MATH 122 PRETEST 2

1. Let \( R(x) \) represent the revenue (in dollars) from renting \( x \) apartments. Interpret each of the following statements using everyday language.
   a. \( R(14) = 32250 \)

   The company earns a revenue of \$32,250 when they rent 14 apartments.

   b. \( R'(14) = 2400 \)

   If the company has already rented 14 apartments, they will earn approximately \$2400 more if they rent one more apartment.

2. Suppose a company’s revenue from car sales, \( R \) (in thousands of dollars), is a function of advertising expenditure, \( a \) (in thousands of dollars), so \( R = f(a) \).
   a. What does the company hope is true about the sign of \( f'' \)? Why?

   The company hopes \( f'' \) is positive. This would mean that the company’s revenue increases as they spend more on advertising.

   b. What does the statement \( f'(50) = 2 \) mean in practical terms? Be sure to include units in your explanation.

   If the company has spent \$50,000 on advertising they can earn approximately \$2000 more in revenue if they spend an additional \$1000 on advertising.

3. Find the points where the function \( f(x) = 3x^3 + 7x^2 - 8x + 15 \) has a horizontal tangent line. Explain your reasoning.

   Horizontal tangent line \( \rightarrow \) slope = 0 \( \rightarrow f'(x) = 0 \)

   \[
   f'(x) = 9x^2 + 14x - 8 = 0 \\
   (9x - 4)(x + 2) = 0 \\
   9x - 4 = 0 \quad x + 2 = 0 \quad 1 \\
   x = \frac{4}{9} \quad x = -2
   \]
4. The following data represent the price and quantity demanded in 2004 for IBM personal computers. The price per computer is given in hundreds of dollars.

<table>
<thead>
<tr>
<th>Price, $p$</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>20</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity, $q$</td>
<td>189</td>
<td>180</td>
<td>176</td>
<td>171</td>
<td>164</td>
<td>159</td>
<td>152</td>
</tr>
</tbody>
</table>

a. Find the average rate of change in quantity demanded between $p = 10$ and $p = 23$.

\[
\frac{D(23) - D(10)}{23 - 10} = \frac{152 - 189}{13} = -2.846 \text{ computers/hundred dollars}
\]

\[D'(15) = -3\]

b. Estimate the instantaneous rate of change when $p = 15$. Interpret your answer.

\[
\frac{D(17) - D(15)}{17 - 15} = \frac{164 - 171}{2} = -3.5 \quad \text{or} \quad -3.5 + 2.5 = -1 \text{ computer/hundred dollars}
\]

If we increase the price from $1500 to $1600, the demand will decrease by approximately 3 personal computers.

5. Suppose that $f(x)$ is a function with $f(30) = 200$ and $f'(30) = -12$.

a. Estimate $f(32)$.

\[f(32) \approx 200 - 12 - 12 = 176\]

b. If the actual value is $f(32) = 180$, what does your answer in (a) tell you about the concavity of $f(x)$ close to $x = 30$?

The actual value is greater than our estimate (on the tangent line) so the graph must be concave up close to $x = 30$.

6. With a yearly inflation rate of 3%, prices are described by $P = P_0(1.03)^t$, where $P_0$ is the price in dollars when $t = 0$ and $t$ is time in years. If $P_0 = 1.2$, how fast are prices rising (in dollars/year) when $t = 15$?

\[P = 1.2(1.03)^t \quad \Rightarrow \quad P'(t) = 1.2(1.03)^t \ln(1.03)\]

\[P'(15) = 1.2(1.03)^{15} \ln(1.03) = 0.0553 \text{ dollars/year}\]
7. Suppose a function is given by a table of values as follows:

<table>
<thead>
<tr>
<th>x</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
<th>1.7</th>
<th>1.9</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>12</td>
<td>15</td>
<td>21</td>
<td>23</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>

a. Estimate $f''(1.7)$.

\[
\frac{f'(1.9)-f'(1.7)}{1.9-1.7} = \frac{24-23}{0.2} = 5
\]

\[
\frac{f'(1.7)-f'(1.5)}{1.7-1.5} = \frac{23-21}{0.2} = 10
\]

\[
\frac{5+10}{2} = 7.5
\]

$\boxed{f''(1.7) \approx 7.5}$

b. Write an equation of the tangent line to $f$ at $x = 1.7$.

Point $(1.7, 23)$ \hspace{1cm} $y - 23 = 7.5(x - 1.7)$

Slope $m = 7.5$ \hspace{1cm} $y - 23 = 7.5x - 12.75$

$\boxed{y = 7.5x + 10.25}$

8. At a production level of 2000 for a product, marginal revenue is $4 per unit and marginal cost is $3.25 per unit. Do you expect maximum profit to occur at a production level above or below 2000? Explain your answer.

At a production level of 2000 items, marginal revenue exceeds marginal cost, creating a marginal profit of $0.75. Since profit is increasing, we expect maximum profit to occur at a production level higher than 2000 items.

9. Fill in the blank with positive, negative, increasing, decreasing, concave up, concave down, or unknown.

If $f'(x)$ is increasing then $f(x)$ is concave up

If $f(x)$ is increasing then $f'(x)$ is positive.

If $f''(x)$ is negative then $f'(x)$ is decreasing.

If $f'(x)$ is decreasing then $f''(x)$ is negative.
10. Given the following graph of $f(x)$, find the graph of $f'(x)$.

\[
\begin{array}{|c|c|c|}
\hline
\text{Interval} & f(x) & f'(x) \\
\hline
(-\infty,-2) & \text{Decreasing} & - \\
\text{x}=-2 & \text{Min} & 0 \\
(-2,0) & \text{Increasing} & + \\
\text{x}=0 & \text{Max} & 0 \\
(0,2) & \text{Decreasing} & - \\
\text{x}=2 & \text{Min} & 0 \\
(2,\infty) & \text{Increasing} & + \\
\hline
\end{array}
\]

11. Given the following graph of $f'(x)$, find the intervals over which $f(x)$ is increasing, decreasing, concave up, and concave down.

$f(x)$ increasing $(-9,6) \cup (15,\infty)$
$\Rightarrow f''(x)$ positive

$f(x)$ decreasing $(-\infty,-9) \cup (6,15)$
$\Rightarrow f''(x)$ negative

$f(x)$ concave up $(-\infty,-3) \cup (11,\infty)$
$\Rightarrow f''(x)$ positive $\Rightarrow f'(x)$ increasing

$f(x)$ concave down $(-3,11)$
$\Rightarrow f''(x)$ negative $\Rightarrow f'(x)$ decreasing

At what point(s) does $f(x)$ have a local maximum? How can you tell?

$f'(x) = 0$
$f''(x) < 0$

$f(x)$ has a maximum when $x=6$. We can tell because $f'(6)=0$ and $f''(x)$ changes from positive to negative when $x=6$. 
12. Find the equation of the tangent line to the graph of \( f(x) = \frac{x^2 - 2}{x+1} \) at the point at which \( x = 1 \).

Point \((1, -0.5)\)  
Slope \( m = 1.25 \)  
\[ y - (-0.5) = 1.25(x - 1) \]

\[ y + 0.5 = 1.25x - 1.25 \]  
\[ y = 1.25x - 1.75 \]

13. The price in dollars of a house during a period of mild inflation is described by the formula \( P(t) = 80000e^{0.05t} \), where \( t \) is the number of years after 1990.

a. What is the value of the house in the year 2000?

\[ P(10) = 80000e^{0.05(10)} \]
\[ = \boxed{131897.70} \]

b. At what rate (in dollars/year) will the value of the house be increasing in the year 2000?

\[ P'(t) = 80000e^{0.05t} \cdot (0.05) \]
\[ = 4000e^{0.05t} \]
\[ P'(10) = 4000e^{0.05(10)} \]
\[ = \boxed{6594.89/\text{year}} \]

c. How long will it take the house to double in value?

\[ 160000 = 80000e^{0.05t} \]
\[ 2 = e^{0.05t} \]
\[ \ln 2 = 0.05t \]
\[ t = \frac{\ln 2}{0.05} \]
\[ t = 13.863 \text{ years} \]

d. When will the house be increasing in value at a rate of \$10,000 per year?

\[ P'(t) = 10000 \]
\[ 4000e^{0.05t} = 10000 \]
\[ e^{0.05t} = 2.5 \]
\[ \ln 2.5 = 0.05t \]
\[ t = \frac{\ln 2.5}{0.05} \]
\[ t = 18.326 \]
14. Find the derivative $f'(x)$.

\[
f(x) = 7\ln(x) + 5^x - \frac{2}{3x^4} \rightarrow f''(x) = 7 \left( \frac{1}{x} \right)^2 + 5^x \ln 5 + \frac{8}{3}x^{-5}
\]

15. Find the derivative $f'(x)$.

\[
f(x) = 7^{4x^2 + 5x - 1} \rightarrow f''(x) = 7^{4x^2 + 5x - 1} \ln 7 (8x + 1)
\]

16. Find the derivative $f'(x)$.

\[
f(x) = \frac{x^5 + e^{2x}}{x^3 - x} \rightarrow f''(x) = \frac{(x^3 - x)(5x^4 + e^{2x}) - (x^5 + e^{2x})(3x^2 - 1)}{(x^3 - x)^2}
\]

17. Find the derivative $f'(x)$.

\[
f(x) = (x^9 - 3x^4 + 4)^7(\ln(x) + 3x)^3 \rightarrow f''(x) = (x^9 - 3x^4 + 4)^2 \cdot 3(\ln(x) + 3x)^2 \left( \frac{1}{x} + 3 \right) + (\ln(x) + 3x)^3 \cdot 7(x^9 - 3x^4 + 4)^6 \cdot 9x^8 - 12x^3)
\]
18. Find the derivative $f'(x)$.

$$f(x) = e^{x^5 - 7} + \ln(x^4 - 5x^3 + 2)$$

$$f'(x) = e^{x^5 - 7} \left( 5x^4 + \frac{1}{x^4 - 5x^3 + 2} \right) (4x^3 - 15x^2)$$

19. Find the derivative $f'(x)$.

$$f(x) = (\sqrt[3]{x} - 4x^6 + 2x^2)^{15}$$

$$f'(x) = 15 \left( \sqrt[3]{x} - 4x^6 + 2x^2 \right)^{14} \left( \frac{1}{3x^{2/3}} - 24x^5 + 4x \right)$$

20. Find the derivative $f'(x)$.

$$f(x) = \left( x + \frac{1}{x^2} \right) \left( \frac{3x + 1}{x^2 - 1} \right)$$

$$f'(x) = \left( x + \frac{1}{x^2} \right) \left( \frac{(x - 1)(3) - (3x + 1)(2x)}{(x^2 - 1)^2} \right) + \left( \frac{3x + 1}{x^2 - 1} \right) (1 - 7x^{-8})$$

21. Find the derivative $f'(x)$.

$$f(x) = \left( \frac{x^3 + 1}{3x^3 + x} \right)^4$$

$$f'(x) = 4 \left( \frac{x^2 + 1}{3x^3 + x} \right)^3 \left( \frac{(3x^3 + x)(2x) - (x^2 + 1)(9x^2 + 1)}{(3x^3 + x)^2} \right)$$
22. Determine the critical points and identify each as a local minimum, local maximum or neither.

\[ f(x) = \frac{1}{3}x^3 - \frac{13}{2}x^2 - 30x + 15 \]

\[ f''(x) = x^2 - 13x - 30 \]

\( (x - 15)(x + 2) = 0 \)

\( x = 15 \quad x = -2 \)

23. Determine the intervals over which \( f(x) \) is increasing, decreasing, concave up, and concave down.

\[ f(x) = x^3 - 6x^2 + 9x + 3 \]

\[ f''(x) = 3(x - 3)(x - 1) \]

Increasing/Decreasing

\( f'(x) = 3x^2 - 12x + 9 \)

\( 3(x^2 - 4x + 3) = 0 \)

\( 3(x - 3)(x - 1) = 0 \)

\( x = 3 \quad x = 1 \)

Concavity

\( f''(x) = 6x - 12 \)

\( 6x - 2 = 0 \quad x = \frac{1}{3} \)

Increasing \((-\infty, 1) \cup (3, \infty)\)

Decreasing \((1, 3)\)

Concave up \((2, \infty)\)

Concave down \((-\infty, 2)\)

24. Determine the global maximum and global minimum of \( f(x) \) over the given interval.

\[ f(x) = 2x^3 - 3x^2 - 12x + 1 \text{ over } [-2, 4] \]

\[ f'(x) = 6x^2 - 6x - 12 \]

\( 6(x^2 - x - 2) = 0 \)

\( 6(x - 2)(x + 1) = 0 \)

\( x = 2 \quad x = -1 \)

Critical Points: -1, 2

Compare: \[ f(-2) = -3 \]

\[ f(-1) = 8 \]

\[ f(2) = -19 \text{ global minimum} \]

\[ f(4) = 33 \text{ global maximum} \]