Schur Functors
(a project for class)

R. Vandermolen

1 Introduction

In this presentation we will be discussing the Schur functor. For a complex vector space V, the Schur functor gives many irreducible representations of $GL(V)$, and other important subgroups of $GL(V)$, it will not be the purpose of this presentation to give this deep of an explanation. We will be concerning ourselves with the basic concepts needed to understand the Schur functor, and finish off by showing that the Schur functor does indeed help us decompose the nth tensor products $V^{\otimes n}$.

We will, for the most part, follow the presentation given in [1], and in turn we will use German gothic font for the symmetric group, and use diagrams in many occasions. With this said, we will not reprove major theorems from [1], instead we will focus our efforts on "filling in" the details of the presentation. We create new lemmas and deliver proofs of these new lemmas in efforts to deliver a fuller exposition on these Schur functors.

Before we can begin a discussion of the *Schur functor* (or *Weyl's module*), we will need a familiarity with multilinear algebra.

1.1 Preliminaries

Now we will deliver the definitions that will be of most interest to us in this presentation. We begin with the definition of the n-th tensor product.

Definition 1. For a finite dimensional complex vector space V, and an natural number n, we define the ***complex n-th tensor product*** as follows

$$V^{\otimes n} := F(V^n)/\sim$$

where $F(V^n)$ is the free abelian group on V^n, considered as a set, and $\sim$ is the equivalence relation defined for $v_1, ..., v_n, \hat{v}_1, ..., \hat{v}_n, w_1, ..., w_n \in V$, as

$$(v_1, ..., v_n) \sim (w_1, ..., w_n)$$

whenever there exists a $k \in \mathbb{C}$ such that $k(v_1, .., v_n) = (w_1, ..., w_n)$, and

$$(v_1, .., v_i, ...v_n) + (v_1, ..., \hat{v}_i, ..., v_n) \sim (v_1, .., v_i + \hat{v}_i, .., v_n)$$

One can easily verify that this is indeed an equivalence relation.

Next we will use the following definitions of some interesting subvector spaces of the n-th tensor product, which will be of the most interest in this paper.

Definition 2. For a complex finite dimensional vector space V, and a natural number n, we define the ***complex n-th symmetric space*** denoted $\mathrm{Sym}^n(V)$, as the subspace of $V^{\otimes n}$, spanned by the following set,

$$\left\{ \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \middle| v_i \in V \right\}$$

and denote a generator $\sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} = v_1 \cdots v_n$.

Definition 3. For a finite dimensional complex vector space V, and an natural number n, we define the ***complex n-th alternating space*** denoted $\bigwedge^n V$, as the subspace of $V^{\otimes n}$, spanned by the following set

$$\left\{ \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \middle| v_i \in V \right\},$$

and denote a generator $\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} = v_1 \wedge \cdots \wedge v_n$.

The experienced and curious reader may be asking themselves why we chose these definitions, rather than more accepted ones, while these definitions, over $\mathbb{C}$, are equivalent it takes some work to show this non-trivial fact, so to avoid the discussion we chose these definitions. Finally, we will define the strongest tool we will use in this presentation.

Definition 4. For a group G, we define the ***complex group algebra of*** G denoted as $\mathbb{C}G$ as, the $|G|$-dimensional complex vector space with the canonical basis indexed by elements of G, that is for $g \in G$ we have a basis element $e_g \in \mathbb{C}G$, where we define the multiplication in $\mathbb{C}G$ on basis elments, for $g_1, g_2 \in G$ as

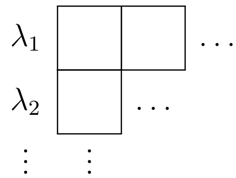
$$e_{g_1} \cdot e_{g_2} = e_{g_1 \cdot g_2}$$

and expand the definiton linearly for the remaining elements. One easily sees that if G is non-abelian then so is $\mathbb{C}G$ and if $|G| = \infty$ then $\mathbb{C}G$ has inifinte dimension.

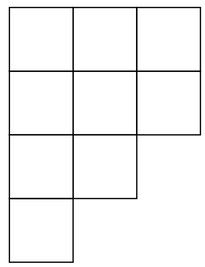
Next, we will discuss partitions of finite sets.

2 Young Tableau

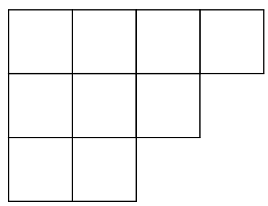
In this section we will discuss the Young tableau as a method of identifying special subgroups of the symmetric group $\mathfrak{S}_n$. We recall, for a partition $\lambda = (\lambda_1, \dots, \lambda_d)$ (we will always follow the convention that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 1$), where $\sum_{i=1}^{d} \lambda_i = n$, we may represent this partition as a **Young diagram**,



with λ_i boxes on the ith row aligned to the left. The **conjugate partition** is the partition $\lambda' = (\lambda'_1, ...)$ defined by swapping the rows and the columns in the Young diagram. For a specific example consider the partition $(3, 3, 2, 1)$ of a 9 element set is drawn as follows:



then the conjugate partition is (4,3,2), that is the following Young diagram.



A **Young Tableau** is a labeling a Young diagram with the numbers $\{1, ..., n\}$, for our example the canonical labeling will be,

1	2	3
4	5	6
7	8	
9		

in general the canoical labeling is labiling the diagram from left to right with the top row 1-λ_1 etc. *we will only consider the canonical labeling*, since for our use any labeling creates isomorphic representations, which we will note when it becomes relevant. Now while Young diagrams and tableaus are a rich area of research, we have all that we will need to proceed.

3 Young's Symmetrizer

Given a Young tableau (that is a partition λ) we can construct the following two subgroups of the group of permutations, $\mathfrak{S}_n$

$$P = P_\lambda = \{g \in \mathfrak{S}_n : g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_\mathfrak{n} : g \text{ preserves each column}\}.$$

Let us now take a moment to, describe what we mean by "preserving a column", (respecitvely "preserving a row") by this we mean that each column (resp. row) when considered as a subset of $\{1, ..., n\}$, (with its labeling) remains stable under the action by Q (resp. P).

We now take a moment to prove the following necessary Lemma.

Lemma 3.1. With the definition above P and Q are both subgroups of $\mathfrak{S}_n$, in particular, $P \cong \bigoplus_{i=1}^{k} \mathfrak{S}_{\lambda_i}$, and $Q \cong \bigoplus_{i=1}^{k'} \mathfrak{S}_{\lambda'_i}$, where k' denotes the number of columns in the Young's diagram.

Proof. First let $\lambda = (\lambda_1, .., \lambda_k)$ be a partition of $n \in \mathbb{N}$, and consider the canonical Young tableau. Now recall all that needs to be done to prove that they are subgroups it suffices verify for $a, b \in P$ then $ab^{-1} \in P$ (this is a standard result in group theory). Yet, this is evident since when $x \in \{1, .., n\}$, and say x is in row i then $b^{-1}(x)$ remains in row i since b has this property and thus $ab^{-1}(x)$ remains in row i, since a has this property, and hence $ab^{-1} \in P$ (and similarly for Q).
To see that last conclusion of the lemma just note that the stabilizer of row i is subgroup of $\mathfrak{S}_n$, and that P is generated by all of these stabilizers (similar for column i and Q). ☐

With these two subgroups we can now define the operators that we are most interested in. Consider the group algebra $\mathbb{C}\mathfrak{S}_n$, and consider the elements

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \operatorname{sgn}(g) e_g.$$

We define for a vector space V the canonical action of $\mathfrak{S}_n$ on the nth tensor product, $V^{\otimes n}$, for $v_1 \otimes ... \otimes v_n$ and $g \in \mathfrak{S}_n$ as $g(v_1 \otimes ... \otimes v_n) = v_{g(1)} \otimes ... \otimes v_{g(n)}$, that is permute the indices, and then expand the definition linearly. The informed reader may be concerned that we did not use the inverse when we permute, yet there is an equivalent definition using the inverse on the coefficients, yet not on the basis elements themselves. We now can interpret a_λ and b_λ as operators on $V^{\otimes n}$. Now, we can look at their respective images, described in the following lemma.

Lemma 3.2. Let V be a finite dimensional complex vector space, λ be a partition of $n \in \mathbb{N}$, and consider the canonical Young tableau. Then

$$a_\lambda(V^{\otimes n}) \cong \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2}(V) \otimes ... \otimes \operatorname{Sym}^{\lambda_d} V$$

and

$$b_\lambda(V^{\otimes n}) \cong \bigwedge^{\lambda'_1} V \otimes \bigwedge^{\lambda'_2} V \otimes ... \otimes \bigwedge^{\lambda'_{d'}} V$$

where λ' is the conjugate partition to λ.

Proof. This follows from Lemma 3.1 and the definition of the symmetric and alternating tensor. ☐

Now, we can define the *Young symmetrizer* as,

$$c_\lambda = a_\lambda b_\lambda \in \mathbb{C}\mathfrak{S}_n.$$

The astute reader may be asking themselves, "does the choice of Young's Tableau make a difference on how these operators work?" The answer turns out to be no! If one finds themselves bewildered by this answer, do not fret just recall the basic result from linear algebra, $V_1 \otimes V_2 \cong V_2 \otimes V_1$, and if this is still not settling the troubling stomach of the reader we provide a helpful example at this point.

Example 3.3.

For an explicit example, consider when $n = 3$, consider the partition (2,1), and consider the canonical Young tableau, on this partition,

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

we can then explicitly describe both P and Q, as

$$P = \{(), (12)\} \cong \mathfrak{S}_2$$

and

$$Q = \{(13), ()\} \cong \mathfrak{S}_2$$

where () denotes the identity element in $\mathfrak{S}_3$ (and we point out the above isomorphisms to observe Lemma (3.1) in action). Thus we see that

$$a_\lambda = e_{()} + e_{(12)}$$

and

$$b_\lambda = e_{()} - e_{(13)}.$$

We will begin by an exploration of the action of a_λ, which is very natural with the canonical labeling of the Young's tableau, we will see an example of the "extra work" that needs to be done to gain isomorphic copies when we do not work in the canonical labeling when we examine b_λ below.

Now let $V = \mathbb{C}$ and consider the nth tensor $V^{\otimes 3}$, let $v = v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$, and let us see the action of a_λ on v.

$$a_\lambda(v) = v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3$$

So the image, $a_\lambda(V^{\otimes 3})$ is generated by elements of the above form. Now by Lemma (3.2) we are suppose to have,

$$a_\lambda(V^{\otimes n}) \cong \mathrm{Sym}^2(V) \otimes V.$$

To see this explicitly we define the natural isomorphism $\varphi : V^{\otimes 3} \to V^{\otimes 2} \otimes V$, defined for a pure tensor $v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$ as $\varphi(v_1 \otimes v_2 \otimes v_3) = (v_1 \otimes v_2) \otimes v_3$. Next, we consider the canonical imbedding $\iota : \mathrm{Sym}^2(V) \otimes V \to V^{\otimes 2} \otimes V$. By composing on a generator we see the following,

$$\begin{aligned}\varphi^{-1}(\iota(v_1 \cdot v_2 \otimes v_3)) &= \varphi^{-1}((v_1 \otimes v_2 + v_2 \otimes v_1) \otimes v_3) \\ &= v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 \\ &= a_\lambda(v_1 \otimes v_2 \otimes v_3).\end{aligned}$$

Thus, we can define the homomorphism $\vartheta : a_\lambda(V^{\otimes 3}) \to \mathrm{Sym}^2(V) \otimes V$, as $\vartheta(a_\lambda(v_1 \otimes v_2 \otimes v_3)) = v_1 v_2 \otimes v_3$, and hence from the preceding calculation we have that the following diagram commutes,

$$\begin{tikzcd} \mathrm{Sym}^2(V) \otimes V \arrow[r, "\iota"] & V^{\otimes 2} \otimes V \arrow[d, "\varphi^{-1}"] \\ a_\lambda(V^{\otimes 3}) \arrow[u, "\vartheta"] & V^{\otimes 3} \arrow[l, "a_\lambda"] \end{tikzcd}$$

and hence we have our desired isomorphism.

Next, the examination of b_λ is a bit trickier, and this consideration shows us an example why choosing another tableau other than the canonical one gives us isomorphic results, since b_λ relies on the conjugate table, and hence is in itself not the canonical labeling. So we begin by examining b_λ, acting on v,

$$b_\lambda(v) = v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1$$

so the image of $b_\lambda(V^{\otimes 3})$ is generated by elements of the above form. Now by Lemma (3.2) we are suppose to have,

$$b_\lambda(V^{\otimes n}) \cong \bigwedge^2(V) \otimes V.$$

This time to see this explicitly, we will need to use the "less natural" isomorphism $\psi : V^{\otimes 3} \to V^{\otimes 2} \otimes V$ defined as $\psi(v_1 \otimes v_2 \otimes v_3) = (v_1 \otimes v_3) \otimes v_2$. This time we again consider the canonical imbedding $i : \bigwedge^2(V) \otimes V \to V^{\otimes 2} \otimes V$, hence by composing we arrive at the following calculation,

$$\begin{aligned}\psi^{-1}(i((v_1 \wedge v_2) \otimes v_3)) &= \psi^{-1}((v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_3) \\ &= v_1 \otimes v_3 \otimes v_2 - v_2 \otimes v_3 \otimes v_1 \\ &= b_\lambda(v_1 \otimes v_3 \otimes v_2)\end{aligned}$$

Thus, we can define the homomorphism $\gamma : b_\lambda(V^{\otimes 3}) \to \bigwedge^2(V) \otimes V$, as $\gamma(b_\lambda(v_1 \otimes v_2 \otimes v_3) = (v_1 \wedge v_3) \otimes v_2$, and hence from the preceding calculation we have that the following diagram commutes,

$$\begin{tikzcd} \bigwedge^2(V) \otimes V \arrow[r, "i"] & V^{\otimes 2} \otimes V \arrow[d, "\psi^{-1}"] \\ b_\lambda(V^{\otimes 3}) \arrow[u, "\gamma"] & V^{\otimes 3} \arrow[l, "b_\lambda"] \end{tikzcd}$$

and hence we have our desired isomorphism. Here the author would like to note that we see that no matter the labeling, we obtain our isomorphic images with the use of an isomorphic twisting of $V^{\otimes n}$, in this example, specifically it was the map ψ.

Finally we explore the action of Young Symmetrizer,

$$c_\lambda = a_\lambda b_\lambda = e_{()} + e_{(12)} - e_{(13)} - e_{(132)}$$

on v, that is

$$c_\lambda(v) = v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2$$

we will wait till the next section to describe this image.

4 Schur Functor

We will denote the image of the Young symmetrizer on $V^{\otimes n}$ as $\mathbb{S}_\lambda V$, that is

$$\mathbb{S}_\lambda V = c_\lambda(V^{\otimes n}).$$

We may consider $\mathbb{S}_\lambda : \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$, as a functor from the category of complex vector spaces to itself. We call this functor the **Schur functor** corresponding to λ. To see that this is a well defined functor recall that the functors $\bigwedge^n : \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$, $\mathrm{Sym}^n : \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$, and $\bigotimes^n : \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$, are all well defined functors. Thus for any vector spaces U, V, W and $\mathbb{C}$-linear maps, $\varphi : V \to W$, $\psi : W \to U$ we have, that the following diagram commutes.

$$\begin{tikzcd}
V^{\otimes n} \arrow[r, "\varphi^{\otimes n}"] \arrow[d, "b_\lambda"] & W^{\otimes n} \arrow[r, "\psi^{\otimes n}"] \arrow[d, "b_\lambda"] & U^{\otimes n} \arrow[d, "b_\lambda"] \\
b_\lambda(V^{\otimes n}) \arrow[r] \arrow[d, "a_\lambda"] & b_\lambda(W^{\otimes n}) \arrow[r] \arrow[d, "a_\lambda"] & b_\lambda(U^{\otimes n}) \arrow[d, "a_\lambda"] \\
a_\lambda(a_\lambda V^{\otimes n}) \arrow[r] \arrow[d, "="] & a_\lambda(a_\lambda W^{\otimes n}) \arrow[r] \arrow[d, "="] & a_\lambda(a_\lambda U^{\otimes n}) \arrow[d, "="] \\
\mathbb{S}_\lambda(V) \arrow[r, "\mathbb{S}_\lambda(\varphi)"] & \mathbb{S}_\lambda(W) \arrow[r, "\mathbb{S}_\lambda(\psi)"] & \mathbb{S}(U)
\end{tikzcd}$$

Thus $\mathbb{S}_\lambda : \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$ is well defined.

To get a better grip of these we will first look at the two easiest examples and then return to example 3.3 from the last section to see a more involved example of $\mathbb{S}_\lambda$.

Example 4.1.

Let $n = 4$ and consider the two partitions let $\lambda_1 = (1,1,1,1)$ and $\lambda_2 = (4)$, that is when we consider the canonical Young tableau of λ_1, and λ_2 we have

$$\lambda_1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \qquad \lambda_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

and thus

$$P_{\lambda_1} = \{()\}, \quad Q_{\lambda_1} = \mathfrak{S}_4$$

and

$$P_{\lambda_2} = \mathfrak{S}_4, \quad Q_{\lambda_2} = \{()\}$$

one quickly calculates

$$c_{\lambda_1} = b_{\lambda_1} = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) e_\sigma, \quad c_{\lambda_2} = a_{\lambda_2} = \sum_{\sigma \in \mathfrak{S}_n} e_\sigma$$

thus if we consider any vector space V over $\mathbb{C}$, by Lemma 3.2 we have that

$$\mathbb{S}_{\lambda_1}(V) = \bigwedge^4(V), \quad \mathbb{S}_{\lambda_2}(V) = \operatorname{Sym}^4(V)$$

Example 4.2.

We recall that in example 3.3 we had for $v = v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$

$$c_\lambda(v) = v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$

Now, as promised we will describe $\mathbb{S}_\lambda = c_\lambda(V^{\otimes 3})$. We claim that $\mathbb{S}_\lambda$ is the subspace of $\bigwedge^2(V) \otimes V$ spanned by the elements

$$(v_1 \wedge v_3) \otimes v_2 + (v_3 \wedge v_2) \otimes v_1.$$

So to see this once again consider the canonical imbedding of $i : \bigwedge^2(V) \otimes V \to V^{\otimes 2} \otimes V$, and recall $\varphi : V^{\otimes 3} \to V^{\otimes 2} \otimes V$, as the canonical isomorphism, and hence we calculate,

$$\begin{aligned}
\varphi^{-1} \circ i((v_1 \wedge v_3) \otimes v_2 + (v_3 \wedge v_2) \otimes v_1) &= \varphi^{-1}((v_1 \otimes v_3 - v_3 \otimes v_1) \otimes v_2 + (v_3 \otimes v_2 - v_2 \otimes v_3) \otimes v_1) \\
&= v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_1 \otimes v_2 + v_3 \otimes v_2 \otimes v_1 - v_2 \otimes v_3 \otimes v_1 \\
&= c_\lambda(v_1 \otimes v_3 \otimes v_2).
\end{aligned}$$

We are now ready for our final section.

5 Decompositions of $V^{\otimes n}$

In this final section we will show how to use the Schur functor to decompose $V^{\otimes n}$. To motivate first we make a some observations. First notice that, using the notation of example 4.1 we have,

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \bigwedge^2(V) = \mathbb{S}_{\lambda_2}(V) \oplus \mathbb{S}_{\lambda_1}(V).$$

This turns out to not just be a happy accident, our final conclusion will show that every tensor power can be decomposed into summands of Shur functors. We will need the following fact to be able to state our final conclusion, which is Theorem 4.3 in [1], where we refer the reader for a proof.

Theorem 5.1. Given a partition of n the image of c_λ (by left multiplication on $\mathbb{C}\mathfrak{S}_n$) is an irreducible representation of, denote it V_λ, of $\mathfrak{S}_n$, and every irreducible representation is obtained in this way.

This following theorem is part (2) of Theorem 6.3 in [1], and again the reader is referred to [1] for the proof.

Theorem 5.2. Let λ be a partition of n, and let m_λ be the dimension of the irreducible representation V_λ of $\mathfrak{S}_n$ corresponding to λ, then

$$V^{\otimes n} \cong \bigoplus_{\lambda \text{ a partition of } n} \mathbb{S}_\lambda(V^{\otimes m_\lambda}).$$

With these theorems and looking at example 4.1 we can see where the first calculation of this section comes from, since there is only two partitions of the number 2. Our final example will use Theorem 5.1 and 5.2 to show a decomposition of $V^{\otimes 3}$.

Example 5.3.

First, notice that the three partitions of the number 3 are the following, (where the corresponding irreducible representation is written next to them)

$$\text{alternating:}(\lambda_1) \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \text{trivial:}(\lambda_2) \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\text{standard:}(\lambda_3) \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Let us now take a moment to prove that these are indeed the corresponding representations. First recall,

$$\mathfrak{S}_3 = \{(), (12), (13), (23), (123), (132)\}$$

Now, as seen in example 3.3 and 4.1 we have that

$$\begin{aligned} c_{\lambda_1} &= \sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) e_\sigma \\ c_{\lambda_2} &= \sum_{\sigma \in \mathfrak{S}_3} e_\sigma \\ c_{\lambda_3} &= e_{()} + e_{(12)} - e_{(13)} - e_{(132)}. \end{aligned}$$

It is easy to see that the images of c_{λ_1} and c_{λ_2} acting on the left of $\mathbb{C}\mathfrak{S}_3$ indeed gives us the expected irreducible representations. What may be unexpected is that $c_{\lambda_3}(\mathbb{C}\mathfrak{S}_n)$ is the irreducible representation corresponding to the standard representation. To unpack what this says, in [1] the standard representation of a permutation is the two-dimensional space spanned by "diagonal" elements and the "trace zero" elements (for more on this representation the reader is referred to [1] page 9). We will use the fact that this is the last of the irreducible representations, to prove our result here. So clearly by inspection $c_{\lambda_3}(\mathbb{C}\mathfrak{S}_3)$ is neither the trivial nor the alternating representation, so we have our result by Theorem 5.1.

There is another result from [1], which we omitted, yet can be observed easily in this example. Notice that we have the following calculation,

$$(c_{\lambda_3})^2 = 3c_{\lambda_3}.$$

To conclude this example, as promised, by Theorem 5.2, and example 4.2 we should have,

$$V^{\otimes 3} \cong \operatorname{Sym}^3(V) \oplus \bigwedge^3(V) \oplus \operatorname{span}\{(v_1 \wedge v_3) \otimes v_2 + (v_3 \wedge v_2) \otimes v_1\}.$$

Yet with the work we have done in this example this is now clear. That is we have

$$\mathbb{C}\mathfrak{S}_3 = \mathbb{C}c_{\lambda_1}(\mathbb{C}\mathfrak{S}_3) \oplus \mathbb{C}c_{\lambda_2}(\mathbb{C}\mathfrak{S}_3) \oplus \mathbb{C}c_{\lambda_3}(\mathbb{C}\mathfrak{S}_3),$$

so in particular we get that $e_{()} = ax + by + cz$ where $x \in c_{\lambda_1}(\mathbb{C}\mathfrak{S}_3)$, $y \in c_{\lambda_2}(\mathbb{C}\mathfrak{S}_3)$, $z \in c_{\lambda_3}(\mathbb{C}\mathfrak{S}_n)$, and $a, b, c \in \mathbb{C}$, and hence

$$v_1 \otimes v_2 \otimes v_3 = ax(v_1 \otimes v_2 \otimes v_3) + by(v_1 \otimes v_2 \otimes v_3) + cz(v_1 \otimes v_2 \otimes v_3).$$

References

[1] William Fulton, Joe Harris *Representation Theory, A First Course* Springer 2000