Denote by Δx_k the length of the interval $[x_{k-1}, x_k]$, i.e., $\Delta x_k = x_k - x_{k-1}$. Then the *mesh* $|\mathcal{P}|$ is by definition equal to $\max_{1 \le k \le n} \Delta x_k$. Given a bounded function f and a partition $\mathcal{P}$ of $[a, b]$ we define the *Riemann upper sum* by

$$\mathcal{U}(\mathcal{P}, f) = \sum_{k=1}^{n} M_k \Delta x_k,$$

where $M_k = \text{lub}\,\{f(x) : x_{k-1} \le x \le x_k\}$. Similarly we define the *Riemann lower sum* by

$$\mathcal{L}(\mathcal{P}, f) = \sum_{k=1}^{n} m_k \Delta x_k,$$

where $m_k = \text{glb}\,\{f(x) : x_{k-1} \le x \le x_k\}$. Since $m_k \le M_k$ we clearly have $\mathcal{L}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}, f)$.

Definition 1.2. Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions of $[a, b]$. The $\mathcal{P}_2$ refines $\mathcal{P}_1$ if as sets $\mathcal{P}_1 \subset \mathcal{P}_2$.

We will write $\mathcal{P}_1 \preceq \mathcal{P}_2$ whenever $\mathcal{P}_2$ refines $\mathcal{P}_1$. Note given any two partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a, b]$ we can find the common refinement $\mathcal{P}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ by taking the partition corresponding to the union of the two point sets which form the partions $\mathcal{P}_1, \mathcal{P}_2$.

Lemma 1.3. *If $\mathcal{P}_1 \preceq \mathcal{P}_2$, then $\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{L}(\mathcal{P}_2, f)$ and $\mathcal{U}(\mathcal{P}_2, f) \le \mathcal{U}(\mathcal{P}_1, f)$.*

Proof. Let $\mathcal{P}_1 = \{a = x_0 < x_1 < \cdots < x_n = b\}$. We will first assume that $\mathcal{P}_2$ is a partition obtained from $\mathcal{P}_1$ by adding a point x between x_{i-1} and x_I. The general case will follow by induction by adding one point at the time. Thus we assume that $\mathcal{P}_2 = \{a = x_0 < x_1 < \cdots < x_{i-1} < x < x_i < \cdots < x_n = b\}$. Then we write as above

$$\mathcal{U}(\mathcal{P}_1, f) = \sum_{k=1}^{n} M_k \Delta x_k$$

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and now

$$\mathcal{U}(\mathcal{P}_2, f) = \sum_{k=1}^{i-1} M_k \Delta x_k + M'(x - x_{i-1}) + M''(x_i - x) + \sum_{k=i+1}^{n} M_k \Delta x_k,$$

where $M' = \text{lub}\ \{f(y) : x_{i-1} \le y \le x\}$ and $M'' = \text{lub}\ \{f(y) : x \le y \le x_i\}$. It is clear that $M', M'' \le M_i$, so we have that

$$M'(x - x_{i-1}) + M''(x_i - x) \le M(x - x_{i-1}) + M(x_i - x) = M\Delta x_i.$$

Hence $\mathcal{U}(\mathcal{P}_2, f) \le \mathcal{U}(\mathcal{P}_1, f)$. Similarly $\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{L}(\mathcal{P}_2, f)$. □

Corollary 1.4. *Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions of $[a, b]$. Then $\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{U}(\mathcal{P}_2, f)$.*

Proof. Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then we have by the above lemma that

$$\mathcal{L}(\mathcal{P}_1, f) \le \mathcal{L}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}_2, f).$$

□

Definition 1.5. The *upper Riemann integral* of f over $[a, b]$ is

$$\overline{\int_a^b} f(x)\, dx = \text{glb}\ \{\mathcal{U}(\mathcal{P}, f) : \mathcal{P} \text{ a partion of } [a, b]\}$$

and the *lower Riemann integral* of f over $[a, b]$ is

$$\underline{\int_a^b} f(x)\, dx = \text{lub}\ \{\mathcal{L}(\mathcal{P}, f) : \mathcal{P} \text{ a partion of } [a, b]\}.$$

From the above corollary we conclude that

$$\underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx.$$

The following definition is Darboux's version of the Riemann integral.

Definition 1.6. A bounded function $f; [a, b] \to \mathbb{R}$ is *Riemann integrable* if

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx.$$

The common value is then called the Riemann integral of f and denoted by $\int_a^b f(x)\, dx$.

Example 1.7. Let $f : [0, 1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \\ 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}. \end{cases}$$

Then for any partition $\mathcal{P}$ of $[0, 1]$ we have $\mathcal{U}(\mathcal{P}, f) = 1$ and $\mathcal{L}(\mathcal{P}, f) = 0$. Hence the upper Riemann integral of f is one, while the lower Riemann integral is equal to zero. Hence f is not Riemann integrable.

Theorem 1.8. *A bounded function $f; [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for all $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon. \tag{*}$$

Proof. Assume first that (*) holds. Let $\epsilon > 0$ and let $\mathcal{P}$ be a partition of $[a, b]$ satisfying (*). Then

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq \mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon$$

for all $\epsilon > 0$, which implies that f is Riemann integrable. Assume now that f is Riemann integrable and let $\epsilon > 0$. Then by the definition of the upper Riemann integral there exists a partition $\mathcal{P}_1$ of $[a.b]$ such that

$$\int_a^b f(x)\,dx \leq \mathcal{U}(\mathcal{P}_1, f) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

Similarly ny the definition of the lower Riemann integral there exists a partition $\mathcal{P}_2$ of $[a.b]$ such that

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < \mathcal{L}(\mathcal{P}_2, f) \leq \int_a^b f(x)\,dx.$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} < \mathcal{L}(\mathcal{P}, f) \leq \int_a^b f(x)\,dx \leq \mathcal{U}(\mathcal{P}, f) < \int_a^b f(x)\,dx + \frac{\epsilon}{2},$$

which implies

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon.$$

□

Proposition 1.9. *Let f and g be Riemann integrable functions on $[a, b]$. Then cf and $f + g$ are Riemann integrable, i.e., the set of Riemann integrable functions on $[a, b]$ form a real vector space. Moreover $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ and $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.*

Proof. For $c > 0$ it is clear that $\mathcal{U}(\mathcal{P}, cf) = c\mathcal{U}(\mathcal{P}, f)$ and $\mathcal{L}(\mathcal{P}, cf) = c\mathcal{L}(\mathcal{P}, f)$ for any partition $\mathcal{P}$. This implies immediately, via an $\frac{\epsilon}{c}$ argument, that cf is Riemann integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for $c > 0$. The case $c = 0$ is obvious and the case $c < 0$ follows from the fact that $\mathcal{U}(\mathcal{P}, -f) = -\mathcal{L}(\mathcal{P}, f)$ and $\mathcal{L}(\mathcal{P}, -f) = -\mathcal{U}(\mathcal{P}, f)$ for any partition $\mathcal{P}$. To prove that $f + g$ is Riemann integrable let $\epsilon > 0$. Then there exists a partition $\mathcal{P}_1$ such that

$$\mathcal{U}(\mathcal{P}_1, f) - \mathcal{L}(\mathcal{P}_1, f) < \frac{\epsilon}{2}.$$

Similarly there exists a partition $\mathcal{P}_2$ such that

$$\mathcal{U}(\mathcal{P}_2, g) - \mathcal{L}(\mathcal{P}_2, g) < \frac{\epsilon}{2}.$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ given by $a = x_0 < x_1 < \cdots < x_n = b$. Then lub $\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\} \leq$ lub $\{f(x) : x_{i-1} \leq x \leq x_i\} +$ lub $\{g(x) : x_{i-1} \leq x \leq x_i\}$ implies that

$$\mathcal{U}(\mathcal{P}, f + g) \leq \mathcal{U}(\mathcal{P}, f) + \mathcal{U}(\mathcal{P}, g).$$

Similarly we have that

$$\mathcal{L}(\mathcal{P}, f) + \mathcal{L}(\mathcal{P}, g) \leq \mathcal{L}(\mathcal{P}, f + g).$$

Combining these inequalities with the previous inequalities we have that

$$\mathcal{U}(\mathcal{P}, f+g) - \mathcal{L}(\mathcal{P}, f+g) < \mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) + \mathcal{U}(\mathcal{P}, g) - \mathcal{L}(\mathcal{P}, g) < \epsilon.$$

This implies that $f+g$ is Riemann integrable and by including $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ in our inequalities above we find also that $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. □

Corollary 1.10. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions and assume $g(x) \leq f(x)$ on $[a,b]$. Then $\int_a^b g(x)\,dx \leq \int_a^b f(x), dx$. In particular if $|f(x)| \leq M$ on $[a,b]$, then $\left|\int_a^b f(x)\,dx\right| \leq M(b-a)$.*

Proof. Let $h = f - g$. Then h Riemann integrable on $[a,b]$ and $h(x) \geq 0$ on $[a,b]$. Since every lower Riemann sum is greater or equal than 0, it follows that $\int_a^b h(x)\,dx \geq 0$, which implies that $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$. If $|f(x)| \leq M$, then $-M \leq f(x) \leq M$ on $[a,b]$ implies that $-M(b-a) = \int_a^b -M\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$, i.e., $\left|\int_a^b f(x)\,dx\right| \leq M(b-a)$. □

2. Riemann integrability of continuous or monotone functions

We start with the Riemann integrability of continuous functions.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable on $[a,b]$.*

Proof. Let $\epsilon > 0$. Then the compactness of $[a,b]$ implies that f is uniformly continuous and bounded on $[a,b]$. Hence there exists $M > 0$ and $\delta > 0$ such that $|f(x)| \leq M$ and such that $|x-y| < \delta$, $x, y \in [a,b]$, implies that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let now $\mathcal{P}$ be any partition of $[a,b]$ with mesh $|\mathcal{P}| < \delta$ given by $a = x_0 < x_1 < \cdots < x_n = b$. Then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ for all $x, y \in [x_{i-1}, x_i]$ implies that $M_i - m_i < \frac{\epsilon}{b-a}$. From this it follows that

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \sum_{i=1}^{n} \frac{\epsilon}{b-a}\Delta x_i = \epsilon.$$

Hence f is Riemann integrable on $[a,b]$. □

Recall now that a function $f : [a,b] \to \mathbb{R}$ is called *monotone*, if f is either non-decreasing on $[a,b]$, i.e. , $f(x) \leq f(y)$ for all $a \leq x < y \leq b$, or non-increasing on $[a,b]$, i.e., $f(x) \geq f(y)$ for all $a \leq x < y \leq b$.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable on $[a,b]$.*

Proof. We can assume without loss of generality that f is non-constant and non-decreasing on $[a,b]$. Then $f(b) > f(a)$. Let $\epsilon > 0$ and put $\delta = \frac{\epsilon}{f(b)-f(a)}$. Let $\mathcal{P}$ be a partition given by $a = x_0 < x_1 < \cdots < x_n = b$ with mesh $|\mathcal{P}| < \delta$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for $i = 1, \cdots, n$. Therefore we have

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta x_i < \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \epsilon.$$

Hence f is Riemann integrable on $[a,b]$. □

To extend the above theorems to piecewise continuous or monotone functions we prove the following theorem.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $a < c < b$. Then f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and $[c,b]$. Moreover we have in that case that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.*

Proof. Assume first that f is Riemann integrable on $[a,b]$. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\mathcal{U}(\mathcal{P},f) - \mathcal{L}(\mathcal{P},f) < \epsilon$. If necessary we can add c to the partition $\mathcal{P}$. Then we can write $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ is a partition of $[a,c]$ and $\mathcal{P}_2$ is a partition of $[c,b]$. Now it is clear that $\mathcal{U}(\mathcal{P}_1,f) + \mathcal{U}(\mathcal{P}_2,f) = \mathcal{U}(\mathcal{P},f)$ and $\mathcal{L}(\mathcal{P}_1,f) + \mathcal{L}(\mathcal{P}_2,f) = \mathcal{L}(\mathcal{P},f)$. Now $\mathcal{U}(\mathcal{P},f) - \mathcal{L}(\mathcal{P},f) < \epsilon$ implies that also $\mathcal{U}(\mathcal{P}_1,f) - \mathcal{L}(\mathcal{P}_1,f) < \epsilon$ and $\mathcal{U}(\mathcal{P}_2,f) - \mathcal{L}(\mathcal{P}_2,f) < \epsilon$. Hence f is Riemann integrable over $[a,c]$ and $[c,d]$. Moreover the inequalities

$$\mathcal{L}(\mathcal{P}_1,f) \le \int_a^c f(x)\,dx \le \mathcal{U}(\mathcal{P}_1,f),$$

$$\mathcal{L}(\mathcal{P}_2,f) \le \int_c^b f(x)\,dx \le \mathcal{U}(\mathcal{P}_2,f),$$

and

$$\mathcal{L}(\mathcal{P},f) \le \int_a^b f(x)\,dx \le \mathcal{U}(\mathcal{P},f)$$

imply that

$$\left| \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \right| < \epsilon$$

for all $\epsilon > 0$. Therefore $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. Now assume that f is Riemann integrable on $[a,c]$ and $[c,b]$. Let $\epsilon > 0$. Then there exist partitions $\mathcal{P}_1$ of $[a,c]$ and $\mathcal{P}_2$ of $[c,b]$ such that $\mathcal{U}(\mathcal{P}_1,f) - \mathcal{L}(\mathcal{P}_1,f) < \frac{\epsilon}{2}$ and $\mathcal{U}(\mathcal{P}_2,f) - \mathcal{L}(\mathcal{P}_2,f) < \frac{\epsilon}{2}$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a,b]$. As above we have $\mathcal{U}(\mathcal{P}_1,f) + \mathcal{U}(\mathcal{P}_2,f) = \mathcal{U}(\mathcal{P},f)$ and $\mathcal{L}(\mathcal{P}_1,f) + \mathcal{L}(\mathcal{P}_2,f) = \mathcal{L}(\mathcal{P},f)$ which implies immediately that $\mathcal{U}(\mathcal{P},f) - \mathcal{L}(\mathcal{P},f) < \epsilon$, so f is Riemann integrable on $[a,b]$. □

Before we can apply the above theorem to piecewise continuous or montone functions we need to prove that for bounded functions the behaviour at the endpoints does not matter for integrability.

Lemma 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and assume either that f is Riemann integrable on $[c,b]$ for all $a < c \le b$ or that f is Riemann integrable on $[a,c]$ for all $a \le c < b$. Then f is Riemann integrable over $[a,b]$.*

Proof. Let M be such that $|f(x)| \le M$ on $[a,b]$ and assume that f is Riemann integrable on $[c,b]$ for all $a < c \le b$. Let $\epsilon > 0$. Then let $a < c \le b$ such that $c - a < \frac{\epsilon}{2M}$. Then we can find a partition $\mathcal{P}_1$ of $[c,b]$ such that $\mathcal{U}(\mathcal{P}_1,f) - \mathcal{L}(\mathcal{P}_1,f) < \frac{\epsilon}{2}$. Asume $\mathcal{P}_1$ is given by the points $c = x_1 < \cdots < x_n = b$. Then define the partition $\mathcal{P}$ of $[a,b]$ by the points $a = x_0 < c = x_1 < \cdots < x_n = b$. Then $M_1 - m_1 \le 2M$ implies that $(M_1 - m_1)\Delta x_1 \le 2M(c-a) < \frac{\epsilon}{2}$. Hence

$$\mathcal{U}(\mathcal{P},f) - \mathcal{L}(\mathcal{P},f) = (M_1 - m_1)\Delta x_1 + \mathcal{U}(\mathcal{P}_1,f) - \mathcal{L}(\mathcal{P}_1,f) < \epsilon.$$

Hence f is Riemann integrable. The result for the case that f is Riemann integrable on $[a,c]$ for all $a \le c < b$ follows similarly.

□

Example 2.5. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin \frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

Then f is Riemann integrable on $[0,1]$, since f is continuous on $[c,1]$ for all $c > 0$ and f is bounded by one in absolute value on $[0,1]$.

We will now call a bounded function $f : [a,b] \to \mathbb{R}$ piecewise continuous or monotone, if there exist a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ such that the restriction of f to the *open* interval (x_{k-1}, x_k) is continuous or monotone for $k = 1, \cdots, n$. Combining all the results of this section we have the following corollary.

Corollary 2.6. *Let $f : [a,b] \to \mathbb{R}$ be a bounded piecewise continuous or monotone function. Then f is Riemann integrable. Moreover, if $a = x_0 < \cdots < x_n = b$ are such that the restriction of f to the open interval (x_{k-1}, x_k) is continuous or monotone for $k = 1, \cdots, n$, then*

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx.$$

We conclude this section with a theorem about compositions of Riemann integrable functions.

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function with range contained in $[c,d]$ and let $g : [c,d] \to \mathbb{R}$ be a continuous function. Then the composition $g \circ f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. As $[c,d]$ is compact, g is bounded and uniformly continuous on $[c,d]$. Let $K = \max\{|g(y)| : c \leq y \leq d\}$ and let $\epsilon > 0$. Then there exists $0 < \delta < \frac{\epsilon}{b-a+2K}$ such that

$$|g(y) - g(z)| < \frac{\epsilon}{b-a+2K}, \tag{1}$$

for all $y, z \in [c,d]$ with $|y - z| < \delta$. Now let $\mathcal{P}$ be a partition of $[a,b]$ such that

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \delta^2.$$

Assume that $\mathcal{P}$ is given by $a = x_0 < \cdots < x_n = b$ and denote as before by M_k the least upper bound of f on $[x_{k-1}, x_k]$ and m_k the corresponding greatest lower bound of f. We denote by M_k^* and m_k^* the corresponding quantities for $g \circ f$. Now we partition $\{1, \cdots, n\} = A \cup B$, where $A = \{k : M_k - m_k < \delta\}$ and $B = \{k : M_k - m_k \geq \delta\}$. If $k \in A$, then $|f(t) - f(s)| \leq M_k - m_k < \delta$ for all $t, s \in [x_{k-1}, x_k]$. Hence by (1) we have for all $t, s \in [x_{k-1}, x_k]$ that

$$|g(f(t)) - g(f(s))| < \frac{\epsilon}{b-a+2K},$$

which implies that $M_k^* - m_k^* \leq \frac{\epsilon}{b-a+2K}$ for all $k \in A$. This implies that

$$\sum_{k \in A} (M_k^* - m_k^*)\Delta x_k \leq \frac{\epsilon}{b-a+2K} \sum_{k \in A} \Delta x_k \leq \frac{\epsilon(b-a)}{b-a+2K}. \tag{2}$$

If $k \in B$ we will use the trivial estimate $M_k^* - m_k^* \le 2K$, so that we will need to estimate $\sum_{k\in B} \Delta x_k$. If $k \in B$, then $\delta_k \le M_k - m_k$, so we have

$$\begin{aligned}\sum_{k\in B} \Delta x_k &\le \frac{1}{\delta} \sum_{k\in B} (M_k - m_k)\Delta x_k \\ &\le \frac{1}{\delta}\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \frac{1}{\delta}\delta^2 = \delta < \frac{\epsilon}{b-a+2K}.\end{aligned}$$

This implies that

$$\sum_{k\in B} (M_k^* - m_k^*)\Delta x_k \le \frac{2\epsilon K}{b-a+2K}. \tag{3}$$

Combining now the inequalities (2) and (3) we have that

$$\mathcal{U}(\mathcal{P}, g\circ f) - \mathcal{L}(\mathcal{P}, g\circ f) < \frac{\epsilon(b-a)}{b-a+2K} + \frac{2\epsilon K}{b-a+2K} = \epsilon.$$

Hence $g \circ f$ is Riemann integrable on $[a, b]$. □

Corollary 2.8. *Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Then $|f|$, f^2, fg, $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann integrable. Moreover $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.*

Proof. By taking $g(x) = |x|$ or $g(x) = x^2$ in the above theorem we see that $|f|$ and f^2 are Riemann integrable. The remaining integrability statements follow now from the identities: $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$, $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$. To prove the integral inequality observe that $\pm f(x) \le |f(x)|$ for all $x \in [a, b]$ implies by Corollary 1.10 that $\pm\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$, i.e., $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

□

3. The Fundamental Theorems of Calculus

The two Fundamental Theorems of Calculus deal with the integral of a derivative and the derivative of an integral.

Definition 3.1. Let $f : [a, b] \to \mathbb{R}$. Then $F : [a, b] \to \mathbb{R}$ is called an *antiderivative* of f if F is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$.

Remark 3.2. We note first that if f has an antiderivative F on $[a, b]$, then F is unique up to a constant. To see this assume f has antiderivatives F and G on $[a, b]$. Then $(F - G)'(x) = f(x) - f(x) = 0$ for all $x \in (a, b)$, which implies that $F(x) - G(x)$ is constant on $[a, b]$.

Theorem 3.3 (First Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and assume F is an antiderivative of f on $[a, b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof. Let $\mathcal{P}$ be a partition of $[a,b]$ given by the points $a = x_0 < \cdots < x_n = b$. Then by the mean value theorem applied to F on the interval $[x_{k-1}, x_k]$ we can find $t_k \in [x_{k-1}, x_k]$ such that $F(x_k) - F(x_{k-1}) = f(t_k)\Delta x_k$. This implies that $\sum_{k=1}^n f(t_k)\Delta x_k = \sum_{k=1}^n F(x_k) - F(x_{k-1}) = F(b) - F(a)$. Hence

$$\mathcal{L}(\mathcal{P}, f) \leq \sum_{k=1}^n f(t_k)\Delta x_k = F(b) - F(a)$$

and

$$\mathcal{U}(\mathcal{P}, f) \geq \sum_{k=1}^n f(t_k)\Delta x_k = F(b) - F(a)$$

for all partitions $\mathcal{P}$. This implies that $\int_a^b f(x)\,dx = F(b) - F(a)$. □

The First Fundamental Theorem of Calculus is used to compute Riemann integrals. In order to do so, we need to be able to find antiderivatives of our function f. If f has an antiderivative F on $[a,b]$, then we can apply the First Fundamental Theorem to the interval $[a,x]$ to find that F satisfies

$$F(x) - F(a) = \int_a^x f(t)\,dt,$$

or

$$F(x) = F(a) + \int_a^x f(t)\,dt. \tag{4}$$

This suggests that for every Riemann integrable function f we can find an antiderivative F by means of formula (4). In general F is however not differentiable everywhere on (a,b) as we will see by means of an example. The Second Fundamental Theorem gives a sufficient condition for f so that F is an antiderivative of f.

Theorem 3.4 (Second Fundamental Theorem of Calculus)**.** *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function and let $F(x) = \int_a^x f(t)\,dt$ for $a \leq x \leq b$. Then F is continuous on $[a,b]$. Moreover, if f is continuous at $c \in (a,b)$, then $F'(x)$ exists at $x = c$ and $F'(c) = f(c)$. In particular, if f is continuous on (a,b), then F is an antiderivative of f.*

Proof. Let $|f(t)| \leq M$ on $[a,b]$. Then for $a \leq x < y \leq b$ we have

$$|F(y) - F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_x^y f(t)\,dt\right| \leq M|y-x|$$

by Corollary 1.10. This inequality shows that F is continuous on $[a,b]$. Now assume that f is continuous at $c \in (a,b)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in [a,b]$ such that $|x - c| < \delta$. It follows that for $c < x \leq b$ with $|x - c| < \delta$ we have

$$\begin{aligned}\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| &= \left|\frac{1}{x-c}\int_c^x f(t)\,dt - f(c)\right| \\ &= \frac{1}{x-c}\left|\int_c^x f(t) - f(c)\,dt\right| < \frac{1}{x-c}\epsilon(x-c) = \epsilon.\end{aligned}$$

This shows that the right derivative $F'_+(c)$ exists and equals $f(c)$, Similarly the left derivative $F'_-(c)$ exists and equals $f(c)$. Hence $F'(c)$ exists and equals $f(c)$. □

Remark 3.5. Note that the above proof actually shows that if f is Riemann integrable on $[a,b]$ and $f(c+)$ exists for $c \in (a,b)$, then $F'_+(c)$ exists and is equal to $f(c+)$. Similarly if $f(c-)$ exists, then $F'_-(c)$ exists and is equal to $f(c-)$.

Example 3.6. Let $f : [0,2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x \le 2. \end{cases}$$

Then f is Riemann integrable on $[0,2]$, since f is piecewise continuous. Now $F(x) = \int_0^x f(t)\,dt$ can be computed as follows. For $0 \le x < 1$ we have $F(x) = x$ by the First Fundamental theorem of Calculus and for $1 \le x \le 2$ by that same theorem we have $F(x) = \int_0^1 t\,dt + \int_1^x t - 1\,dt = \frac{1}{2}x^2 - x + \frac{3}{2}$. Note F is continuous at $x = 1$, but $F'_+(1) = 0$, while $F'_-(1) = 1$ by the above remark. Hence F is not differentiable at $x = 1$.

In the above example F is an antiderivative of f on the closed subintervals $[0,1]$ and $[1,2]$. In the following example we will provide a Riemann integrable function f on $[0,1]$, which has no antiderivative on any closed subinterval $[a,b]$ of $[0,1]$ with $a < b$.

Example 3.7. Let $\{r_n\}$ be an enumeration of $(0,1) \cap \mathbb{Q}$. Then define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{r_n \le x} \frac{1}{2^n},$$

where $\sum_{r_n \le x}$ is a shorthand for the sum over those n for which $r_n \le x$. Then f is Riemann integrable on $[0,1]$, since f is strictly increasing on $[0,1]$. Let $F(x) = \int_0^x f(t)\,dt$. Then F has a derivative $F'(x) = f(x)$ at every irrational number $x \in (0,1)$, since f is continuous at each irrational x. On the other hand we have that $f(r_n) = f(r_n+)$ and $f(r_n-) = f(r_n) - \frac{1}{2^n}$ for all n. From the above remark it follows that $F'_+(r_n) = f(r_n)$ and $F'_-(r_n) = f(r_n) - \frac{1}{2^n}$, so that F' fails to exist at every rational number $x \in (0,1)$.

In the above examples F' failed to exists at some points. In the next example we see that even if F' exists at all points of (a,b), that it might fail to be an antiderivative of f at the discontinuities of f.

Example 3.8. Let $\{r_n\}$ be an enumeration of $[0,1] \cap \mathbb{Q}$. Then define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \ne r_n \text{ for all } n \\ \frac{1}{n} & \text{if } x = r_n. \end{cases}$$

Then f is continuous at every irrational $x \in [0,1]$ and discontinuous at every rational $x \in [0,1]$. For rational x this follows from the fact that there exist a sequence $\{x_n\}$ of irrational numbers with limit x, so that $\lim_{n\to\infty} f(x_n) = 0 \ne f(x)$. The continuity at irrational x follows from the fact that if a sequence $\{x_k\}$ of rational numbers converges to x, then $x_k = r_{n_k}$ where $n_k \to \infty$. Hence $\lim_{k\to\infty} f(x_k) = 0 = f(x)$. Next we will show that f is Riemann integrable and that $\int_0^1 f(x)\,dx = 0$. Since every interval contains irrational numbers, it is clear that every lower Riemann sum $\mathcal{L}(\mathcal{P}, f) = 0$. Hence it suffices to show that for every $\epsilon > 0$ there exist a partition $\mathcal{P}$ such that $\mathcal{U}(\mathcal{P}, f) < \epsilon$. Let therefore $\epsilon > 0$ be given. Then we can find

N such that $\frac{1}{n} < \frac{\epsilon}{2}$ for all $n \geq N$. Now let $\mathcal{P} = \{0 = x_0 < \cdots < x_m = 1\}$ be a partition of $[0,1]$ with mesh $|\mathcal{P}| < \frac{\epsilon}{2N}$ such that each r_n with $1 \leq n < N$ is in exactly one subinterval $[x_{k-1}, x_k]$. Let now $A = \{k : r_n \in [x_{k-1}, x_k]$ for some n with $1 \leq n < N\}$ and put $B = \{1, \cdots, m\} \setminus A$. Then A contains at most $N-1$ elements so that

$$\sum_{k \in A} M_k \Delta x_k \leq \sum_{k \in A} \Delta x_k \leq (N-1)\frac{\epsilon}{2N} < \frac{\epsilon}{2}.$$

For $k \in B$ we have that $M_k < \frac{\epsilon}{2}$ so that

$$\sum_{k \in B} M_k \Delta x_k < \frac{\epsilon}{2} \sum_{k \in B} \Delta x_k \leq \frac{\epsilon}{2}.$$

Combining the above inequalities we get that $\mathcal{U}(\mathcal{P}, f) < \epsilon$. Hence f is Riemann integrable and $\int_0^1 f(x)\,dx = 0$. Since $f(x) \geq 0$, this implies that $F(x) = \int_0^x f(t)\,dt = 0$ for all $0 \leq x \leq 1$. In particular $F'(x) = 0$ for all $0 < x < 1$.

The Second Fundamental Theorem and the above examples raise the question whether a Riemann integrable function f on $[a,b]$ must have points in $[a,b]$ where f is continuous. Lebesgue characterized Riemann integrable functions in a way, which shows that f must actually continuous at "most" points of $[a.b]$. To make "most" more precise we need a definition.

Definition 3.9. A set $E \subset \mathbb{R}$ has (Lebesgue) *measure zero* if for all $\epsilon > 0$ there exists a countable collection $\{I_n\}$ of open intervals with $E \subset \cup_{n=1}^{\infty} I_n$ such that $\sum_{n=1}^{\infty} m(I_n) < \epsilon$, where $m(I_n)$ denotes the length of the interval I_n

We say that a property P holds almost everywhere (abbreviated by a.e.) on $[a,b]$, if the set $\{x \in [a,b] : P$ fails for x$\}$ has measure zero. Now Lebesgue's Theorem is as follows.

Theorem 3.10 (Lebesgue)**.** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e. on $[a,b]$.*

One can derive from Lebesgue's Theorem that any Riemann integrable function on $[a,b]$ must be continuous at uncountably many points.

4. Limits of Riemann integrable functions

We begin with an example which shows that pointwise limits of Riemann integrable functions do not have to be Riemann integrable.

Example 4.1. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n} \text{ for } k = 0, \cdots, 2^n \\ 0 & \text{elsewhere.} \end{cases}$$

Then f_n is Riemann integrable for each n, and it converges pointwise to the non-Riemann integrable function f given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n} \text{ for } 0 \le k \le 2^n, n = 1, \cdots \\ 0 & \text{elsewhere.} \end{cases}$$

Theorem 4.2. *Let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable functions and assume that $\{f_n\}$ converges uniformly to f on $[a,b]$. Then f is Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. First proof. From the inequalities $|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| \le \|f - f_n\| + \|f_n\|$ we get that f is also bounded on $[a,b]$. Let $\epsilon > 0$. Then there exists N such that $\|f_N - f\| < \frac{\epsilon}{4}$. Now f_N is Riemann integrable, so there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\mathcal{U}(\mathcal{P}, f_N) - \mathcal{L}(\mathcal{P}, f_N) < \frac{\epsilon}{2}$. Now $\|f_N - f\| < \frac{\epsilon}{4}$ implies that $\text{lub}\{f(x) : x_{k-1} \le x \le x_k\} < \text{lub}\{f_N(x) : x_{k-1} \le x \le x_k\} + \frac{\epsilon}{4}$ and $\text{glb}\{f_N(x) : x_{k-1} \le x \le x_k\} - \frac{\epsilon}{4} < \text{glb}\{f(x) : x_{k-1} \le x \le x_k\}$ for each interval $[x_{k-1}, x_k]$ of $\mathcal{P}$. This implies that

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \mathcal{U}(\mathcal{P}, f_N) - \mathcal{L}(\mathcal{P}, f_N) + \frac{\epsilon}{2} < \epsilon.$$

Hence f is Riemann integrable. This implies that $f - f_n$ is Riemann integrable and

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx \le (b-a)\|f_n - f\| \to 0.$$

Second proof. The proof that f is bounded is as in the first proof. Let $\epsilon_n = \|f - f_n\|$. Then $\epsilon_n \to 0$ as $n \to \infty$ and $|f(x) - f_n(x)| \le \epsilon_n$ on $[a,b]$ implies that $f_n(x) - \epsilon_n \le f(x) \le f_n(x) + \epsilon_n$ on $[a,b]$. This implies that

$$(5) \quad \int_a^b f_n(x)\,dx - \epsilon_n(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le \int_a^b f_n(x)\,dx + \epsilon_n(b-a),$$

which implies that

$$\left| \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \right| \le 2\epsilon_n(b-a).$$

This implies that f is Riemann integrable on $[a,b]$ as $\epsilon_n \to 0$ as $n \to \infty$. The proof that the integrals of f_n converge to the integral of f can now be given as in the first proof, or by derived from the inequalities (5). □

If we replace uniform convergence by pointwise convergence, then the above example shows that the limit function f does not have to be Riemann integrable. Therefore the above theorem is not true if we replace uniform convergence by pointwise convergence. There is however a version of the above theorem for pointwise convergence if we add the hypothesis that the limit function is Riemann integrable. This theorem is called Arzela's Theorem for the Riemann integral, which is a special case of the Bounded Convergence Theorem of Lebesgue for the Lebesgue integral.

Theorem 4.3 (Arzela's Theorem)**.** *Let $f_n, f : [a,b] \to \mathbb{R}$ be Riemann integrable functions and assume that $\{f_n\}$ converges pointwise to f on $[a,b]$. If there exists M such that $|f_n(x)| \leq M$ for all $n \geq 1$. then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$