Applications of Integration I: Areas Between Curves

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Overview

This lab will help to develop your understanding of the definite integral as area for non-negative functions. In this lab you will explore how to use Maple to visualize a given region, determine points of intersection if necessary, and determine its area. You will discover that for some regions it is best to regard x as a function of y.

Maple Essentials

New Maple commands introduced in this lab include:

Command	Description
`fsolve`	numerically solves one or more equations for their unknowns `fsolve(x^2-4=0);` returns the values -2.0, 2.0 *To find only real solutions in an interval, specify the interval. `fsolve(x^2-4=0, x=-3..-1);` returns only -2.0
`int(f(x), x=a..b);`	evaluates $\int_a^b f(x)dx$

The *Area of a Region by Slicing* maplet is available from the course website:

http://www.math.sc.edu/calclab/142L-S11/labs → Area of a Region by Slicing

This maplet demonstrates finding the area of a region by slicing. The student may generate a random example or modify or make their own problem.

Preparation

In the case where both $f(x)$ and $g(x)$ are positive, it is very easy to see that if we want to find the area between $f(x) \geq g(x)$ (both continuous on $[a, b]$) we would do the following:

$$A = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b (f(x) - g(x))\ dx.$$

In general, for a given region with top curve y_T and bottom curve y_B we can draw a typical approximating rectangle with height $(y_T - y_B)$ and width Δx and calculate the area as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (y_T - y_B)\Delta x = \int_a^b (y_T - y_B)\ dx$$

Sometimes regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ then its area is

$$A = \int_c^d (f(y) - g(y))\ dy.$$

In general, for a given region with right boundary x_R and left boundary x_L we can draw a typical approximating rectangle with height Δy and width $(x_R - x_L)$ and calculate the area as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (x_R - x_L)\Delta y = \int_c^d (x_R - x_L)\ dy$$

Related Course Material

Section 6.1 in Stewart. Section 7.1 in CalcLabs.

Activity 1

Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

- We will first assign our functions as $f(x)$ and $g(x)$.
  > `f:=x-> x^2;`
  > `g:=x-> x+6;`
- Next, we plot the functions to get an idea of the region we are considering.
  > `plot([f(x),g(x)], x=-5..5, y=-10..10, color=[red, blue]);`
- We find the intersection points using `fsolve`.
  > `fsolve(f(x)=g(x));`
- We then evaluate the integral to find the area. Notice for this example $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  > `Area:=int(g(x)-f(x), x=-2..3);`

Activity 2

Find the area of the region enclosed between $y = -0.128x^3 + 1.728x^2 - 5.376x + 2.864$ and $y = \ln x$.

- We will first assign our functions as $f(x)$ and $g(x)$.
  > `f:=x-> -0.128*x^3+1.728*x^2-5.376*x+2.864;`
  > `g:=x-> ln(x);`
- Next, we plot the functions to get an idea of the region we are considering. Be careful with the window this time. There are two separate areas enclosed by the functions.
  > `plot([f(x),g(x)], x=0..10, y=-10..10, color=[red, blue]);`
- We can see from the graph that there are three points of intersection, but `fsolve` (without a specified interval) will only give one, so we will write a separate command to find and assign each intersection point.
  > `a:= fsolve(f(x)=g(x), x=0..2);`
  > `b:= fsolve(f(x)=g(x), x=2..6);`
  > `c:= fsolve(f(x)=g(x), x=8..10);`
- For our first area, notice that between a and b, $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  > `A1:=int(g(x)-f(x), x=a..b);`
- For our second area, notice that between b and c, $f(x)$ is the top curve and $g(x)$ is the bottom curve.
  > `A2:=int(f(x)-g(x), x=b..c);`
- We find our total area by adding $A1$ and $A2$.
  > `Area:= A1 + A2;`

Activity 3

Find the area of the region enclosed between $x = y^2$ and $x = y + 2$.

- We will first assign our expressions as $f(y)$ and $g(y)$.
  > `f:=y-> y^2;`
  > `g:=y-> y+2;`
- We can use the `fsolve` command to find the points of intersection.
  > `f(y)=g(y);`
- Once you have the points of intersection, you can use the *Area of a Region by Slicing* maplet to get a clear picture of the region.
- We then evaluate the integral to find the area. Notice for this example $g(y)$ is the right boundary and $f(y)$ is the left boundary. Also, we are integrating with respect to y.
  > `Area:=int(g(y)-f(y), y=-1..2);`

Assignment

Stewart page 420, exercises 23, 36, and 38.