# Sequences and Series

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## Overview

Sequences and series are the objects of interest for the next few weeks. The intent of this lab is to provide additional practice determining the convergence or divergence of a sequence of numbers. Ways to generate sequences and series in Maple are also introduced.

## Maple Essentials

- New Maple commands introduced in this lab:

| Command/Example | Description |
|---|---|
| `evalf(`*expression*`);`<br>Example:<br>`evalf(Pi);` | numerically evaluates expressions involving constants |
| `seq(f(n), n=i..j);`<br>Example:<br>`seq(1/n,n=1..10);` | creates a finite sequence of values<br>$f(i)$, $f(i+1)$, $\cdots$ $f(j)$,<br>where $f(n)$ is a maple function and $i \leq j$ are integers. |
| `seq([n,f(n)],n=i..j);`<br>Example:<br>`f:=x->x^2;`<br>`seq([n,f(n)],n=1..10);` | creates a finite sequence of points on the graph of $y = f(x)$. |
| `sum(f(n), n=i..j)`<br>Example:<br>`sum(n^2, n=1..10);` | creates and evaluates a finite or infinite sum, that is, series $\sum_{n=i}^{j} f(n)$, where `f(n)` is a maple function or expression and $i \leq j$ can be integers, variables, or infinity. For a finite or convergent infinite series, it automatically evaluates the sum and returns a value or formula. If you don't want the automatic evaluation, use `Sum` instead of `sum`. |
| `for n from i to j do...end do;`<br>Example:<br>`s[1]:=1; for n from 1 to 9 do`<br>`s[n+1]:= s[n]+n end do;` | A typical for-loop (`for` and `do` statement) used in general programming languages. It executes whatever between `''do''` and `''end do''` repeatedly for a counted number of times (`''for n from i to j''`). It hence can be used to work with sequences in much more general ways than what the command `seq` could. |

- A link to the *SequenceDrill* maplet can be found on the course website:

`http://www.math.sc.edu/calclab/142L-S12/labs/`→ SequenceDrill

## Preparation

Sections 11.1 and 11.2 in Stewart. Sections 9.1 and 9.2 in CalcLabs. In addition, review the basic qualitative properties of logarithms, powers, exponentials, and so on. For example, exponentials grow faster (at $\infty$) than polynomials, factorials grow faster than exponentials, and so on.

## Assignment

With the help of Maple, work out the problems assigned by your lab instructor. Clearly identify your answers on your Maple worksheet. Make sure you answer each question completely. Your assignment is due at the **beginning** of next week's lab.

## *Activities*

1. For each of the following sequences: (a) Generate the first 10 terms. (b) Determine whether the sequence diverges or converges to a limit. (c) Graph a sequence of points to verify your answer. **Note:** *Let $p$ be a parameter.*

   (a) $\{(-1)^n \arctan(n)\}_{n=1}^{\infty}$ (b) $\left\{\sqrt{n^2+pn}-n\right\}_{n=1}^{\infty}$ (c) $\left\{\dfrac{10^n}{n!}\right\}_{n=1}^{\infty}$

   (d) $\left\{n \sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (e) $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$ (f) $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{k^2}\right\}_{n=1}^{\infty}$

   (g) $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{1+(k/n)}\right\}_{n=1}^{\infty}$

   **Note:** *You may use the* SequenceDrill *maplet. However, it does not work well with sequences involving parameters since it involves* `plot`*. We will work out some examples using explicit commands.*

2. A typical format for a recursively-defined sequence is $a_{n+1} = f(a_n)$, $n = 2, 3, \cdots$ (with $a_1$ given explicitly). Under the assumptions that (i) $\{a_n\}$ converges to $L$ and (ii) $f$ is continuous function (at $L$), we have that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ and

   $$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

   Thus, $L$ must be a solution to $L = f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (exactly, numerically, or graphically).

   (a) (See Exercise 68 on page 685) Consider the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$, $n = 1, 2, 3, \cdots$. Use Maple to verify that it is a bounded monotone sequence and hence converges to a limit. Explain how a plot containing the graphs of $y = x$ and $y = \sqrt{2+x}$ confirms this limit.

   (b) Consider the sequence $\{x_n\}$ produced by Newton's Method to approximate $\sqrt{2}$ as a zero of $f(x) = x^2 - 2$, where $x_1 = 1$, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$, $n = 1, 2, 3, \cdots$. Use Maple to verify that the limit is indeed $\sqrt{2}$.

## *Example: Activity 1a*

```
> with(plots):
> f:= n-> (-1)^n arctan(n);
> evalf(seq(f(n), n=1..10));
> limit(f(n), n=infinity);
> points:=evalf(seq([n,f(n)], n=1..10));
> P1:=plot([points], style=point):
> P2:=plot([-1/2*Pi, 1/2*Pi]):
> display([P1,P2]);
```

## *Example: Activity 2a*

```
> a[1]:=sqrt(2);
> for n from 1 to 9 do a[n+1]:=sqrt(2+a[n]); evalf(a[n+1]) end do;
> plot([x,sqrt(2+x)],x=-4..4);
> solve(x=sqrt(2+x),x);
```