Implicit Differentiation

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Overview
This lab provides experience working with functions defined implicitly.

Maple Essentials

- The new Maple commands introduced in this lab are:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
<th>Example</th>
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<tbody>
<tr>
<td>display</td>
<td>combine one or more plots in a single plot; part of the plots package</td>
<td>display([P1,P2], title=&quot;My Graph&quot;);</td>
</tr>
<tr>
<td>implicitdiff</td>
<td>compute derivatives for implicitly-defined functions</td>
<td>Finding ( \frac{dy}{dx} ): implicitdiff(eq, y, x); Finding ( \frac{d^n y}{dx^n} ): implicitdiff(eq, y, x$n$);</td>
</tr>
<tr>
<td>implicitplot</td>
<td>create graph of function defined implicitly; part of the plots package</td>
<td>implicitplot(eq, x=a..b, y=c..d);</td>
</tr>
<tr>
<td>pointplot</td>
<td>plots a single point; part of the plots package</td>
<td>pointplot([a,b], symbolsize=15);</td>
</tr>
<tr>
<td>fsolve</td>
<td>compute a solution of equations numerically</td>
<td>fsolve({eq1,eq2}, {x,y});</td>
</tr>
<tr>
<td>with</td>
<td>loads the contents of a Maple package</td>
<td>with(plots):</td>
</tr>
<tr>
<td>eval</td>
<td>evaluates a given expression at a given point</td>
<td>eval(eq, {x=a, y=b});</td>
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</table>

- The ImplicitDifferentiation maplet is available from the course website:


Preparation
Review §3.5 Implicit Differentiation in Stewart and §4.4 in Calclabs with Maple.

Activities

1. The curve with equation \( 2(x^2 + y^2)^2 = 25(x^2 - y^2) \) is called a lemniscate.
   - Find the equation of the tangent line to this curve at the point \((3, 1)\). Then graph the curve, the point, and the tangent line together on one plot with a viewing window of \([-5, 5] \times [-4, 4]\).
   (Ex. 29 on page 213)
   - Find all points on the lemniscate where the tangent line is horizontal or vertical.
   (Ex. 39 on page 214)

2. Find \( \frac{d^2 y}{dx^2} \) and \( \frac{d^3 y}{dx^3} \) if \( y \) is defined implicitly by \( y + \sin y = x \).
Example Problem
We will solve Example 2 on page 209 of Stewart together using Maple:

- Use implicit differentiation to find $\frac{dy}{dx}$ for the Folium of Descartes $x^3 + y^3 = 6xy$.
- Find an equation of the tangent line to the Folium of Descartes at the point $(3, 3)$. (Then graph the curve, the point, and the tangent line with a viewing window of $[-5, 5] \times [-5, 5]$ as shown in Figure 4 on page 210.)
- At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal? (At what points is the tangent line vertical?)

Steps:

1. First, load the Maple plots package. Without the contents of this package, much of what we do today will not work.
   > with(plots):
2. Assign our equation using ‘:=’.
   > eq := x^3 + y^3 = 6*x*y;
3. Find (and assign) the derivative using implicit differentiation. Since we want $\frac{dy}{dx}$, we input $y$ and then $x$.
   > dydx := implicitdiff(eq, y, x);
4. Find (and assign) the slope of the tangent line at the point $(3, 3)$.
   > m := eval(dydx, {x=3, y=3});
5. Find (and assign) the equation of the tangent line. Remember: $y = m(x - x_1) + y_1$.
   > L := m*(x - 3) + 3;
6. Next, write (and assign) commands to plot the curve, the point, and the tangent line. Write the commands separately using ‘:’ so Maple does not display the output yet. (In the first plot command, the option numpoints=10000 will insure a smooth curve.)
   > P1 := implicitplot(eq, x=-5..5, y=-5..5, numpoints=10000):
   > P2 := pointplot([3,3], color=green, symbolsize=15):
   > P3 := plot(L, x=-5..5, y=-5..5, color=blue, linestyle=dash):
7. Use the display command to display the curve, point, and tangent line on a single plot.
   > display([P1, P2, P3], title='Figure 1');
8. From the graph, we can see that the tangent line would be horizontal at a point located approximately at $(2.5, 3.1)$. To find the point exactly, we need to find a point on the curve where $\frac{dy}{dx} = 0$. We can find this point using fsolve.
   > fsolve({eq, dydx=0}, {x=2..3, y=3..4});
9. From the graph, we can see that the tangent line would be vertical at a point located approximately at $(3.1, 2.5)$. To find the point exactly, we need to find a point on the curve where $\frac{dy}{dx}$ is undefined. That is, a point where the denominator of $\frac{dy}{dx}$ is 0. We can find this point using fsolve.
   > fsolve({eq, denom(dydx)=0}, {x=3..4, y=2..3});

Additional Notes
The ImplicitDifferentiation maplet provides additional practice finding the slope of a curve at a point.

Assignment
Exercises 31 and 38 on pages 213-214 of Stewart. For problem 38, you will need to zoom to get a good look at the horizontal tangents. For example, $y=1.6..1.7$ is a good view to distinguish the necessary ranges for the top of the wagon. Exercise 12 on page 66 of CalcLabs.