Applications of Integration I: Areas Between Curves

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Overview
This lab will help to develop your understanding of the definite integral as area for non-negative functions. In this lab you will explore how to use Maple to visualize a given region, determine points of intersection if necessary, and determine its area. You will discover that for some regions it is best to regard $x$ as a function of $y$.

Maple Essentials
New Maple commands introduced in this lab include:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
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<tbody>
<tr>
<td>fsolve</td>
<td>numerically solves one or more equations for their unknowns</td>
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<tr>
<td>$\text{fsolve}(x^2-4=0)$; returns the values -2.0, 2.0</td>
<td></td>
</tr>
<tr>
<td>*To find only real solutions in an interval, specify the interval. $\text{fsolve}(x^2-4=0, x=-3..-1)$; returns only -2.0</td>
<td></td>
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<tr>
<td>$\text{int}(f(x), x=a..b)$;</td>
<td>evaluates $\int_{a}^{b} f(x) dx$</td>
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The Area of a Region by Slicing maplet is available from the course website:
http://www.math.sc.edu/calclab/142L-S12/labs → Area of a Region by Slicing

This maplet demonstrates finding the area of a region by slicing. The student may generate a random example or modify or make their own problem.

Preparation
In the case where both $f(x)$ and $g(x)$ are positive, it is very easy to see that if we want to find the area between $f(x) \geq g(x)$ (both continuous on $[a, b]$) we would do the following:

$$A = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx = \int_{a}^{b} (f(x) - g(x)) \, dx.$$ 

In general, for a given region with top curve $y_T$ and bottom curve $y_B$ we can draw a typical approximating rectangle with height $(y_T - y_B)$ and width $\Delta x$ and calculate the area as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B) \Delta x = \int_{a}^{b} (y_T - y_B) \, dx$$

Sometimes regions are best treated by regarding $x$ as a function of $y$. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where $f$ and $g$ are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ then its area is

$$A = \int_{c}^{d} (f(y) - g(y)) \, dy.$$ 

In general, for a given region with right boundary $x_R$ and left boundary $x_L$ we can draw a typical approximating rectangle with height $\Delta y$ and width $(x_R - x_L)$ and calculate the area as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (x_R - x_L) \Delta y = \int_{c}^{d} (x_R - x_L) \, dy$$
Related Course Material
Section 6.1 in Stewart. Section 7.1 in CalcLabs.

Activity 1
Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

- We will first assign our functions as $f(x)$ and $g(x)$.
  
  ```maple
  > f:=x-> x^2;
  > g:=x-> x+6;
  ```

- Next, we plot the functions to get an idea of the region we are considering.
  
  ```maple
  > plot([f(x),g(x)], x=-5..5, y=-10..10, color=[red, blue]);
  ```

- We find the intersection points using `fsolve`.
  
  ```maple
  > fsolve(f(x)=g(x));
  ```

- We then evaluate the integral to find the area. Notice for this example $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  
  ```maple
  > Area:=int(g(x)-f(x), x=-2..3);
  ```

Activity 2
Find the area of the region enclosed between $y = -0.128x^3 + 1.728x^2 - 5.376x + 2.864$ and $y = \ln x$.

- We will first assign our functions as $f(x)$ and $g(x)$.
  
  ```maple
  > f:=x-> -0.128*x^3+1.728*x^2-5.376*x+2.864;
  > g:=x-> ln(x);
  ```

- Next, we plot the functions to get an idea of the region we are considering. Be careful with the window this time. There are two separate areas enclosed by the functions.
  
  ```maple
  > plot([f(x),g(x)], x=0..10, y=-10..10, color=[red, blue]);
  ```

- We can see from the graph that there are three points of intersection, but `fsolve` (without a specified interval) will only give one, so we will write a separate command to find and assign each intersection point.
  
  ```maple
  > a:= fsolve(f(x)=g(x), x=0..2);
  > b:= fsolve(f(x)=g(x), x=2..6);
  > c:= fsolve(f(x)=g(x), x=8..10);
  ```

- For our first area, notice that between $a$ and $b$, $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  
  ```maple
  > A1:=int(g(x)-f(x), x=a..b);
  ```

- For our second area, notice that between $b$ and $c$, $f(x)$ is the top curve and $g(x)$ is the bottom curve.
  
  ```maple
  > A2:=int(f(x)-g(x), x=b..c);
  ```

- We find our total area by adding $A1$ and $A2$.
  
  ```maple
  > Area:= A1 + A2;
  ```

Activity 3
Find the area of the region enclosed between $x = y^2$ and $x = y + 2$.

- We will first assign our expressions as $f(y)$ and $g(y)$.
  
  ```maple
  > f:=y-> y^2;
  > g:=y-> y+2;
  ```

- We can use the `fsolve` command to find the points of intersection.
  
  ```maple
  > fsolve(f(y)=g(y));
  ```

- Once you have the points of intersection, you can use the *Area of a Region by Slicing* maplet to get a clear picture of the region.

- We then evaluate the integral to find the area. Notice for this example $g(y)$ is the right boundary and $f(y)$ is the left boundary. Also, we are integrating with respect to $y$.
  
  ```maple
  > Area:=int(g(y)-f(y), y=-1..2);
  ```

Assignment
With the help of Maple, work out the problems assigned by your lab instructor. Clearly identify your answers on your Maple worksheet. Make sure you answer each question completely.