Chapter 3 Tor and Ext

Section 1: Tor for abelian groups

→ So why the notation/name \( \text{Tor} \)?

The answer to this question comes from the example of abelian groups, let's see the "prime example"

**Calculation 3.1.1:**

Let's calculate \( \text{Tor}_n(\mathbb{Z}/p\mathbb{Z}, B) \) for any abelian group \( B \).

To do so recall it's independent of our choice of projective resolution of \( \mathbb{Z}/p\mathbb{Z} \), so let's choose the obvious one...

\[
0 \rightarrow \mathbb{Z} \overset{p}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0
\]

For ease of notation we denote:

\[ p_i: \mathbb{Z} \rightarrow \mathbb{Z} \]

Recall we denoted \( T(A) = A \otimes \mathbb{Z} \)

and defined \( \text{Tor}_n(\mathbb{Z}/p\mathbb{Z}, B) = (\text{Ln} T)(A) = H_i(T(A)) \) with

\[ T(p_i): 0 \rightarrow \mathbb{Z} \otimes \mathbb{Z} B \rightarrow \mathbb{Z} \otimes \mathbb{Z} B \rightarrow 0 \]

and using elementary properties of tensor products [see ref. see Ch. 6 by Atiyah], we get

\[ T(p_i): 0 \rightarrow B \overset{p}{\rightarrow} B \rightarrow 0 \]

and hence \( \text{Tor}_0(\mathbb{Z}/p\mathbb{Z}, B) = B/pB = \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z} B \) [as discussed before!]

and \( \text{Tor}_1(\mathbb{Z}/p\mathbb{Z}, B) = pB = \mathbb{Z} \beta B \) [known as the \( p \)-torsion of \( B \)]

and \( \text{Tor}_2(\mathbb{Z}/p\mathbb{Z}, B) = \mathbb{Z} \)
Proposition 3.1.2: For all abelian groups $A$ and $B$:

1. $\text{Tor}_1^\mathbb{Z}(A,B)$ is a torsion abelian group.
2. $\text{Tor}_n^\mathbb{Z}(A,B) = 0$ for $n \geq 2$.

Proof:

First note that $A$ is the direct limit of its finitely generated subgroups, check them $A_k$ Thm 2.6.1.7 in Weibel or Thm 19 in kelley notes from last week, we have $A = \text{direct limit of } A_k$. Since $A_k$ is finitely generated it is of the form $A_k = \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n \mathbb{Z}$ (again recall for $A_k$ is also a direct limit from Corollary 1.1.10 in kelley notes also it is in a direct limit).

We have $\text{Tor}_1^\mathbb{Z}(A_k,B) = \text{Tor}_1^\mathbb{Z}(\mathbb{Z}^m,B) \oplus \text{Tor}_1^\mathbb{Z}(\mathbb{Z}/p_i \mathbb{Z},B) \oplus \cdots \oplus \text{Tor}_1^\mathbb{Z}(\mathbb{Z}/p_n \mathbb{Z},B)$. and since $\mathbb{Z}^m$ is projective we get $\text{Tor}_1^\mathbb{Z}(\mathbb{Z}^m,B) = \begin{cases} B^m & \text{for } m = 0 \\ 0 & \text{otherwise} \end{cases}$. Hence from the previous calculation we get

$$\text{Tor}_n^\mathbb{Z}(A_k,B) = \begin{cases} B^m \oplus \mathbb{Z}/p_1 B \oplus \cdots \oplus \mathbb{Z}/p_n B & n = 0 \\ B^m \oplus \cdots \oplus \mathbb{Z}/p_n B & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

So clearly the direct limit of 0 is clearly 0 hence we have part (1).

One also easily sees (check out the chart above...) that the direct limit of torsion is also torsion giving us part (2).

Note: abelian groups that is an infinite direct sum of infinite direct sum if every finite sum of these infinite direct sum are not only finite sum of finite direct sum at hence still have finite order.
**Proposition 3.1.3:** \( \text{Tor}_n^\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, B) \) is the torsion subgroup of \( B \) for any abelian group \( B \).

**Proof:**

Clearly by what the torsion of \( \mathbb{Q}/\mathbb{Z} \) is \([\text{i.e. a direct sum of } \mathbb{Z}/n\mathbb{Z} \text{ for } n \in \mathbb{Z}]\)

and by previous prop and what the defn of the torsion subgroup...

**Proposition 3.1.4:** If \( A \) is a torsion-free abelian group then \( \text{Tor}_n^\mathbb{Z}(A, B) = 0 \) for all \( n \neq 0 \) and all abelian groups \( B \).

Follows from work shown in prop 3.1.2

**Corollary 3.1.5:** For every abelian group \( A \)

\[ \text{Tor}_1^\mathbb{Z}(A, -) = 0 \iff A \text{ is torsion-free} \iff \text{Tor}_1^\mathbb{Z}(-, A) = 0 \]

Follows from 3.1.4 and a result we have seen about the symmetric of \( \text{Tor} \).

**Calculation 3.1.6:** All this fails if we replace \( \mathbb{Z} \) with another ring

For example let's take \( R = \mathbb{Z}/d\mathbb{Z} \) and \( A = \mathbb{Z}/d\mathbb{Z} \) so that \( d \mid n \) then we can use the following (apparently called \( \text{the periodic free resolution} \))

\[ 0 \rightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \]

Again denote \( P_n: \cdots \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \rightarrow 0 \)

So for any \( \mathbb{Z}/m\mathbb{Z}-\text{module } B \), we have

\[ T(P)_n : \cdots \xrightarrow{d} B/mB \xrightarrow{d} B/mB \rightarrow 0 \]

and hence

\[ \text{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, B) = \begin{cases} B/dB & n = 0 \\ \{ b \in B : d(b) = 0 \} / (d^2)(B) & \text{if } n \text{ is odd } n > 0 \\ \{ b \in B : (d/b) = 0 \} / d \cdot B & \text{if } n \text{ is even } n > 0 \end{cases} \]
Example 3.1.7: This example shows that not all hope is lost!

Let \( r \) be a non-zero divisor (i.e., \( r \neq 0 \)).

Let's calculate \( \text{Tor}_n^R \left( R/r, B \right) \) for an \( R \)-module \( B \). Consider the obvious resolution:

\[
0 \to R \xrightarrow{r} R \to R/r \to 0
\]

as my name denote:

\[
p_r : 0 \to R \xrightarrow{r} R \to 0
\]

Then \( T(p_r) : 0 \to B \xrightarrow{r} B \to 0 \)

and again we recover:

\[
\text{Tor}_0^R \left( R/r, B \right) = B/rB
\]

\[
\text{Tor}_1^R \left( R/r, B \right) = rB
\]

\[
\text{Tor}_2^R \left( R/r, B \right) = 0 \quad \text{for} \quad n \geq 2
\]

Exercise 3.1.1: When \( r \neq 0 \), all we have is the non-projective resolution

\[
0 \to R \to R \xrightarrow{r} R \to R/r \to 0
\]

Show that there is a short exact sequence

\[
0 \to \text{Tor}_2^R \left( R/r, B \right) \to R/r \xrightarrow{rB} B \to \text{Tor}_1^R \left( R/r, B \right) \to 0
\]

and that \( \text{Tor}_n^R \left( R/r, B \right) = \text{Tor}_{n-2}^R \left( rB, B \right) \) for \( n \geq 3 \).

Solution: First note from the s.e.s. \( 0 \to R \to R \to R/r \to 0 \) we get a long exact sequence of \( \text{Tor} \) (by thm 2.4.7)

\[
\cdots \to \text{Tor}_n^R \left( R/r, B \right) \to \text{Tor}_n^R \left( R/r, B \right) \to \text{Tor}_n^R \left( rB, B \right) \to \text{Tor}_{n-1}^R \left( R/r, B \right) \to \cdots
\]

and from the s.e.s. \( 0 \to R \to R \to R/r \to 0 \) we get for \( n \geq 3 \)

\[
\cdots \to \text{Tor}_n^R \left( R/r, B \right) \to \text{Tor}_n^R \left( rB, B \right) \to \text{Tor}_{n-1}^R \left( R/r, B \right) \to \cdots
\]

So we get from these two terms:

\[
\text{Tor}_n^R \left( R/r, B \right) \xrightarrow{rB} \text{Tor}_{n-2}^R \left( rB, B \right) \xrightarrow{rB} \text{Tor}_{n-3}^R \left( R/r, B \right) \xrightarrow{rB} \cdots
\]

and is the last part.
To see the last part note that again from the SES $0 \to \mathbb{R} \to \mathbb{R}_B \to B$ we get the following is exact:

$\Tor^R_1(\mathbb{R}, B) \to \Tor^R_2(\mathbb{R}/\mathbb{R}, B) \to \mathbb{R} \otimes B \to B$ \ldots

and hence $\Tor^R_1(\mathbb{R}/\mathbb{R}, B) \cong \ker(\mathbb{R} \otimes B \to B)$.

And note that $\mathbb{R}/\mathbb{R}$ acts on $\mathbb{R}$ via its just multiplication and two

by noting for $T(\mathbb{R}/\mathbb{R} \to \mathbb{R})$ we have its just multiplication and two

that the kernel is \{ $a \otimes b | b \in B$ \} \cong $\Tor^R_1(\mathbb{R}/\mathbb{R}, B)$. \ldots

Next we claim \{ $a \otimes b | b \in B$ \} \cong $\mathbb{R}/\mathbb{R}$ where $a \otimes b = \inf(\mathbb{R} \otimes B \to \mathbb{R})$.

First check that if $x \in \mathbb{R}$ then $\mathbb{R} \otimes b = \mathbb{R} \otimes B = 0$ and hence we have the claim.

We want i.e.,

$\mathbb{R} \otimes B \to B \Rightarrow \Tor^R_1(\mathbb{R}/\mathbb{R}, B) \to 0$ is exact.

So all that is left to show is that

$0 \to \Tor^R_2(\mathbb{R}/\mathbb{R}, B) \to \mathbb{R} \otimes B \to B$ is exact.

And hence $\Tor^R_2(\mathbb{R}/\mathbb{R}, B) \cong \ker(\mathbb{R} \otimes B \to B)$.

Yet recall from the previous page we have that

$\Tor^R_1(\mathbb{R}/\mathbb{R}, B) \cong \Tor^R_1(\mathbb{R}, B)$ and from the SES $0 \to \mathbb{R} \to \mathbb{R}_B \to B$ we have the following is exact:

$0 \to \Tor^R_2(\mathbb{R}, B) \to \Tor^R_1(\mathbb{R}, B) \to \mathbb{R} \otimes B \to B \Rightarrow \mathbb{R} \otimes B \to B$.

Yet note the image of $\mathbb{R} \otimes B \to B$ is in $\mathbb{R}$ hence this gives us what we want.
**Definition 3.2.1:** A left $R$-module $B$ is flat if the functor $- \otimes_R B$ is exact.

**Remark:** Notice that non-projective modules can be flat. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}$.

Yet note that modules need not be projective for example, consider the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$.

**Theorem 3.2.2:** If $S$ is a central multiplicatively closed set in a ring $R$ then $S^{-1}R$ is a flat $R$-module.

**Proof:**

Claim: $S^{-1}R$ is a directed limit of free modules and hence (as we have already recalled) $\text{Tor}_1^R(S^{-1}R, A)$ is a direct limit of $0$ (free $\otimes_R S$-projective) and hence $0$; the limit is defined to be the colimit of all objects of the scheme of $S$ with $\text{Hom}(s^{-1}R, A) = \{ S \mid S \subseteq S^{-1}R \}$.

This is a clear limit since $S_1, S_2 \subseteq S_3 \Rightarrow S_3^{-1}R \subseteq S_1^{-1}R$ and $S_1^{-1}R \subseteq S_2^{-1}R \subseteq S_3^{-1}R$.

Let $F : I \to R$-mod be defined by $F(s) = R$ and $F(s^{-1}r) = r$ for each $s \in S$.

By $S$, $F$ is a $R$-mod satisfies the universal property.

Then define $G : S^{-1}R \to R$ as needed to show that $F$ is a left $R$-module. However, $F(s^{-1}r) = r$ for each $s \in S$.

Thus, $G$ is well-defined.
Exercise 3.2.1: Show $T(F,A,E)$

1. $B$ is flat
2. $\text{Tor}_n^R(A, B) = 0 \quad \forall n \in \mathbb{N}$ and $A$
3. $\text{Tor}_0^n(A, B) = 0$ and $A$

\text{Proof:}

First, $3 \Rightarrow 1$

Let $M \to N \to A \to 0$

and $H^n \to \text{Tor}_n^R(A, B) \to M \otimes B \to 0$. (Same reasoning as $1 \Rightarrow 1$)

Note that $H^n$ is flat.

\text{2} \Rightarrow \text{3}

\text{Oh!}

\text{3} \Rightarrow \text{2}

\text{Next follow from induction}

\text{Exercise 3.2.2: Show that if $0 \to A \to B \to C \to 0$ is exact}

\text{and $B \otimes C$ is flat then $\forall n \in A$}

\text{Oh! note $\text{Tor}_{n+1}(C, D) \to \text{Tor}_{n+1}(B, D)$ is exact in $\text{Tor}_n(A, D) = 0}$

\text{Hence, follow from previous exercise.}

\text{Exercise 3.2.3: Show $0 \to R \overset{f}{\longrightarrow} R^2 \overset{g}{\longrightarrow} R \to 0$ is a resolution over a field}

and have $\text{Tor}_1(f, k)$ and $\text{Tor}_2(f, k)$ but $k$ not flat!

\text{Oh!}
**Definition 3.2.3:** The **Pontryagin dual** $B^{\text{op}}$, over a left $R$-module $B$ is defined as $\text{Hom}_{R^{\text{op}}}(B, \mathbb{Z})$; a coherent $R$-module via $(f)(n) = f(nb)$ (i.e., $ne \in \text{lim} R_{-n}$)

**Proposition 3.2.4:** TFAE for every left $R$-module $B$

1. $B$ is flat $(o, e, R)$
2. $B^{\text{op}}$ is an injective $R$-module
3. $I \otimes_{R} B = I \otimes_{R} B = 0$, for any ideal $I$ of $R$, $I 
4. Tor_{1}^{R}(E, I \otimes_{R} B) = 0$ for any ideal $I$ of $R$

**Proof:** (3$\Rightarrow$4) Consider $0 \to \text{Tor}_{1}(L, I \otimes_{R} B) \to I \otimes_{R} B \to 0 \to \cdots$.

So, $\text{Im}(B \otimes_{R} I) = I \otimes_{R} B$ and $\text{ker}(\text{Tor}_{1}(L, I \otimes_{R} B)) = 0$.

(4$\Rightarrow$3) follows from Exercise.

Now, for $A, I, B$ as above, consider $I \otimes_{R} B$ and $\text{Hom}(\cdot, I \otimes_{R} B)$ give a commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}(A, B^{\text{op}}) & \to & \text{Hom}(A', B^{\text{op}}) \\
\downarrow & & \downarrow \\
\text{Hom}(A \otimes_{R} I, B^{\text{op}}) & \to & \text{Hom}(A' \otimes_{R} I, B^{\text{op}})
\end{array}
$$

Using Lemma [2.1.1] we see $B^{\text{op}}$ is left $I$-injective. Hence $I \otimes_{R} B$ is left $I$-flat.

Since $B^{\text{op}}$ is left $I$-injective and $I \otimes_{R} B$, we have

- $B^{\text{op}}$ injective $\Rightarrow (A \otimes_{R} B)^{\text{op}} \Rightarrow (A' \otimes_{R} B)^{\text{op}}$

- $B^{\text{op}}$ injective $\Rightarrow (L \otimes_{R} B)^{\text{op}} \Rightarrow (L' \otimes_{R} B)^{\text{op}}$

Hence $B^{\text{op}}$ is left $I$-injective, which implies $I \otimes_{R} B$ is left $I$-flat.
**Lemma 3.2.5:** A map $f: A \rightarrow B$ is injective iff the dual map $f^*: B^* \rightarrow A^*$ is surjective.

**Proof:**

First recall from 2.3.3, we get that $\text{Hom}(\_, \mathbb{Q}/\mathbb{Z})$ is actually exact! and thus for example $A^* = 0$ if $C = 0$ and for $O \rightarrow k \rightarrow b \rightarrow c \rightarrow 0$ we get $O \rightarrow (b^*) \rightarrow c^* \rightarrow b^* \rightarrow k^* \rightarrow 0$

Here $k^*$ is the cokernel of $f^*$ and hence this gives us what we want!

**Recall:** we say a $R$-module $M$ is finitely presented when its finitely generated and that generator only satisfies a finite # of relations, more specifically we say it has a presentation $M$ generator $\{e_i \}_{i=1}^m$ and a finite number of relations $\sum_{k=1}^m r_{ij} e_j = 0$.

Therefore there is a matrix $\alpha = (\alpha_{ij})_{m \times n}$ s.t. the following is exact:

$$ R^m \xrightarrow{\alpha} R^n \rightarrow M \rightarrow 0 $$

**Note:** For any $R$-module $A$, $M$ there is a canonical mapping

$$\sigma: A^* \otimes_R M \rightarrow \text{Hom}_R(M, A)^*$$

Defined by

$$\sigma(f \otimes m): h \mapsto f(h(m))$$

Clearly for $f \in A^*$, $m \in M$, $h \in \text{Hom}_R(M, A)$

*also note $M = \mathbb{Q}/\mathbb{Z}$ makes $\sigma$ not an iso. Since not in general surjective!*
Lemma 3.2.6: The map $\sigma$ is an isomorphism for every finitely presented $M$ and all $A$.

Proof: First, let's see that $\text{Hom}(L, A)$ is an isomorphism. Well note $\text{Hom}(L, A) = A$ and $A^* \otimes A = A^*$; thus it's super clear and hence it's true for $M = R^n$ for any $n > 0$ (by additional of $\sigma$).

Now consider the diagram:

$$
\begin{align*}
A^* \otimes \mathbb{R}^n &\rightarrow A^* \otimes \mathbb{R}^n \\
\sigma &\downarrow \cong \\
\text{Hom}(L, A) &\rightarrow \text{Hom}(\mathbb{R}^n, A) \\
\end{align*}
$$

Exactness is the same as noted in previous lemma, hence our result follows from 5-lemma.

Theorem 3.2.7: Every finitely presented flat $\mathbb{R}$-module $M$ is projective.

Proof: We do this by saying $\text{Hom}(M, -)$ is exact. (Recall this is an equivalent statement to proj.) Let $A \rightarrow B$ be a surj. Two (as stated before) $0 \rightarrow B^\ast$ is an injection, so if $M$ is flat, we get

$$
\begin{align*}
0 &\rightarrow (C^\ast) \otimes M \\
\text{and } &\rightarrow C^\ast \\
\text{exact since the top arrow is an injection, and the vertical arrows are surj.}
\end{align*}
$$

Since the top arrow is a surjection and the vertical arrows are isos, it follows the bottom is also surjective.
**Flat Resolution Lemma 3.2.8:** The groups $\text{Tor}_n(A, B)$ may be computed using resolutions by flat modules! That is, if $F \rightarrow A$ is a resolution of $A$ while $F_n$ is a flat module, then $\text{Tor}_n(A, B) \cong H_n(A \otimes F')$.

**Proof:** The proof uses induction and dimension shifting. I refer the reader to the proof in Weibel pg 71-72.

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**Proposition 3.2.9:** (Flat base change for $\text{Tor}$) Suppose $R \rightarrow T$ is a ring map such that $T$ is flat as an $R$-module. Then for all $R$-modules $A$, all $T$-modules $C$ and all $n$,

$$\text{Tor}_n^R(A, C) \cong \text{Tor}_n^T(A \otimes_R T, C)$$

**Proof:** Choose an $R$-module projective resolution $P \rightarrow A$. Then $\text{Tor}_n^R(A, C)$ is the homology of $P \otimes_R C$. Since $T$ is $R$-flat, and each $P_n \otimes_R T$ is a projective $T$-module (while still becoming a summand of free since we now keep freedom), $P \otimes_R T \rightarrow A \otimes_T T$ is a $T$-module projective resolution. Thus $\text{Tor}_n^T(A \otimes_R T, C)$ is the homology of the complex $(P \otimes_T T) \otimes_T C \cong P \otimes_R C$ as well.

---

**Corollary 3.2.10:** If $R$ is commutative and $T$ is a flat $R$-algebra, then for all $R$-modules $A$ and $B$, and for all $n$,

$$T \otimes_R \text{Tor}_n^R(A, B) \cong \text{Tor}_n^T(A \otimes_R T, T \otimes_R B)$$

**Proof:** Setting $C = T \otimes_R B$, it is enough to show that $\text{Tor}_n^R(A, T \otimes_R B) = T \otimes_R \text{Tor}_n^R(A, B)$.

As $T \otimes_R -$ is an exact functor, $T \otimes_R \text{Tor}_n^R(A, B)$ is the homology of $T \otimes_R (P(A, B))$ (see in Kelly's notes) in a complex whose homology is $\text{Tor}_n^R(A, T \otimes_R B)$.
For what remaining assume $R$ is commutative so that $\text{Tor}^R_n(A, B)$ are actually $R$-modules. So we can see how for local $R$.

**Lemma 32.11:** If $\mu : A \to A$ is multiplication by a central element of $R$, so are the induced maps $M_\mu : \text{Tor}^R_n(A, B) \to \text{Tor}^R_n(A_\mu B)$ for all $n$ and $B$.

**Proof:** Choose a projective module $P \to A$. Multiplying by $\mu$ on the chain map $P \to P$ over $A$ (it's additive since $\mu$ is central) we get $P \to P$ over $A$ and $P \to P$. The induced map $M_\mu$ therefore we can pull it out of having gotten what we want.

**Corollary 32.12:** If $A$ is an $R/I, R$-module, then for every $R$-module $P$, the $R$-modules $\text{Tor}_n^R(A, P)$ are actually $R/I$-modules that are annihilated by the ideal $I, R$.

**Corollary 32.13:** (Localization of Tor) If $R$ is commutative so $A$ and $B$ are $R$-modules, then the following are equivalent for each $n$:

1. $\text{Tor}_n^R(A, B) = 0$

2. For every prime ideal $p$ of $R$, $\text{Tor}_n^R(A_p, B_p) = 0$

3. For every maximal ideal $m$ of $R$, $\text{Tor}_n^R(A_m, B_m) = 0$

Proof: Recall for every module $M$, $M = 0 \Rightarrow M_p = 0$ for any prime $p$ if $M$ is finitely generated. In the case $M = \text{Tor}_n^R(A, B)$ we have

$$M_p = \text{Tor}^R_p(A_p, B_p).$$