2.2 Projective Resolutions

**Def.** An object $P$ in an abelian category $A$ is projective if it satisfies the following: for each surjection $g: B \to C \to 0$, $x: P \to C$, we have

\[
\begin{array}{c}
P \\ \downarrow \rho \\
\downarrow \left\downarrow \rightdownarrow \end{array}
\]

\[
B \to C \to 0
\]

**Rem.**
1. $P$ may not be unique.
2. $P$ is projective $\iff \text{Hom}(P, -)$ is right exact.

**Prop.** An $R$-module is projective $\iff$ it's a direct summand of a free $R$-module.

(2.2.1) $R$-module $\Rightarrow$

**Proof:**

\[
\begin{array}{c}
P \\ \downarrow \text{id} \\
F \xrightarrow{\pi} P \to 0
\end{array}
\]

so $P$ is a summand of $F$

\[
F \xrightarrow{\pi} P \to 0
\]

**Suppose** $F = P \oplus Q$.

\[
P \oplus Q
\]

\[
\begin{array}{c}
P \oplus Q \\ \downarrow \iota \\
F \\ \downarrow \rho \\
P \\ \downarrow \pi \\
B \to C \to 0
\end{array}
\]

Easy to check: $f = \rho \pi$.
E.g.: Over some rings (like \( \mathbb{Z} \), \( \mathbb{R} \), division rings), projective \( \Rightarrow \) free.

(2.2)

However, it's not always true.

1. If \( R = R_1 \oplus R_2 \), then \( P = R_1 \oplus R_2 \) (summands) are projective,
   but they are not free, since \( (0,1)P = 0 \).

2. Let \( R = M_n(k) \), \( V = \mathbb{R}^n \), then \( \text{dim}_R R = n^2 \), \( \text{dim}_R V = n \).
   Moreover, \( V \) is an \( R \)-module, where \( R = V \oplus R^n \).
   Thus \( V \) is \( R \)-projective.

Claim: \( V \) is not \( R \)-free. If not, \( aV \cong \mathbb{R}^d \) (also \( aV \cong \mathbb{R}^d \))

So \( \text{dim}_R V = d \cdot n^2 \), but \( \text{dim}_R V = n \), a contradiction.

Remark: The category \( A \) of finite abelian groups is an example of an

abelian category with no \( \mathbb{Z} \)-projective objects. (No free objects)

We say an abelian category \( A \) has enough projectives if for
every object \( A \) of \( A \), there is a surjection \( P \twoheadrightarrow A \) with \( P \) projective.

Def: A chain complex \( P \) with each \( P_n \) projective is called a

chain complex of projectives. Note: \( P \) may not be a projective
object of \( Ch(A) \).

E.g.: A chain complex \( P \) is a projective object in \( Ch \) \( \iff \) it's a

split exact complex of projectives.

(2.2.1)

\[ P^2: \]

\[ \Rightarrow: \text{claim 1: every } P_n \text{ is projective.} \]

\[ Pt: \text{ consider } 0 \to B_0 \to C_0 \to \cdots \]

\[ \to 0 \to C_n \to 0 \to \cdots \]

\[ \to 0 \to 0 \to 0 \to \cdots \]

So \( P_n \) is projective.

Claim 2: \( P \) is split exact.

\[ Pt: \text{ consider the surjection } \text{cone} (P) \to P[-1] \to 0 \]

\[ P \text{ is projective so is } P[-1]. \]

\[ P[-1]: \]

\[ 0 \to P \to \text{cone}(P) \to P[-1] \to 0 \]

So \( \text{cone}(P) \cong P \oplus P[-1] \).

\[ \Leftrightarrow: P \text{ is split exact, so } P_n \cong B_n \oplus B_{n-1}, \text{ and } B_n \text{ are projective} \]

Let \( P(n) \) be the chain complex: \( \cdots \to 0 \to B_{n-1} \to B_n \to 0 \to \cdots \)

\[ \Rightarrow: P \cong P[n]. \]

Consider the maps

\[ X \twoheadrightarrow Y \to 0, \]

Where \( X = \bigoplus \mathbb{T}_n \)

For each \( n \), \( P(n) \) is a split exact chain complex of projectives.

So projective, thus we get \( \bar{f}_n : P(n) \to X \).

Define \( f = \bigoplus \bar{f}_n \), we get \( \overline{\overrightarrow{f}} = f \).
E.x If \( A \) has enough projectives, then so does \( \text{Ch}(A) \).

(2.2.2) \( \text{Pr} : \)

\[
\begin{align*}
P_n & \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \\
& \xrightarrow{\partial_n} B_n \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0
\end{align*}
\]

First, we get \( d_0 : P_n \rightarrow P_{n-1} \).

Second, we want to get a split exact complex from \(-\bullet\rightarrow P_0 \rightarrow \cdots \rightarrow P_n \rightarrow P_{n+1} \rightarrow \cdots \).

Indeed, cone \((P, 0)\) works.

\[
\begin{align*}
P_n \oplus P_0 & \xrightarrow{(x, 0)} P_{n+1} \oplus P_{n+1} \\
& \xrightarrow{(d_{n+1}, 0)} P_{n+2} \oplus P_{n+2} \\
& \xrightarrow{\partial_{n+1}} B_{n+1} \oplus B_{n+1} \\
& \xrightarrow{\partial_{n+1}} B_{n+2} \oplus B_{n+2}
\end{align*}
\]

Def: Let \( M \) be an object of \( A \). A left resolution of \( M \) is an exact complex \(-\bullet \rightarrow P_0 \rightarrow \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow \text{O} \). It's a projective resolution if each \( P_i \) is projective.

(2.2.4)

Lemma: If an Abelian category \( A \) has enough projectives, then every object \( M \) in \( A \) has a projective resolution.

(2.2.5)

Pt:

\[
\begin{align*}
P_1 & \xrightarrow{d_1} P_0 \\
& \xrightarrow{\partial_1} B_0
\end{align*}
\]

E.x If \( P_i \) is a chain complex of projectives with \( \partial_i = 0 \) for \( i < 0 \), then a map \( E : P_i \rightarrow M \) giving a resolution for \( M \) is the same thing as a quasi-isomorphism \( E : P_\bullet \rightarrow M_\bullet \) where \( M_\bullet \) is the complex concentrated in degree 0.

(2.2.3) \( \text{Pr} : \)

\[
\begin{align*}
P_n & \rightarrow \cdots \rightarrow P_0 \\
& \rightarrow \text{cons. map} \rightarrow M \rightarrow \text{O}
\end{align*}
\]

is equivalent to say \( b_i = 0 \) for \( i \geq 0 \), and \( P_i / \text{Im} \partial_i \cong M_i \), which is equivalent to say the following chain map

\[
\begin{align*}
P_n & \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \\
& \rightarrow \text{cons. map} \rightarrow M \rightarrow \text{O}
\end{align*}
\]

is a quasi-isomorphism.

Thm: (Comparison Theorem) Let \( P_\bullet \rightarrow M \) be a projective resolution of \( M \) and \( f : M \rightarrow N \) a map in \( A \). Then for every resolution \( Q_\bullet \rightarrow N \), there is a chain map \( f : P_\bullet \rightarrow Q_\bullet \) such that \( g \circ f = f' \). The chain map is unique up to chain homotopy equivalence.

(2.2.6)

Pt:

\[
\begin{align*}
P_1 & \xrightarrow{d_1} P_0 \\
& \xrightarrow{\partial_1} B_0
\end{align*}
\]

1. Suppose we have \( + \) and \( g \) satisfying the above conditions.

\[
\begin{align*}
g f_0 &= f' \\
h_0 &= h'
\end{align*}
\]

So we have \( P_0 \rightarrow B_0 \rightarrow M \rightarrow O \).

Similarly, get \( S_2 \) and \( S_1 \).
Lemma: (Horseshoe Lemma) Suppose we have the following projective resolutions:

\[ \cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow P_0 \rightarrow A \rightarrow 0 \]

\[ \cdots \rightarrow Q_i \rightarrow Q_{i-1} \rightarrow Q_0 \rightarrow A' \rightarrow 0 \]

Then \( P \oplus Q \) is a projective resolution of \( A \).

\[ P : \]

\[ P_i \rightarrow A \rightarrow 0 \quad P_i \oplus Q_i \text{ is projective} \]

\[ \text{since it's a summand} \]

\[ \text{of a free module} \]

First, let's define \( d_0 : P_0 \oplus Q_0 \rightarrow A \)

\[ \pi : A \rightarrow A' \text{ is surjective, } Q_0 \text{ is projective} \Rightarrow \exists \gamma : Q_0 \rightarrow A. \]

Define \( d_0 : P_0 \oplus Q_0 \rightarrow A \) as \( d_0 = I \circ \gamma \).

Easy to check \( d_0 \) is surjective, and the commutativity.

Next, we can get the following diagram:

\[ \cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow P_0 \rightarrow A \rightarrow 0 \]

\[ P_i \oplus Q_i \rightarrow A \rightarrow 0 \]

\[ P \oplus Q \rightarrow A \rightarrow 0 \]

Similarly, we get \( d_1 \) and \( d_1 \) is surjective, i.e., \( P \oplus Q \) is exact at \( P_0 \oplus Q_0 \). Repeat this, we get a projective resolution \( P \oplus Q \) of \( A \).

2.3 Injective Resolutions

Def: An object \( I \) in an abelian category \( A \) is injective if it satisfies the following universal lifting property: Given an injection \( f : A \rightarrow B \) and a map \( i : A \rightarrow I \), we have \( B \rightarrow A \rightarrow I \).

We say \( A \) has enough injectives if for every object \( A \) in \( A \), there is an injection \( A \rightarrow I \) with \( I \) injective.

Prop: If \( \{ I_\alpha \} \) is a family of injectives, then \( \bigoplus I_\alpha \) is also injective.

\[ P : \]

\[ 0 \rightarrow A \rightarrow B \]

\[ \pi_I \alpha \]

\[ I_\alpha \]

(2.3.1) Baer's Criterion: A right \( R \)-module \( E \) is injective if for every right ideal \( J \) of \( R \), every map \( J \rightarrow E \) extends to a map \( R \rightarrow E \).
Cor: An \( \mathbb{Z} \)-module \( M \) (more generally, a PID-module \( M \)) is injective if and only if \( M \) is divisible.

(2.3.1) $\Rightarrow$ \( M \) is divisible.

Proof: Consider \( 0 \rightarrow (n) \rightarrow \mathbb{Z} \).

Lemma: Every abelian group \( G \) can be embedded into a divisible group.

Proof: If \( G \) is free, then \( G = \bigoplus \mathbb{Z} \), let \( D = \bigoplus \mathbb{Q} \), then \( D \) is divisible and \( 0 \rightarrow G \rightarrow D \).

If \( G \) is not free, then we have \( 0 \rightarrow k \rightarrow F \rightarrow G \rightarrow 0 \) so \( G = \overline{F/k} \leftrightarrow \overline{D/k} \), where \( D \) is divisible.

\( D \) is divisible \( \Rightarrow \overline{D/k} \) is divisible.

Lemma: Let \( D \) be an \( \mathbb{Z} \)-injective module, \( R \) a ring, then \( \text{Hom}_R(R, D) \) is an \( R \)-injective module.

Proof: \( 0 \rightarrow I \rightarrow R \)

\[ \begin{array}{c}
\text{Let } x, y \in M, \text{ then } x = ny.
\end{array} \]
Thm: R-mod has enough injectives, i.e., every R-module can be embedded into an injective R-module.

Pt: \[
M \hookrightarrow \text{Hom}_R(R,M) \hookrightarrow \text{Hom}_R(R,M) \hookrightarrow \text{Hom}_R(R,0)
\]

Rmk: The above theorem is the only one that we can not get from dualizing the projective result.

Def: A pair of functors \( L: A \to B \) and \( R: B \to A \) are adjoint if there is a natural bijection for all \( A \) in \( A \) and \( B \) in \( B \):\[
\text{Nat} : \text{Hom}_A(A, R(B)) \rightleftharpoons \text{Hom}_B(L(A), B)
\]
Here, "natural" means for \( f: A \to A' \) and \( g: B \to B' \):
\[
\text{Hom}_A(A', R(B)) \to \text{Hom}_A(A, R(B)) \to \text{Hom}_A(A, R(B')) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Hom}(L(A), B) \to \text{Hom}(L(A), B) \to \text{Hom}(L(A), B')
\]
We call \( L \) the left adjoint and \( R \) the right adjoint.

Eg:
- Forgetful functor: \( R\text{-mod} \to \text{Ab} \) is left adjoint
- \( \text{Hom}(R, -) : \text{Ab} \to R\text{-mod} \) is right adjoint

Prop: Suppose \( A, B \) are abelian categories, \( (L, R) \) is an adjoint pair, \( L: A \to B \) is exact, and \( I \) is an injective object in \( B \).
Then \( R(I) \) is an injective object in \( A \).
Pt: It suffices to prove \( \text{Hom}_A(-, R(I)) \) is right exact.
\[
\begin{align*}
0 & \to A \to A' \\
\Rightarrow & \ 0 \to L(A) \to L(A') \\
\Rightarrow & \ \text{Hom}(L(A'), I) \to \text{Hom}(L(A), I) \to 0 \\
\Rightarrow & \ \text{Hom}(A', R(I)) \to \text{Hom}(A, R(I)) \to 0
\end{align*}
\]