Implicit Differentiation

Differentiation Rules

Let $y = f(x)$ and $y = g(x)$ be functions which are differentiable at x. Let a and b be constants.

Linearity:

$$D_x\left[af(x) + bg(x)\right] = af'(x) + bg'(x)$$

Product Rule:

$$D_x\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule:

In the case $g(x) \neq 0$

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$$

Chain Rule:

In the case that f is diferentiable at x and g is differentiable at $f(x)$

$$D_x\left[g\Big(f(x)\Big)\right] = g'\Big(f(x)\Big)f'(x)$$

To this point weve done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately, not all the functions that were going to look at will fall into this form.

Lets take a look at an example of a function like this.

Example 1

Find y' for $xy = 1$.

There are actually two solution methods for this problem.

Solution 1 : This is the simple way of doing the problem. Just solve for y to get the function in the form that were used to dealing with and then differentiate.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, thats easy enough to do. However, there are some functions for which this cant be done. Thats where the second solution technique comes into play.

Example 1

Find y' for $xy = 1$.
Solution 2 : In this case were going to leave the function in the form that we were given and work with it in that form. However, lets recall from the first part of this solution that if we could solve for y then we will get y as a function of x. In other words, if we could solve for y (as we could in this case but wont always be able to do) we get $y = y(x)$. Lets rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y(x)$ we dont mean y times x. What we are noting here is that y is some (probably unknown) function of x. This is important to recall when doing this solution technique.
The next step in this solution is to differentiate both sides with respect to x as follows,

$$\frac{d}{dx}(x\,y(x)) = \frac{d}{dx}(1)$$

The right side is easy. Its just the derivative of a constant. The left side is also easy, but weve got to recognize that weve actually got a product here, the x and the $y(x)$. So, to do the derivative of the left side well need to do the product rule. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}(y(x)) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}(y(x)) = \frac{dy}{dx} = y'$$

Using this we get the following,

$$y + xy' = 0$$

Note that we dropped the (x) on the y as it was only there to remind us that the y was a function of x and now that weve taken the derivative its no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.
So, lets now recall just what were we after. We were after the derivative, y', and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'.

$$y' = -\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesnt look like the same derivative that we got from the first solution. Recall however, that we really do know what y is in terms of x and if we plug that in we will get,

$$y' = -\frac{{}^1/_x}{x} = -\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section. In the previous example we were able to just solve for y and avoid implicit differentiation. However, in the remainder of the examples in this section we either wont be able to solve for y or, as well see in one of the examples below, the answer will not be in a form that we can deal with.

In the second solution above we replaced the y with $y(x)$ and then did the derivative. Recall that we did this to remind us that y is in fact a function of x. Well be doing this quite a bit in these problems, although we rarely actually write $y(x)$. So, before we actually work anymore implicit differentiation problems lets do a quick set of simple derivatives that will hopefully help us with doing derivatives of functions that also contain a $y(x)$.

Example 2

Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution:
Now, this is just a circle and we can solve for y which would give,

$$y = \pm\sqrt{9 - x^2}$$

Prior to starting this problem, we stated that we had to do implicit differentiation here because we couldnt just solve for y and yet thats what we just did. So, why cant we use normal differentiation here? The problem is the $\pm$. With this in the solution for y we see that y is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.
So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example well do the same thing we did in the first example and remind ourselves that y is really a function of x and write y as $y(x)$. Once weve done this all we need to do is differentiate each term with respect to x.

$$\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d}{dx}(9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the part (a) of the second example. All we need to do for the second term is use the chain rule.
After taking the derivative we have,

$$2x + 2[y(x)]^1 y'(x) = 0$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for y'.

$$\begin{aligned} 2x + 2yy' &= 0 \\ y' &= -\frac{x}{y} \end{aligned}$$

Unlike the first example we cant just plug in for y since we wouldnt know which of the two functions to use. Most answers from implicit differentiation will involve both x and y so dont get excited about that when it happens.

Problem 1. For each of the following

a. Find y' by solving the equation for y and differentiating directly.

b. Find y' by implict differentiation

c. Check that the derivatives in (a) and (b) are the same.

i. $x^2y^9 = 2$

ii. $\dfrac{6x}{y^7} = 4$

iii. $1 = x^4 + 5y^3$

iv. $8x - y^2 = 3$

v. $4x - 6y^2 = xy^2$

vi. $\ln(x\,y) = x$

Problem 2. For the following find y' by implicit differentiation.

i. $y^2 - 12x^3 = 8y$

ii. $3y^7 + x^{10} = y^{-2} - 6x^3 + 2$

iii. $y^{-3} + 4x^{-1} = 8y^{-1}$

iv. $10x^4 - y^{-6} = 7y^3 + 4x^{-3}$

v. $\sin(x) + \cos(y) = \mathbf{e}^{4y}$

vi. $x + \ln(y) = \sec(y)$

vii. $y^2\left(4 - x^2\right) = y^7 + 9x$

viii. $6x^{-2} - x^3y^2 + 4x = 0$

ix. $8xy + 2x^4y^{-3} = x^3$

x. $yx^3 - \cos(x)\sin(y) = 7x$

xi. $\mathbf{e}^x\cos(y) + \sin(xy) = 9$

xii. $x^2 + \sqrt{x^3 + 2y} = y^2$

xiii. $\tan(3x + 7y) = 6 - 4x^{-1}$

xiv. $\mathbf{e}^{x^2+y^2} = \mathbf{e}^{x^2y^2} + 1$

xv. $\sin\left(\frac{x}{y}\right) + x^3 = 2 - y^4$