# Student Homological Algebra seminar

## Chapter 3 Tor and Ext

### Section 1: Tor for abelian groups

→ So why the notation/name $\boxed{\text{Tor}}$ ?

The answer to this question comes from the example of abelian groups... let's see the "prime example"

**Calculation 3.1.1:**

Let's calculate $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B)$ for any abelian group $B$

to do so recall it's independent of our choice of projective Resolution of $\mathbb{Z}/p\mathbb{Z}$... So let's choose the obvious one...

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

In general a torsion abelian group is one which every element has finite order!

for ease of notation we denote:

$$P_\cdot : \underset{\text{"}P_1\text{"}}{0 \to \mathbb{Z}} \xrightarrow{p} \underset{\text{"}P_0\text{"}}{\mathbb{Z}} \to 0$$

Recall we denoted $T(A) = A \otimes_{\mathbb{Z}} B$

and defined $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = (L_n T)(A) = H_i(T(P_\cdot))$

with

$$T(P_\cdot): \quad 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} B \xrightarrow{p \otimes Id} \mathbb{Z} \otimes_{\mathbb{Z}} B \to 0$$

and by elementary properties of Tensor products [for a ref see Ch.0 by Aluffi]

we get

$$T(P_\cdot): \quad 0 \to B \xrightarrow{p} B \to 0$$

and hence $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = B/p \cdot B = \mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} B$ [as discussed before!] $= T(\mathbb{Z}/p\mathbb{Z})$

and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = {}_pB = \{b \in B \mid p \cdot b = 0\} = \text{kernel of "}\cdot p\text{"}$

↖ This is subgroup!

[known as the p-torsion of B]

Man so many Reasons to call it Torsion!

and $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = 0 \quad \forall n \geq 2$

**Proposition 3.1.2:** for all abelian groups $A$ and $B$:

(a) $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ is a torsion abelian group

(b) $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0$ for $n \geq 2$

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R.V.

$$\begin{tikzcd} & A & \\ \langle a_1 \rangle \arrow[ur] & \cdots & \langle a_1, a_2, a_3 \rangle \arrow[ul] \end{tikzcd}$$

or perhaps better: $A = \sum_{a \in A} \langle a_i \rangle$

clear?

proof:

First note that $A$ is the direct limit of it's finitely generated subgroups, denote them $A_\alpha$

Thm 2.6.1.7 in Weibel or Thm 1.9 in Keller's notes from last week, we have

$\mathrm{Tor}_n(A,B)$ is the direct limit of $\mathrm{Tor}_n(A_\alpha, B)$. Since $A_\alpha$ is finitely generated

it is of the form $A_\alpha = \mathbb{Z}^m \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_r\mathbb{Z}$ (again a ref for this is Ch. 0 Awfi)

from Corollary 1.10 in Keller's notes (also it's just a direct limit)

we have $\mathrm{Tor}_n(A_\alpha, B) = \mathrm{Tor}_n(\mathbb{Z}^m, B) \oplus \mathrm{Tor}_n(\mathbb{Z}/p_1\mathbb{Z}, B) \oplus \cdots \oplus \mathrm{Tor}_n(\mathbb{Z}/p_r\mathbb{Z}, B)$

and since $\mathbb{Z}^m$ is projective we get $\mathrm{Tor}_n(\mathbb{Z}^m, B) = \begin{cases} B^m & \text{for } n=0 \\ 0 & \sim \end{cases}$

[since $P_\cdot : 0 \to \mathbb{Z}^m \to 0$] and hence from the previous calculation

we get

$$\mathrm{Tor}_n(A_\alpha, B) = \begin{cases} B^m \oplus B/p_1 B \oplus \cdots \oplus B/p_r B & n=0 \\ {}_{p_1}B \oplus \cdots \oplus {}_{p_r}B & n=1 \\ 0 & \sim \end{cases}$$

So clearly the direct limit of $0$ is clearly $0$ hence we have part (b)

one are also fairly easily sees [check out the claim above...] that

the direct limit of torsion is also torsion giving us part (a)

note abelian groups thus is the notion of infinite direct sum where elements are merely finite sums of this infinite mess and hence still have finite order!

*read Hovi, Yerger...

**Proposition 3.1.3:** $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion subgroup of $B$ for any abelian group $B$.

(SAS RV)

proof:
Clear by what the torsion of $\mathbb{Q}/\mathbb{Z}$ is [i.e. a direct limit of $\mathbb{Z}/n\mathbb{Z}$ $\forall n \in \mathbb{Z}$]
and by previous prop and what the defn of the torsion subgroup...

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**Proposition 3.1.4:** If $A$ is a Torsion free Abelian group then $\mathrm{Tor}_n^{\mathbb{Z}}(A,B)=0$ for all $n \neq 0$ and all abelian groups $B$.

$\rightarrow$ follows from work shown in prop 3.1.2

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**Corollary 3.1.5:** for every abelian group $A$

$$\mathrm{Tor}_1^{\mathbb{Z}}(A,-)=0 \iff A \text{ is torsion free} \iff \mathrm{Tor}_1^{\mathbb{Z}}(-,A)=0$$

$\rightarrow$ follows from 3.1.4 and a result we have seen about the symmetry of Tor...

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**Calculation 3.1.6:** All this fails if we replace $\mathbb{Z}$ w/ another ring
for example let's take $R = \mathbb{Z}/m\mathbb{Z}$ and $A = \mathbb{Z}/d\mathbb{Z}$ so that $d|m$ then we can use the following (apparently called: the periodic free resolution)

$$\cdots \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/d\mathbb{Z} \to 0$$

$\leftarrow$ Can we see this is a Resolution? [Hint: Follows from 3rd iso Thm!]

again denote $P_\cdot$: $\cdots \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\frac{m}{d}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \to 0$

So for any $\mathbb{Z}/m\mathbb{Z}$-Module $B$, we have

$$T(P_\cdot): \cdots \xrightarrow{d} B/mB \xrightarrow{\frac{m}{d}} B/mB \xrightarrow{d} B/mB \to 0$$

and thus

$$\mathrm{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, B) = \begin{cases} B/dB & n=0 \\ \{b \in B : db=0\}/(\frac{m}{d})\cdot B & \text{if } n \text{ is odd } n>0 \\ \{b \in B : (\frac{m}{d})b=0\}/d\cdot B & \text{if } n \text{ is even } n>0 \end{cases}$$

(SAJ RV)

**Example 3.1.7:** This example shows that not all hope is lost!

Let $r \in R$ be a left non-zero divisor [i.e. ${}_rR = \{0\}$]

Let's calculate $\mathrm{Tor}_n^R(R/r.R, B)$ for an $R$-module $B$!

consider the obvious resolution:

$$0 \to R \xrightarrow{r} R \to R/r.R \to 0$$

this works again since since the kernel of multiplying on the left by $r$ is zero!

as my thing denote

$$P_\bullet : \quad 0 \to R \xrightarrow{r} R \to 0$$

then $T(P_\bullet)$: $\quad 0 \to B \xrightarrow{r} B \to 0$

and again we recover:

$$\mathrm{Tor}_0^R(R/r.R, B) = B/r.B$$

$$\mathrm{Tor}_1^R(R/r.R, B) = {}_rB$$

$$\mathrm{Tor}_n^R(R/r.R, B) = 0 \quad \text{for } n \geq 2$$

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**Exercise 3.1.1:** when ${}_rR \neq \{0\}$, all we have is the non-projective resolution

$$0 \to {}_rR \to R \xrightarrow{r} R \to R/r.R \to 0$$

Show that there is a short exact sequence

$$0 \to \mathrm{Tor}_2^R(R/r.R, B) \to {}_rR \otimes_R B \xrightarrow{\text{multiplo}} {}_rB \to \mathrm{Tor}_1^R(R/r.R, B) \to 0$$

and that $\mathrm{Tor}_n^R(R/rR, B) \cong \mathrm{Tor}_{n-2}^R({}_rR, B)$ for $n \geq 3$

Sol'n: first note from the s.e.s $0 \to r.R \to R \to R/r.R \to 0$ we get a long exact sequence of Tor (by thm 2.4.7)

$$\cdots \to \underbrace{\mathrm{Tor}_n^R(R, B)}_{=0} \to \underbrace{\mathrm{Tor}_n^R(R/r.R, B) \to \mathrm{Tor}_{n-1}^R(r.R, B)}_{\text{since } \mathrm{Tor}_n^R(R/r.R, B) \cong \mathrm{Tor}_{n-1}^R(r.R, B)} \to \underbrace{\mathrm{Tor}_{n-1}^R(R, B)}_{=0}$$

for $n \geq 2$

now from the s.e.s $0 \to {}_rR \to R \xrightarrow{r} r.R \to 0$ we get for $n \geq 3$

$$\cdots \to \underbrace{\mathrm{Tor}_n^R(R, B)}_{=0} \to \mathrm{Tor}_n^R(r.R, B) \to \mathrm{Tor}_{n-1}^R({}_rR, B) \to \underbrace{\mathrm{Tor}_{n-1}^R(R, B)}_{=0} \to \cdots \Rightarrow \mathrm{Tor}_n^R(r.R, B) \cong \mathrm{Tor}_{n-1}^R({}_rR, B)$$

So we get from these two things

$$\mathrm{Tor}_n^R(R/rR, B) \cong \mathrm{Tor}_{n-1}^R(r.R, B) \cong \mathrm{Tor}_{n-2}^R({}_rR, B)$$ as is the last part

To see the first part note that again from the ses $0 \to r.R \to R \to R/r.R \to 0$ we get that the following is exact

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$$\underbrace{\mathrm{Tor}_1^R(R,B)}_{=0} \to \mathrm{Tor}_1^R(R/r.R,B) \hookrightarrow r.R \otimes B \to B \to \cdots$$

and hence $\mathrm{Tor}_1^R(R/r.R,B) \simeq \ker(r.R \otimes B \to B)$

and note this is just $T(r.R \to R)$ and hence it's just multiplication and thus

by notes for $\sum r x_i \otimes b_i = r \otimes (\sum x_i b_i) \mapsto r.(\sum x_i b_i)$ and clear $r \otimes b \mapsto 0$ if $b \in {}_rB$

that the kernel is $\{r \otimes b \mid b \in {}_rB\} \simeq \mathrm{Tor}_1^R(R/rR,B)$

next we claim $\{r \otimes b \mid b \in {}_rB\} \simeq {}_rB/{}_rRB$ where ${}_rRB = \mathrm{Im}({}_rR \otimes B \xrightarrow{\text{mult.}} {}_rB)$

first clearly note if $x \in {}_rR$ then $r \otimes xb = r.x \otimes b = 0$ and hence we have the relation we want i.e.

$${}_rR \otimes B \to {}_rB \to \mathrm{Tor}_1^R(R/rR,B) \to 0 \text{ is exact}$$

So all that is left to show is that

$$0 \to \mathrm{Tor}_2^R(R/rR,B) \to {}_rR \otimes B \to {}_rB \text{ is exact}$$

and hence $\mathrm{Tor}_2^R(R/rR,B) \simeq \ker({}_rR \otimes B \to {}_rB)$

yet recall from the previous page we have that

$\mathrm{Tor}_2^R(R/rR,B) \simeq \mathrm{Tor}_1^R(r.R,B)$ and from the s.e.s $0 \to {}_rR \to R \xrightarrow{r} r.R \to 0$

we have the following is exact

$$\cdots \to \underbrace{\mathrm{Tor}_2^R(R,B)}_{=0} \to \mathrm{Tor}_1^R(r.R,B) \to {}_rR \otimes B \to B \to r.R \otimes B \to 0$$

yet note the image of ${}_rR \otimes B \to B$ is in ${}_rB$ hence

this gives us what we want!

# Section 2 Tor and Flatness

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**Def$^{n}$ 3.2.1:** a left $R$-module $B$ is flat if the functor $-\otimes_R B$ is exact
dim. a right $R$-module $A$ is flat if the functor $-\otimes_R B$ is exact

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Remark: notice this means projective modules are flat
since given s.e.s $0 \to M \to N \to Q \to 0$
then for a projective $P$

$$\mathrm{Tor}_1(M,P) = 0 \to M\otimes P \to N\otimes P \to Q\otimes P \to 0$$

↖ hence exact

always right exact!

yet note flat modules need not be projective
for example consider the $\mathbb{Z}$-module, $\mathbb{Q}$
flat by the next thm, and not projective since

$$\begin{tikzcd} & \mathbb{Q} \arrow[dl, dashed, "?"'] \arrow[dr] & \\ \mathbb{Z}^{\omega} \arrow[rr, two heads] & & \mathbb{Q} \end{tikzcd}$$

yet $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$ proj
and $\mathbb{Q} \to \mathbb{Z}^{\omega} \to \mathbb{Z}$ $\to \leftarrow$

where would $\frac{1}{2}$ go then
$x$ is divisible by
even # $\to \leftarrow$

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**Theorem 3.2.2:** If $S$ is a central multiplicatively closed set in a ring $R$, then
$S^{-1}R$ is a flat $R$-module

proof:

$\boxed{\text{Claim: } S^{-1}R \text{ is a directed limit of free modules}}$ and hence (as we have already recalled)
Tor is a direct limit of $0$ (free $\Rightarrow$ projective) and hence $0$, the limit is

define $I$ to be the cat w/ objects are elements of $S$ and $\mathrm{Hom}(s_1,s_2) = \{s \mid s_1 s = s_2\}$

$s_1 \xrightarrow{s_2} s_3 = s_1 s_2$
$s_2 \xrightarrow{s_1} s_3$

thus is clear filtered since for $s_1, s_2$ $\exists$ $s_3$ s.t.
and for $s_1 \underset{s_4}{\overset{s_3}{\rightrightarrows}} s_2$ $\exists$ $s_2 \xrightarrow{s_6} s_5$ s.t. $(s_1 \xrightarrow{s_3} s_2 \xrightarrow{s_6} s_5) = (s_1 \xrightarrow{s_4} s_2 \xrightarrow{s_5} s_6)$
since is $s_1 \cdot s_3 = s_1 \cdot s_4$
then $s_1 \cdot s_3 \cdot s_5 = s_1 \cdot s_3 \cdot s_5$

and $F: I \to R\text{-mod}$ is defined by $F(s) = R$ and $F(s_1 \xrightarrow{s} s_2) =$ multiplication
by $s$, so let $Z$ be a $R$-mod satisfies the canonical commutation
then define $G: S^{-1}R \to Z$ as (only need to show when $\frac{1}{s}$ gets sent to map sm $G(\frac{x}{s}) = x \cdot G(\frac{1}{s})$)..
$G(\frac{1}{s}) = P_s(1)$ yet assu $\frac{1}{s} = \frac{1}{s'} \Rightarrow \exists s_1$ s.t. $s_1 s = s_1 s'$ this implies
$P_s(1) = P_{ss_1}(1) = P_{s'}(1)$ as desired
[shows its well defined!]

ok show first $\pi_s: F(s) \to S^{-1}R$
note we have
Localization for $S^{-1}R$
$\pi_s(x) = \frac{x}{s}$
$s_1 \to s$

$\frac{x}{s} = \frac{s_1 x}{s_1 s} \to$ since
then $\pi_s$ $x \to s_1 x$ $\pi_{s_1 s}$

$s \cdot s_1 = s_1 s$ so
since central!
$s_1 x - s s_1 x = s_1 x - s_1 s x = 0$ as desired!

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## Exercise 3.2.1: Show T.F.A.E

1. $B$ is flat
2. $\mathrm{Tor}_n^R(A,B)=0 \quad \forall\, n\neq 0$ and $A$
3. $\mathrm{Tor}_1^R(A,B)=0 \quad \forall\, A$

Sol$^n$/ first $\boxed{3 \Rightarrow 1}$ is done since $0 \to M \to N \to Q \to 0$

and then $\cdots \to \underbrace{\mathrm{Tor}_1^R(Q,B)}_{\substack{= 0 \\ \text{by ass}}} \to M\otimes B \to \cdots$ (same reasoning gives $1 \Rightarrow 3$)

$M \otimes B$ ← now flat!

$\boxed{2 \Rightarrow 3}$ Duh!

$\boxed{3 \Rightarrow 2}$ $0 \to K \to F \to A \to 0$ ($F$ is free!) ← we can use $R^{(A)}$

$n \neq 0$ $\quad \underbrace{\mathrm{Tor}_{n+1}(F,B)}_{0} \to \mathrm{Tor}_{n+1}(A,B) \to \underbrace{\mathrm{Tor}_n(K,B)}_{0}$

always! $\qquad$ by induction

now follows from induction

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## Exercise 3.2.2: Show that if $0 \to A \to B \to C \to 0$ is exact

and $B$ & $C$ are flat then so is $A$

Sol$^n$: note $n \neq 0$ $\quad \underbrace{\mathrm{Tor}_{n+1}(C,D)}_{0} \to \mathrm{Tor}_n(A,D) \to \underbrace{\mathrm{Tor}_n(B,D)}_{0}$ is exact so $\mathrm{Tor}_n(A,D)=0$ $\forall\, n \neq 0$

hence follows from previous exercise

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## Exercise 3.2.3: Show $0 \to R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{(x,y)} R \to k \to 0$ is a resolution of $k$ field

and hence $\mathrm{Tor}_1(I,k) \cong \mathrm{Tor}_2^R(k,R) \cong k$ so $I$ not flat!

$R = k[x,y]$

$I = \langle x,y \rangle$

Sol$^n$:

**Definition 3.2.3:** The <u>Pontrjagin dual</u> $B^*$ of a left $R$-mod $B$ (SAS)

1) defined as $\mathrm{Hom}_{Ab}(B, \mathbb{Q}/\mathbb{Z})$; a element $r$ of $R$ acts via $(fr)(b) = f(rb)$
(i.e. is a right $R$-mod)

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**Proposition 3.2.4:** TFAE for every left $R$-mod $B$

1. $B$ is flat (as $R$-mod)
2. $B^*$ is an injective $R$-mod
3. $I \otimes_R B \cong IB = \{x_1 b_1 + \dots + x_n b_n \in B : x_i \in I, b_i \in B\} \subseteq B$ for every right ideal of $R$
4. $\mathrm{Tor}_1^R(R/I, B) = 0$ $\forall$ right ideal of $R$

<u>proof</u>: $(3 \Rightarrow 4)$ consider $0 \to \mathrm{Tor}_1(R/I, B) \to I \otimes B \to B \to B/IB \to 0$ ← com for $0 \to I \to R \to R/I \to 0$

$\mathrm{Tor}_1(R, B) = 0$

So $\mathrm{Im}(I \otimes B \to B) = IB$ and then $\mathrm{Tor}(R/I, B) = \ker(I \otimes B \to B) = 0$
and by ass. $I \otimes B = IB$

$(4 \Rightarrow 1)$ follows from exercise

Now for $A' \subset A$ the adjoint functor $- \otimes L$ and $\mathrm{Hom}(-, B)$ give a comm diagram

$$\begin{tikzcd}
\mathrm{Hom}(A, B^*) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(A', B^*) \arrow[d, "\cong"] \\
(A \otimes B)^* = \mathrm{Hom}(A \otimes B, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Hom}(A' \otimes B, \mathbb{Q}/\mathbb{Z}) = (A' \otimes B)^*
\end{tikzcd}$$

using lemma below and $B$ acts critm 2.3.1 we see that

$B^*$ inj $\Leftrightarrow$ $(A \otimes B)^* \to (A' \otimes B)^*$ surjective
$\Leftrightarrow$ $A' \otimes B \to A \otimes B$ $\Leftrightarrow$ $B$ – flat
} $1 \Leftrightarrow 2$

$B^*$ inj $\Leftrightarrow$ $(R \otimes B)^* \to (I \otimes B)^*$ surjective
$\Leftrightarrow$ $(I \otimes B)^* \to R \otimes B$ injective
$\Leftrightarrow$ $I \otimes B \cong IB$
} $3 \Leftrightarrow 2$

**Lemma 3.2.5:** A map $f: B \to C$ is injective iff the dual map $f^*: C^* \to B^*$ is surjective

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proof

First recall from 2.3.3. we get that $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is actually exact! and thus for example $A^* = 0$ iff $A = 0$ and for

$$0 \to K \to B \xrightarrow{f} C \to \mathrm{Cok} \to 0$$

we get

$$0 \to (\mathrm{Cok})^* \to C^* \xrightarrow{f^*} B^* \to K^* \to 0$$

here $K^*$ is the cokernel of $f^*$ and hence this gives us what we want!

* for more info on $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ check out Section 3 Ch. 2 it's actually pretty good!

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**Recall:** we say a $R$-module $M$ is finitely presented when it's finitely generated and these generators only satisfy a finite # of relations more specifically we say it has a presentation w/ generators $\{e_1, \dots, e_n\}$ and a finite number of relations $\{\sum_{j=1}^{n} a_{ij} e_j \mid a_{ij} \in R\}_{i=1}^{m<\infty}$ therefore there is a matrix $\alpha = (\alpha_{ij})_{m \times n}$ s/t the following is exact

$$R^m \xrightarrow{\alpha} R^n \to M \to 0$$

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**Note:** for any $R$-modules $A$ & $M$ there is a canonical mapping

$$\sigma: A^* \otimes_R M \to \mathrm{Hom}_R(M, A)^*$$

defined by

$$\sigma(f \otimes m): h \longmapsto f(h(m))$$

(Clearly) for $f \in A^*$, $m \in M$, $h \in \mathrm{Hom}_R(M, A)$

* also note $M = \bigoplus^{\infty} R$ makes $\sigma$ not an iso. since not in general surjective! *

**Lemma 3.2.6:** The map $\sigma$ is an isomorphism for every finitely presented $M$ and all $A$

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proof: first let's see that $M=R$ makes $\sigma$ an isomorphism

well note $\mathrm{Hom}(R,A)\cong A$ and $A^*\otimes_R R\cong A^*$ thus it's super clear

and hence it's true for $M=R^n$ for any $n\geq 0$ (by additivity of $\sigma$)

now consider the diagram

$$\begin{tikzcd}
A^*\otimes R^m \arrow[r] \arrow[d, "\sigma", "\cong"'] & A^*\otimes R^n \arrow[r] \arrow[d, "\sigma", "\cong"'] & A^*\otimes M \arrow[r] \arrow[d, "\sigma"'] & 0 \\
\mathrm{Hom}(R^m,A)^* \arrow[r, "\alpha^*"] & \mathrm{Hom}(R^n,A)^* \arrow[r] & \mathrm{Hom}(M,A)^* \arrow[r] & 0
\end{tikzcd}$$

exactness is the same as noted in previous lemma, hence our result follows from 5-lemma.

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**Theorem 3.2.7:** Every finitely presented flat $R$-module $M$ is projective

proof: we do this by proving $\mathrm{Hom}_R(M,-)$ is exact (recall this is an equivalent statement to proj.)

let $B\twoheadrightarrow C$ be a surj. thus (as stated before) $C^*\to B^*$ is an injection, so if $M$ is flat, we get

$$\begin{tikzcd}
0 \arrow[r] & (C^*)\otimes_R M \arrow[r, hook] \arrow[d, "\cong"'] & (B^*)\otimes_R M \arrow[d, "\cong"'] \\
 & \mathrm{Hom}(M,C)^* \arrow[r] & \mathrm{Hom}(M,B)^*
\end{tikzcd}$$

$\leftarrow$ follows from continuance as discussed before

prev lem $\rightarrow$

$\Rightarrow$ since the top arrow is an injection and the vertical arrows are iso's it follows the bottom is also injective

and as noted before this implies

$\mathrm{Hom}(M,B)\twoheadrightarrow \mathrm{Hom}(M,C)$ is a surjection as required!

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**Flat Resolution lemma 3.2.8:** The groups $\mathrm{Tor}_*(A,B)$ may be computed using resolutions by flat modules! that is if $F \to A$ is a resolution of $A$ while $F_n$ being flat modules, then $\mathrm{Tor}_*(A,B) \cong H_*(A \otimes F')$

proof: The proof uses induction and dimension shifting, I refer the reader to the proof in Weibel pg 71 – 72

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**proposition 3.2.9:** (flat base change for Tor) Suppose $R \to T$ is a ring map such that $T$ is flat as an $R$-module. Then for all $R$-modules $A$, all $T$-modules $C$ and all $n$

$$\mathrm{Tor}_n^R(A,C) \cong \mathrm{Tor}_n^T(A \otimes_R T, C)$$

proof: Choose an $R$-module projective resolution $P \to A$. Then $\mathrm{Tor}_n^R(A,C)$ is the homology of $P \otimes_R C$. Since $T$ is $R$-flat, and each $P_n \otimes_R T$ is a projective $T$-module (notice still becomes a summand of free since tensor keeps freedom)

$P \otimes T \to A \otimes T$ is a $T$-module projective resolution. Thus $\mathrm{Tor}_*^T(A \otimes_R T, C)$ is the homology of the complex $(P \otimes_R T) \otimes_T C \cong P \otimes_R C$ as well.

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**Corollary 3.2.10:** If $R$ is commutative and $T$ is a flat $R$-algebra, then for all $R$-modules $A$ and $B$, and for all $n$

$$T \otimes_R \mathrm{Tor}_n^R(A,B) \cong \mathrm{Tor}_n^T(A \otimes_R T, T \otimes_R B)$$

proof: setting $C = T \otimes_R B$ it is enough to show that $\mathrm{Tor}_*^R(A, T \otimes B) = T \otimes \mathrm{Tor}_*^R(A,B)$

As $T \otimes_R$ is an exact functor, $T \otimes \mathrm{Tor}_*^R(A,B)$ is the homology of $T \otimes_R (P \otimes_R B)$ $\cong P \otimes_R (T \otimes_R B)$

(see in Keller's notes) the complex whose homology is

$$\mathrm{Tor}_*^R(A, T \otimes B)$$

For what remains assume $R$ is commutative, so that $\mathrm{Tor}_n^R(A,B)$ are actually $R$-modules, so we can see how Tor localizes...

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**Lemma 3.2.11:** If $\mu: A \to A$ is multiplication by a central element $r \in R$ so are the induced maps $\mu_*: \mathrm{Tor}_n^R(A,B) \to \mathrm{Tor}_n^R(A,B)$ for all $n$ and $B$

proof: Choose a proj. res. $P \to A$. Multiplic by $r$ is an $R$-mod chain map $\tilde{\mu}: P \to P$ over $\mu$ (it's additive since $r$ is central!) and $\tilde{\mu} \otimes B$ is multiplication by $r$ on $P \otimes B$. The induced map $\mu_*$ therefore we can pull it out of homology getting what we want

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**Corollary 3.2.12:** If $A$ is an $R/r.R$-module, then for every $R$-module $B$ the $R$-modules $\mathrm{Tor}_n^R(A,B)$ are actually $R/R$-modules, that is annihilated by the ideal $r.R$.

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**Corollary 3.2.13:** (localization of Tor) If $R$ is commutative and $A$ and $B$ are $R$-modules, then the following are equivalent for each $n$:

1. $\mathrm{Tor}_n^R(A,B) = 0$
2. for every prime ideal $p$ of $R$ $\mathrm{Tor}_n^{R_p}(A_p, B_p) = 0$
3. for every max'l ideal $m$ of $R$ $\mathrm{Tor}_n^{R_m}(A_m, B_m) = 0$

proof. Recall for every module $M$, $M = 0 \iff M_p = 0$ $\forall$ primes $p$ iff $M_m = 0$ $\forall$ max'l $m$

In the case $M = \mathrm{Tor}_n^R(A,B)$ we have

$$M_p = R_p \otimes_R M = \mathrm{Tor}_n^{R_p}(A_p, B_p).$$