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# HOMOLOGICAL ALGEBRA NOTES

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## 1. Chain Homotopies

Consider a chain complex $C$ of vector spaces

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots$$

At every point we may extract the short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow C_n/Z_n \longrightarrow 0$$

$$0 \longrightarrow d(C_{n+1}) \longrightarrow Z_n \longrightarrow Z_n/d(C_{n+1}) \longrightarrow 0$$

Since $Z_n$ and $d(C_n)$ are vector subspaces, in particular they are injective modules, giving that

$$C_n = Z_n \oplus B'_n$$

$$Z_n = B_n \oplus H'_n$$

with $B'_n := C_n/Z_n$, $H'_n := H_n(C)$, and $B_n := d(C_{n+1})$. This decomposition allows for a way to move backward along our complex via a composition of projections and inclusions:

$$C_n \cong Z_n \oplus B'_n \to Z_n \cong B_n \oplus H'_n \to B_n \cong B'_{n+1} \hookrightarrow Z_{n+1} \oplus B'_{n+1} \cong C_{n+1}$$

If we denote by $s_n : C_n \to C_{n+l}$ the composition of the above, then one sees $d_n s_n d_n = d_n$ (more succinctly, $dsd = d$), and we have the

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commutative diagram

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[d] \arrow[dl, "s_n"] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "s_{n-1}"] & {} \\
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & {}
\end{tikzcd}$$

**Definition 1.1.** A complex $C$ is called *split* is there are maps $s_n : C_n \to C_{n+1}$ such that $dsd = d$. The $s_n$ are called the splitting maps. If in addition $C$ is acyclic, $C$ is called *split exact*.

The map $d_{n+1}s_n + s_{n-1}d_n$ is particularly interesting. We have the following:

**Proposition 1.2.** *If $id = d_{n+1}s_n + s_{n-1}d_n$, then the chain complex $C$ is acyclic.*

*Proof.* Let $z$ be an $n$-cycle. Then, $\text{id}(z) = d_{n+1}s_n(z) \in B_n(C)$, so that the induced map

$$\text{id}_* : H_p(C) \to H_p(C)$$

is equivalent to the 0 map. Since $\text{id}_*$ must be an isomorphism, we conclude that $H_p(C) = 0$. $\square$

This motivates the following:

**Definition 1.3.** Let $f$, $g : C \to C'$ be two morphisms of complexes. $f$ and $g$ are called chain homotopic if $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$. If $g \equiv 0$ in the above, then $f$ is called *null-homotopic*.

Why do we care about such maps? If we follow the proof of the previous proposition with $f_n - g_n$ substituted for the identity, we see that the induced homology maps coincide. That is,

$$f_{n*} = g_{n*} : H_n(C) \to H_n(C')$$

In particular, $f$ is null-homotopic when the induced homology maps are trivial. Additionally, we see that $f$ must commute with our differentials in this case.

**Proposition 1.4.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. If $f, g : C \to C'$ are chain homotopic, then so are $F(f)$ and $F(g)$*

*Proof.* Note that additivity of our functors guarantees that $F(d)$ remains a differential. Since functors preserve commutativity, we see

$$F(f) - F(g) = F(d)F(s) + F(s)F(d)$$

□

## 2. Mapping Cones

Let $f : \mathcal{B} \to \mathcal{C}$ be a morphism of chain complexes. We define a new complex, the mapping cone of $f$ denoted cone$(f)$ by complex whose degree $n$ part is

$$B_{n-1} \oplus C_n$$

with differential

$$d = \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}$$

It remains to show that this is actually a chain complex. We see:

$$\begin{aligned} d \circ d &= \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} \\ &= \begin{pmatrix} -d_B^2 & 0 \\ f d_B - d_C f & d_C^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Where, by definition of a morphism of complexes, we have that $f d_B = d_C f$.

**Exercise 2.1.** Let cone($C$) denote the mapping cone of the identity map on $C$. Show that cone($C$) is split exact.

*Proof.* Define our splitting map $s_n : C_{n-1} \oplus C_n \to C_n \oplus C_{n-1}$ by $s_n(b,c) = (-c, 0)$. Then, at every point of our complex, $s$ is represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

We first show that $d = dsd$:

$$\begin{aligned}
\begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} &= \begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} \begin{pmatrix} \mathrm{id} & -d_C \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -d_C \circ \mathrm{id} & d_C^2 \\ -\mathrm{id}^2 & \mathrm{id} \circ d_C \end{pmatrix} \\
&= \begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} = d
\end{aligned}$$

So this sequence is split. It remains to show exactness. Suppose that $(b,c) \in \mathrm{Ker}(d)$, so that

$$d(b,c) = (-d_C(b), d_C(c) - b) = (0,0)$$

So that $b \in \mathrm{Ker}(d_C) \cap \mathrm{Im}(d_C) = \mathrm{Im}(d_C)$. Then,

$$\begin{aligned}
(b,c) &= (d_C(c), c) \\
&= (-d_C(-c), d_C(0) - (-c)) \\
&= d(-c, 0) \in \mathrm{Im}(d)
\end{aligned}$$

So that $\mathrm{Ker}(d) = \mathrm{Im}(d)$, yielding exactness.

$\square$

**Exercise 2.2.** Let $f : C \to D$ be a map of complexes. Show that $f$ is null-homotopic if and only if $f$ extends to a map $\overline{f} : \mathrm{cone}(f) \to D$.

*Proof.* Assume first that $f$ is null-homotopic. We have mappings $s_n : C_n \to D_{n+1}$, and by definition of extension we should have that $\overline{f}(0,c) =$

$f(c)$. For each $n$, define

$$\overline{f}_n(c', c) := f_n(c) - s_{n-1}(c')$$

For convenience, we may represent $\overline{f}_n = (-s_{n-1}\ f_n)$ as a row vector. Then, we only need show that $fd - df = 0$:

$$\begin{aligned} \begin{pmatrix} -s_{n-1} & f_n \end{pmatrix} \begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} &= \begin{pmatrix} s_{n-1}d_C - f_n & f_n d_C \end{pmatrix} \\ &= \begin{pmatrix} -d_D s_{n-1} & d_D f_n \end{pmatrix} \\ &= d_D \begin{pmatrix} -s_{n-1} & f_n \end{pmatrix} \end{aligned}$$

Where the final steps used the assumption that $f$ is null-homotopic and a chain map. Conversely, if such an extension $\overline{f}$ exists, we may construct splitting maps $s_{n-1} : C_{n-1} \to D_n$ by defining $s_{n-1}(c) := -\overline{f}(c, 0)$. It remains to show that $f$ is null-homotopic with our $s_n$ defined in this manner.

$$\begin{aligned} f(c) &= \overline{f}(0, c) \\ &= \overline{f}(d_C(c), c) + \overline{f}(-d_C(c), 0) \\ &= -\overline{f}\left( \begin{pmatrix} -d_C & 0 \\ -\mathrm{id} & d_C \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} \right) + s_n \circ d_C(c) \\ &= -\overline{f} \circ d(c) + s_n \circ d_C(c) \\ &= d_D s_n(c) + s_n d_C(c) \end{aligned}$$

Where the final equality follows from the assumption that $\overline{f}$ is a chain map. The above of course shows that $f$ is null-homotopic, completing the proof.

□

**Proposition 2.3.** *We have a short exact sequence of chain complexes*

$$0 \longrightarrow C \longrightarrow \mathit{cone}(f) \xrightarrow{\delta} B[-1] \longrightarrow 0$$

*Where the left map takes* $c \mapsto (0, c)$*, and* $\delta(b, c) = -b$

*Proof.* The inclusion $\iota(c) = (0, c)$ is clearly injective, so we have exactness at $C$. We also see that $\delta(0, c) = 0$, so $\operatorname{Im} \iota \subset \operatorname{Ker} \delta$. The reverse inclusion follows immediately by noting $\delta(b, c) = 0 \iff b = 0$. Finally, $\delta$ is certainly surjective, so our sequence is exact. $\square$

Whenever we have a short exact sequence, one should imagine the induced long exact sequence of homology groups. In this case, the mapping cone induces a particularly nice property of the connecting morphism constructed in the Snake Lemma.

**Lemma 2.4.** *Let*

$$H_{n+1}(cone(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \longrightarrow H_n(cone(f))$$

*be the induced long exact sequence of homology groups. Then, in the above,* $\partial \equiv f_*$

*Proof.* Choose $b \in B_n$ to be some cycle. Then, $\delta(-b, 0) = b$, and applying our cone differential to $(-b, 0)$ gives $(d_B(b), f(b)) = (0, f(b))$. By definition of the map $\partial$, this implies that

$$\partial[b] = [fb] = f_*[b]$$

$\square$

**Corollary 2.5.** $f : B \to C$ *is a quasi-isomorphism if and only if* $\operatorname{Cone}(f)$ *is exact.*

**Example 2.6.** Let $X$ be a chain complex of $R$-modules (assume $R$ is commutative/Noetherian). For any $r \in R$, the homothety map $\mu^r : X \to X$ (which I'll denote by just $r$ when context is clear) is a chain map, where $\mu^r_i(m) = rm$. We inductively define the *Koszul complex*

$K(\underline{x})$ of a sequence $\underline{x} = x_1, \dots, x_n$ by setting $K(x_1)$ to be the complex

$$0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$$

For $n \geqslant 2$, set $\underline{x}' = x_1, \dots, x_{n-1}$. Then, $K(\underline{x}) = \operatorname{cone}(x_n)$, where we are viewing $x_n$ as the homothety $x_n : K(\underline{x}') \to K(\underline{x}')$. We then have the exact sequence

$$0 \longrightarrow K(\underline{x}') \longrightarrow K(\underline{x}) \longrightarrow K(\underline{x}')[-1] \longrightarrow 0$$

The previous lemma then immediately tells us that the induced map $x_{n*} : H_p(K(\underline{x}')) \to H_n(K(\underline{x}'))$ is our connecting morphism. This definition of the Koszul complex has the advantage that certain properties are easily obtained. For instance, if the first element of $\underline{x}$ is a unit, $K(\underline{x})$ is exact.

## 3. Mapping Cylinder

**Definition 3.1.** The mapping cylinder $\operatorname{cyl}(f)$ of a chain map $f : B \to C$ is the complex whose degree $n$ part is $B_n \oplus B_{n-1} \oplus C_n$ with differential given by

$$d = \begin{pmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix}$$

One checks that

$$\begin{aligned} d^2 &= \begin{pmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix} \begin{pmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix} \\ &= \begin{pmatrix} d_B^2 & d_B - d_B & 0 \\ 0 & d_B^2 & 0 \\ 0 & f d_B - d_C f & d_C^2 \end{pmatrix} = 0 \end{aligned}$$

And we have the following exercise, similar to the previous exercise for the cone case.

**Exercise 3.2.** Let $\mathrm{cyl}(C)$ denote the mapping cylinder of the identity. Show that two maps $f$, $g$ are chain homotopic if and only if they extend to a map $f, s, g) : \mathrm{cyl}(C) \to D$.

*Proof.* Assume first that $f$ and $g$ are chain homotopic. There exist $s_n : C_n \to D_{n+1}$, and we may define an extension $h$ of $f$ and $g$ by $h(a, b, c) = f(a) + s(b) + g(c)$. As a row vector, we may say

$$h = \begin{pmatrix} f & s & g \end{pmatrix}$$

It remains to show that this extension is a chain map, that is, our differentials commute with it. We have:

$$\begin{aligned} hd_{\mathrm{cyl}(C)} &= \begin{pmatrix} f & s & g \end{pmatrix} \begin{pmatrix} d_C & \mathrm{id} & 0 \\ 0 & -d_C & 0 \\ 0 & -\mathrm{id} & d_C \end{pmatrix} \\ &= \begin{pmatrix} fd_C & f - sd_C - g & g & gd_C \end{pmatrix} \\ &= \begin{pmatrix} d_D f & d_D s & d_D g \end{pmatrix} \\ &= d_D \begin{pmatrix} f & s & g \end{pmatrix} \end{aligned}$$

Where the second to last step in the above used that $f$ and $g$ are both chain maps, and by assumption $f - g = sd + ds$. Hence this extension is indeed a chain map.

Conversely, suppose such an extension $h$ exists. Then we may define splitting maps $s : C_n \to D_{n+1}$ by $s(c) = h(0, c, 0)$. It remains to show that this implies $f$ and $g$ are chain homotopic. Since $h$ is an extension, we see for any $c \in C_n$:

$$\begin{aligned} f(c) - g(c) &= h(c, 0, -c) \\ &= h(c, -d_C(c), -c) + h(0, d_C(c), 0) \\ &= h\left( \begin{pmatrix} d_C & \mathrm{id} & 0 \\ 0 & -d_C & 0 \\ 0 & -\mathrm{id} & d_C \end{pmatrix} \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix} \right) + s \circ d_C(c) \\ &= h(d_{\mathrm{cyl}(C)}(0, c, 0)) + s d_C(c) \\ &= d_D h(0, c, 0) + s d_C(c) \\ &= d_D s(c) + s d_C(c) \end{aligned}$$

Where the third to fourth equality uses the assumption that $h$ is a chain map. We then see that $f$ and $g$ are chain homotopic, as asserted.

□

**Lemma 3.3.** *The inclusion $\alpha : C \to cyl(f)$ is a quasi-isomorphism.*

*Proof.* This follows from observing that

$$0 \longrightarrow C \xrightarrow{\alpha} \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(-\mathrm{id}_B) \longrightarrow 0$$

is exact, where the left map is our inclusion and the right map is a transposition and projection, that is, $(b, b', c) \mapsto (b', b)$ with induced differential

$$d = \begin{pmatrix} d_B & 0 \\ \mathrm{id} & -d_B \end{pmatrix}$$

We also have the exact sequence

$$0 \longrightarrow B \longrightarrow \mathrm{cone}(-\mathrm{id}_B) \longrightarrow B[-1] \longrightarrow 0$$

Which induces the exact sequence

$$H_n(B) \xrightarrow{\mathrm{id}_*} H_n(B) \longrightarrow H_n(\mathrm{cone}(-\mathrm{id}_B)) \longrightarrow H_{n-1}(B) \longrightarrow \cdots$$

Since $\mathrm{id}_*$ is an isomorphism, exactness yields that $H_n(\mathrm{cone}(-\mathrm{id}_B)) = 0$ for every $n$. Looking at the induced long exact sequence of our first short exact sequence, we have

$$H_{n+1}(\text{cone}(-\text{id}_B)) \longrightarrow H_n(C) \xrightarrow{\alpha_*} H_n(\text{cyl}(f)) \longrightarrow H_n(\text{cone}(-\text{id}_B))$$

Since our mapping cone homology groups vanish, we conclude that $\alpha_*$ is an isormorphism, that is, $\alpha$ is a quasi-isomorphism. $\square$

**Exercise 3.4.** Suppose $f : B \to C$ is a chain map. Define $\beta : \text{cyl}(f) \to C$ by $\beta(b, b', c) = f(b) + c$. Show that $\beta$ is a chain map and $\beta\alpha = \text{id}$.

Additionally, show that $s$ defined by $s(b, b', c)$ defines a chain homotopy from the identity to $\alpha\beta$, and conclude that $\alpha$ is a chain homotopy equivalence between $C$ and $\text{cyl}(f)$.

*Proof.* Firstly, given $c \in C_n$,

$$\beta(\alpha(c)) = \beta(0, 0, c) = c$$

So that $\beta\alpha = \text{id}$. It remains to show that $\beta$ is a chain map, that is, it commutes with our differentials. We see:

$$\begin{aligned} \beta\left( \begin{pmatrix} d_B & \text{id} & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix} \begin{pmatrix} b \\ b' \\ c \end{pmatrix} \right) &= \beta\left( \begin{pmatrix} d_B(b) + b' \\ -d_B(b') \\ -f(b') + d_C(c) \end{pmatrix} \right) \\ &= f(d_B(b)) + f(b') - f(b') + d_C(c) \\ &= d_C(f(b)) + d_C(c) \\ &= d_C\beta(b, b', c) \end{aligned}$$

So that $\beta$ is indeed a chain map. Let $s$ be defined as in the problem statement. We wish to show that $\text{id}_{\text{cyl}(f)} - \alpha\beta = ds + sd$. To this end, compute:

$$\begin{aligned}
(b,b',c)-\alpha\beta(b,b',c) &= (b,b',c)-\alpha(f(b)+c)\\
&= (b,b',-f(b))\\
&= (b,-d_B(b),-f(b))+(0,d_B(b)+b',0)\\
&= d_{\mathrm{cyl}(f)}(0,b,0)+s(d_B(b)+b',-d_B(b),-f(b)+d_C(c))\\
&= d_{\mathrm{cyl}(f)}s(b,b',c)+sd_{\mathrm{cyl}(f)}(b,b',c)
\end{aligned}$$

So that $1-\alpha\beta = ds+sd$, as desired. By definition, $\alpha$ is a chain homotopy equivalence.

$\square$

Given a short exact sequence

$$0 \longrightarrow B \xrightarrow{f} C \xrightarrow{g} D \longrightarrow 0$$

of complexes, we can form the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[d] & B[-1] \arrow[r] & 0\\
0 \arrow[r] & B \arrow[r] \arrow[d] & \mathrm{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{cone}(f) \arrow[r] \arrow[d, "\phi"] & 0 & \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0 &
\end{tikzcd}$$

Where $\phi(b,c) := g(c)$ and $\alpha$, $\beta$ are the maps considered in the previous exercise. It is also clear by the definition of our mapping cylinder that $\mathrm{cyl}(f)/B = \mathrm{cone}(f)$. We then have the following:

**Lemma 3.5.** *In the following induced commutative diagram (with exact rows):*

$$\begin{tikzcd}
H_n(B) \arrow[r] \arrow[d] & H_n(cyl(f)) \arrow[r] \arrow[d] & H_n(cone(f)) \arrow[d] \\
H_n(B) \arrow[r, "f_*"] & H_n(C) \arrow[r, "g_*"] & H_n(D)
\end{tikzcd}$$

*All vertical arrows are isomorphisms.*

The proof of this is largely a collection of the previous results in these notes, and is left as an exercise to the reader.