Mathematical Models: Designing a Roller Coaster

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Overview
There are three objectives in this lab:

- understand the mathematical reasoning associated with a real-world example,
- learn to define a piecewise-defined function in Maple, and
- learn to set up and solve a system of equations in Maple.

Maple Essentials
Important Maple commands introduced in this lab are:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>piecewise</td>
<td>define a piecewise-defined function</td>
</tr>
<tr>
<td>diff</td>
<td>diff(f(x), x); finds the derivative of f(x) with respect to x.</td>
</tr>
<tr>
<td>solve</td>
<td>solve(eqn, var); solves an equation, eqn, for one variable, var.</td>
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<tr>
<td></td>
<td>solve({eqn1, eqn2}, {var1, var2}); solves a system of two equations for two variables.</td>
</tr>
<tr>
<td>assign</td>
<td>assign(values); assigns a set of values</td>
</tr>
</tbody>
</table>

Maple does not recognize double inequalities, so if your condition is \( a \leq x < b \) you would write \( x \geq a \) and \( x < b \).

Preparation
§2.8 and §4.1 of Stewart. Review properties of the graph of the first derivative.

Assignment
This week’s assignment is to design a larger roller coaster that meets given specifications and prepare a neat and complete project report. **Project 1 will be due at the beginning of the next lab.**

The Problem: Design a Roller Coaster
Suppose we are asked to design a simple ascent and drop roller coaster with an overall horizontal displacement of 200 feet. By studying pictures of our favorite roller coasters, we decide to create our roller coaster using a line, a parabola and a cubic. We begin the ascent along a line \( y = f_1(x) \) of slope \( \frac{4}{3} \) for the first 20ft horizontally. We continue the ascent and begin the drop along a parabola \( y = f_2(x) = ax^2 + bx + c \) for the next 100ft horizontally. Finally, we begin a soft landing at 40ft above the ground along a cubic \( y = f_3(x) = dx^3 + cx^2 + fx + g \) for the last 80ft.

Here are our tasks:

1. Find a system of 7 equations with the 7 unknowns \( \{a,b,c,d,e,f,g\} \) that will ensure that the track is smooth at transition points.
2. Solve the equations in (1) to find our functions. (We should get a unique solution as we have the same number of equations and unknowns.)
3. Plot the graph to see the design.
4. Find the maximum height of the roller coaster.
Solving the Problem

1. Start your Maple session with
   \[
   \texttt{restart;
   }
   \]
   This clears the internal memory so that Maple acts (almost) as if just started and is very helpful if you
   make a mistake and want to start over.

2. We begin by defining our functions in Maple. If we choose the origin as our starting point, our first
   function \( y = f_1(x) \) is a line of slope \( \frac{4}{3} \) that passes through \((0,0)\), and we have:
   \[
   \begin{align*}
   &f_1:=x\rightarrow 4/3\cdot x; \\
   &f_2:=x\rightarrow a\cdot x^2+b\cdot x+c; \\
   &f_3:=x\rightarrow d\cdot x^3+e\cdot x^2+f\cdot x+g;
   \end{align*}
   \]

3. We will also need the first derivatives of our functions. (If you do not see why, you will soon.) To find and
   assign the derivatives, right-click over the function and choose differentiate. Then right-click over the
   derivative function and choose assign to a name. Name the derivatives \( d\, f_1, d\, f_2, \) and \( d\, f_3 \), respectively.

4. Since our roller coaster consists of three curves, it can be set up mathematically as a piecewise-defined
   function:
   \[
   F(x) = \begin{cases}
   f_1(x), & 0 \leq x \leq 20 \\
   f_2(x), & 20 < x < 120 \\
   f_3(x), & 120 \leq x \leq 200
   \end{cases}
   \]
   We assign \( F \) as a function in Maple as follows:
   \[
   F:= x \rightarrow \text{piecewise}(x<20, f_1(x), x>20 \text{ and } x<120, f_2(x), x\geq120 \text{ and } x<200, f_3(x));
   \]
   Note: You can verify your piecewise-defined function by typing \( F(x) \);

5. Obviously, we want \( F(x) \) to be continuous (so our passengers do not perish). This means that our
   functions should be equal at transition points. So we get the following equations:
   \[
   \begin{align*}
   &\text{eq1:=} f_1(20)=f_2(20); \\
   &\text{eq2:=} f_2(120)=f_3(120);
   \end{align*}
   \]

6. If we are to have a smooth track, we cannot have abrupt changes in direction, so the first derivative
   \( F'(x) \) should also be continuous. That is, the first derivatives of our functions should also be equal at transition
   points. So we get:
   \[
   \begin{align*}
   &\text{eq3:=} d\, f_1(20)=d\, f_2(20); \\
   &\text{eq4:=} d\, f_2(120)=d\, f_3(120);
   \end{align*}
   \]

7. To start our landing at 40ft above the ground for the last 80ft, we would have:
   \[
   \begin{align*}
   &\text{eq5:=} f_3(120)=40; \\
   &\text{eq6:=} f_3(200)=0; \\
   &\text{eq7:=} d\, f_3(200)=0;
   \end{align*}
   \]

8. Finally, in order to have a soft landing, the track should be tangent to the ground at the end:
   \[
   \begin{align*}
   &\text{eq8:=} f_3(120)=40; \\
   &\text{eq9:=} d\, f_3(120)=0;
   \end{align*}
   \]

9. We now have a system of 7 equations and 7 unknowns. We solve using the solve command and assign
   the solutions as follows:
   \[
   \begin{align*}
   &\text{values:=solve}\left\{\text{eq1,eq2,eq3,eq4,eq5,eq6,eq7}\right\}, \{a,b,c,d,e,f,g\}; \\
   &\text{assign(values)};
   \end{align*}
   \]

10. You can view your completed piecewise-defined function by typing
    \[
    \text{F(x)};
    \]
    Note: If I were preparing a project report about this roller coaster, I would definitely include this function.

11. We can see what our coaster looks like with the following plot command:
    \[
    \text{plot(F(x), x=0..200, scaling=constrained);}
    \]
    Note: The option \texttt{scaling=constrained} scales the \texttt{x} and \texttt{y} dimensions equally.

12. To find the maximum height, we need to find where the graph has a horizontal tangent line (where
    \( F'(x) = 0 \)) and evaluate \( F(x) \) at each point. The largest is the maximum height of the coaster.
    \[
    \begin{align*}
    &\text{diff(F(x),x)}; \\
    &\text{dF:= x \rightarrow label;} \\
    &\text{solve(dF(x)=0, x)};
    \end{align*}
    \]
    Note: You can get this by right-clicking over \( F(x) \) above also.

Note: We know that \( F'(x) = 0 \) when we have a horizontal tangent line (a slope of 0). This occurs at both local
maximums and minimums.