We start with a lemma, whose proof contains the most ingenious part of Banach’s open mapping theorem. Given a norm \( \| \cdot \|_i \) we denote by \( B_i(x, r) \) the open ball \( \{ y \in X : \| y - x \|_i < r \} \).

**Lemma 1.** Let \( X \) be a vector space with two norms \( \| \cdot \|_1, \| \cdot \|_2 \) such that \((X, \| \cdot \|_1)\) is a Banach space and assume that the identity map \( I : (X, \| \cdot \|_1) \to (X, \| \cdot \|_2) \) is continuous. If \( B_2(0, 1) \subset B_1(0, r)^{\| \cdot \|_2} \), then \( B_2(0, 1) \subset B_1(0, 2r) \) and the two norms are equivalent.

**Proof.** From the hypothesis we get \( B_2(0, 1) \subset B_1(0, r) + B_2(0, \frac{r}{2}) \), so by scaling we get that \( B_2(0, \frac{1}{2^n}) \subset B_1(0, \frac{r}{2^n}) + B_2(0, \frac{r}{2^{n+1}}) \) for all \( n \geq 1 \). Let \( \| y \|_2 < 1 \). Then we can write \( y = x_1 + y_1 \), where \( \| x_1 \|_1 < r \) and \( \| y_1 \|_2 < \frac{1}{2} \). Assume we have \( \| y_n \|_2 < \frac{1}{2^n} \) we can write \( y_n = x_{n+1} + y_{n+1} \), where \( \| x_{n+1} \|_1 < \frac{r}{2^n} \) and \( \| y_{n+1} \|_2 < \frac{1}{2^{n+1}} \). By completeness of \((X, \| \cdot \|_1)\) there exists \( x \in X \) such that \( x = \sum_{n=1}^{\infty} x_n \), where the series converges with respect to the norm \( \| \cdot \|_1 \). By continuity of the identity map \( I : (X, \| \cdot \|_1) \to (X, \| \cdot \|_2) \) it follows that the same series also converges to \( x \) with respect to \( \| \cdot \|_2 \). On the other hand the equation \( y = \sum_{k=1}^{k+1} x_k + y_{n+1} \) shows that the series \( \sum_{n=1}^{\infty} x_n \) converges to \( y \) with respect to \( \| \cdot \|_2 \). Hence \( y = x \) and thus \( \| y \|_1 = \| x \|_1 \leq \sum_{n=1}^{\infty} \| x_n \|_1 < 2r \). It follows that \( B_2(0, 1) \subset B_1(0, 2r) \) and thus \( \| y \|_1 \leq 2r \| y \|_2 \) for all \( y \in X \). As the continuity of \( I \) gives that there exists \( C \) such that \( \| y \|_2 \leq C \| y \|_1 \) for all \( y \in X \), we get that the two norms are equivalent.

**Theorem 2.** Let \( X \) be a vector space with two norms \( \| \cdot \|_1, \| \cdot \|_2 \) such that \((X, \| \cdot \|_1)\) and \((X, \| \cdot \|_2)\) are Banach spaces. Assume that the identity map \( I : (X, \| \cdot \|_1) \to (X, \| \cdot \|_2) \) is continuous. Then the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent.

**Proof.** Applying the Baire Category theorem in \((X, \| \cdot \|_2)\) to \( X = \bigcup_{n=1}^{\infty} B_1(0, n) \) we can find \( n_0, x_0 \) and \( r_0 > 0 \) such that \( B_2(x_0, r_0) \subset B_1(0, n_0)^{\| \cdot \|_2} \). Translating over \(-x_0\) we get that \( B_2(0, r_0) \subset B_1(-x_0, n_0)^{\| \cdot \|_2} \). Now by the triangle inequality we get that \( B_2(0, r_0) \subset B_1(0, n_0 + \| x_0 \|_1)^{\| \cdot \|_2} \). By the above lemma the two norms are equivalent.

**Theorem 3** (Bounded Inverse Theorem). Let \( X, Y \) be Banach spaces and assume \( T : X \to Y \) is an one-to-one, onto continuous linear operator. Then \( T^{-1} : Y \to X \) is continuous.

**Proof.** Define \( \| y \|_{T^{-1}} = \| T^{-1} y \| \). Then \( \| \cdot \|_T \) is a norm on \( Y \) and \( \| y \| \leq \| T \| \| y \|_{T^{-1}} \) for \( y \in Y \), so \( I : (Y, \| \cdot \|_{T^{-1}}) \to (Y, \| \cdot \|) \) is continuous. Moreover, if \( \sum_{n=1}^{\infty} \| y_n \|_T < \infty \), then \( x_0 + \sum_{n=1}^{\infty} T^{-1} y_n \) exists in \( X \) and \( \| Tx_0 - \sum_{n=1}^{N} y_n \|_T = \| x_0 - \sum_{n=1}^{N} T^{-1} y_n \| \to 0 \) as \( N \to \infty \). Hence \((Y, \| \cdot \|_T)\) is also a Banach space. By the above Theorem the two norms
on $Y$ are equivalent, so there exists $C$ such that $\|T^{-1}(y)\| \leq C\|y\|$ for all $y \in Y$, i.e. $T^{-1}$ is continuous.

We recall now that a linear map $T : X \to Y$ is called open if $T(O)$ is open for all open $O \subset X$. It is easy to see that an open linear map is surjective. The Open Mapping theorem gives a converse to that statement. Before stating and proving that theorem, we recall a few basic facts about quotient maps. Let $X$ be a Banach space and $M \subset X$ a closed subspace. Then $X/M$ is a Banach space with respect to the quotient norm $\|x\| = \inf\{\|y\|; y \in [x]\}$.

Denote by $Q$ the quotient map $Q(x) = [x]$. Then $Q$ is open. In fact, it is easy to see from the definition of the quotient norm that $Q(\{x : \|x\| < 1\}) = \{[x] : \|[x]\| < 1\}$.

**Theorem 4** (Open Mapping Theorem). Let $X$, $Y$ be Banach spaces and assume $T : X \to Y$ is an onto continuous linear operator. Then $T$ is an open map.

**Proof.** Let $Q : X \to X/\ker(T)$ be the quotient map. Then by the above remarks $Q$ is an open mapping. Let $\hat{T} : X/\ker(T) \to Y$ be the induced map such that $T = \hat{T} \circ Q$. Then $\hat{T}$ is one to one and onto, so by the above Theorem $T^{-1}$ is continuous, so $\hat{T}$ is open and thus $T$ is open.

Let now $A : D(A) \to Y$ be a linear operator, where $D(A)$ is a (not necessarily closed) linear subspace of the Banach space $X$. The subspace $D(A)$ is called the domain of $A$. Given a linear operator $A : D(A) \to Y$ we define the graph

$$\Gamma(A) = \{(x,Ax) : x \in D(A)\},$$

It is clear that $\Gamma(A)$ is linear subspace of $X \times Y$. We can equip $X \times Y$ with the product norm $\|(x,y)\| = \|x\| + \|y\|$. Then we say that $A$ has a closed graph (or is a closed operator), if $\Gamma(A)$ is a closed subspace of $X \times Y$.

**Example 5.** Let $X = Y = C[0,1]$ with the supremum norm. Let $D(A) = C'[0,1]$ the subspace of $X$ consisting of continuously differentiable functions and define $A : D(A) \to Y$ by $Af = f'$. One can can see that $A$ is bounded, by taking $f_n(t) = t^n$, and noting that $\|f_n\| = 1$ and $\|Af_n\| = n$. On the other hand $A$ has a closed graph. To see that $A$ has a closed graph, let $(f_n,f'_n) \to (f,g)$ in $X \times Y$. Then by the Fundamental Theorem of Calculus $f_n(t) - f_n(0) = \int_0^t f'_n(s) \, ds \to \int_0^t g(s) \, ds$. It follows that $f(t) = f(0) + \int_0^t g(s) \, ds$. Hence $f \in D(A)$ and $f' = g$, i.e., $(f,g) \in \Gamma(A)$.

The following proposition is immediate from the definition.

**Proposition 6.** Let $X$ and $Y$ be Banach spaces and assume $A : D(A) \to Y$ is a linear operator, where $D(A)$ is a subspace of $X$. Then the following are equivalent.

1. $A$ has a closed graph.
2. If $x_n \in D(A)$, $x_n \to x \in X$, and $Ax_n \to y \in Y$, then $x \in D(A)$ and $Ax = y$.
3. $D(A)$ is a Banach space with respect to the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

**Theorem 7** (Closed Graph Theorem). Let $X$ and $Y$ be Banach spaces and assume $A : X \to Y$ is a closed linear operator. Then $A$ is bounded.
Proof. Define $P : \Gamma(A) \to X$ by $P(x, Ax) = x$. Then $P$ is clearly a bounded, one-to-one, onto linear operator, so by the Bounded Inverse Theorem the inverse operator $P^{-1} : X \to \Gamma(A)$ is bounded. Hence there exists a constant $C$ such that $\|x\| + \|Ax\| \leq C\|x\|$, i.e., $\|Ax\| \leq (C - 1)\|x\|$ for all $x \in X$. 

□

We now present a proof of the Uniform Boundedness Principle, based on the Closed Graph Theorem.

Theorem 8 (Banach-Steinhaus). Let $X$ and $Y$ be Banach spaces and assume $A_\alpha \in L(X,Y)$ ($\alpha \in \mathcal{F}$) is a pointwise bounded family of bounded operators, i.e., for all $x \in X$ there exists a constant $C_x$ such that $\|A_\alpha x\| \leq C_x$ for all $\alpha \in \mathcal{F}$. Then there exists a constant $C$ such that $\|A_\alpha\| \leq C$ for all $\alpha \in \mathcal{F}$.

Proof. Define the space $\oplus_\alpha Y = \{(y_\alpha) : y_\alpha \in Y, \sup_\alpha \|y_\alpha\| < \infty\}$ with norm $\|(y_\alpha)\| = \sup_\alpha \|y_\alpha\|$. It is straightforward to verify that $\oplus_\alpha Y$ is also a Banach space. Now define $T : X \to \oplus_\alpha Y$ by $Tx = (A_\alpha x)$. Note $Tx \in \oplus_\alpha Y$, since the collection $A_\alpha$ is pointwise bounded. Clearly $T$ is linear and we claim that $T$ is closed. To see this, let $x_n \to 0$ and $Tx_n \to (y_\alpha)$. Then $A_\alpha x_n \to y_\alpha$ for all $\alpha \in \mathcal{F}$, but also $A_\alpha x_n \to 0$ for all $\alpha$. Hence $(y_\alpha) = (0)$ and $T$ is closed. By the Closed Graph Theorem $T$ is bounded, i.e., there exists a constant $C$ such that for all $\|x\| \leq 1$ we have $\sup_\alpha \|A_\alpha x\| \leq C$. Hence $\|A_\alpha\| \leq C$ for all $\alpha \in \mathcal{F}$. 

□