

KERNELS FOR GRASSMANN FLOPS

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ABSTRACT. We develop a generalization of the Q -construction of the first author, Diemer, and the third author for Grassmann flops over an arbitrary field of characteristic zero. This generalization provides a canonical idempotent kernel on the derived category of the associated global quotient stack. This idempotent kernel, after restriction, induces a derived equivalence over any twisted form of a Grassmann flop. Furthermore its image, after restriction to the geometric invariant theory semistable locus, “opens” a canonical “window” in the derived category of the quotient stack. We check this window coincides with the set of representations used by Kapranov to form a full exceptional collection on Grassmannians. Even in the well-studied special case of standard Atiyah flops, the arguments yield a new proof of the derived equivalence.

CONTENTS

Introduction	1
Acknowledgements	4
Notation	4
2. Background	4
3. The kernel	5
3.1. The functor	7
3.2. The integral kernel	15
3.3. Bousfield localizations	16
4. A geometric resolution	17
4.1. A sketch of Kempf	18
4.2. The resolution	18
5. Analyzing the integral transform	20
5.1. Windows from a resolution	20
5.2. Truncation operator	21
5.3. The equivalence	24
References	26

INTRODUCTION

Derived categories, once viewed as a mere technical book-keeping device, have flourished as a topic of investigation with volumes of literature exposing their geometric nature. Derived categories of coherent sheaves on algebraic varieties bind algebraic geometry to commutative algebra, representation theory, symplectic geometry, and theoretical physics in deep and surprising ways. Viewing the derived category of a variety X as a primary

invariant of X requires one to face a basic but highly non-trivial question: when are two derived categories of coherent sheaves equivalent?

A central motivating conjecture of Bondal and Orlov [BO95] extended by Kawamata [Kaw02], identifies a source for nontrivial derived equivalences.

Conjecture (Bondal-Orlov 1995). Assume that Z and Z' are smooth complex varieties. If Z and Z' are related by a flop, then there is a \mathbb{C} -linear triangulated equivalence of their bounded derived categories of coherent sheaves

$$D^b(Z) \cong D^b(Z').$$

Flops themselves figure importantly in the minimal model program and arise via phase transitions in string theory. Consequently, this conjecture offers a precise meeting ground for for three areas of research (birational geometry, derived categories and theoretical physics). Like many enticing but challenging questions, the conjecture provides an interesting conclusion without suggesting a direct method of assault. This begs the question - how can one cook up an equivalence from a flop in general?

The Bondal-Orlov Conjecture is known to hold in many specific examples, including when Z and Z' are three-dimensional [Bri99]. One specific example of interest here is the class of Grassmann flops studied by Donovan and Segal [DS14]. To construct the flop, one begins with the $GL(V)$ action on $Z := \text{Hom}(V, W) \oplus \text{Hom}(W, V)$ and passes to the two associated GIT quotients, which we denote by X^- and X^+ . In [DS14], Donovan and Segal show that $D^b(X^-) \cong D^b(X^+)$ by using a particular set of representations of $GL(V)$ shown by Kapranov [Kap88] to constitute a full strong exceptional collection on the Grassmannian $\text{Gr}(d, W)$, where $d = \dim V$ and comparing their associated vector bundles on X^+ and X^- .

The first step seeking to cook up an equivalence from a flop is to recognize that one should not be looking for a functor but rather a kernel. Given two smooth and projective varieties X and Y and an object of the derived category of their product $\mathcal{K} \in D^b(X \times Y)$, one gets an exact functor

$$\Phi_{\mathcal{K}} := \pi_{2*}(\mathcal{K} \otimes_{\mathcal{O}_{X \times Y}} \pi_1^*(-)) : D^b(X) \rightarrow D^b(Y).$$

where

$$\begin{array}{ccc} & X \times Y & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & & Y \end{array}$$

are the two projections. The object \mathcal{K} is called the kernel of the integral transform $\Phi_{\mathcal{K}}$. If $\Phi_{\mathcal{K}}$ is an equivalence, one calls \mathcal{K} a Fourier-Mukai kernel. The main questions transforms to - how can one cook up a Fourier-Mukai *kernel* from a flop in general?

Not all examples that satisfy the Bondal-Orlov Conjecture directly answer this question. In particular, no explicit Fourier-Mukai kernel is given for Grassmann flops. While an appeal can be made Orlov's representability result [Orl97], this discards the geometry one should focus on to make progress.

Even when an abstract equivalence is known to exist, having an explicit kernel built from actual geometry is still vital for:

- studying the induced maps on invariants such as algebraic or topological K-theory,

- looking for identities that multiple equivalences might satisfy, e.g. groupoid representations in equivalences,
- and understanding how the equivalence behaves under base change, in particular whether it descends.

In [BDF17], the first author, Diemer, and the third author gave a means of constructing a kernel on $Z \times Z'$ for any D -flip (Z, Z') . For flops of smooth projective varieties, the associated integral transform is conjectured to be the desired equivalence of Bondal and Orlov. This provides a single unified approach to constructing equivalences from flops.

As part of this “ Q -construction” in [BDF17], one uses a functorial construction of Drinfeld which yields partial compactifications of \mathbb{G}_m -actions on schemes. Morally speaking, this is given by a moduli space of specific types of orbit degenerations coming from an affine monoidal partial compactification of \mathbb{G}_m , i.e., \mathbb{A}^1 with multiplication.

One may generalize Drinfeld’s construction to the case of a linear algebraic monoid M with unit group G . A natural place to begin investigating this generalization is $M = \text{End}(V)$ for a \mathbb{k} -vector space V .

Here we give a thorough examination of the generalized Q -construction for twisted Grassmann flops. As a consequence, we give an explicit Fourier-Mukai kernel.

Theorem 1. Let \mathbb{k} be an (arbitrary) field of characteristic zero. The structure sheaf of the fiber product of a twisted Grassmann flop diagram

$$\begin{array}{ccc} X^+ & \overset{\dashrightarrow}{\longleftarrow} & X^- \\ & \searrow & \swarrow \\ & X_0 & \end{array}$$

induces an equivalence $D^b(X^+) \cong D^b(X^-)$, ie $\mathcal{O}_{X^+ \times_{X_0} X^-}$ is a Fourier-Mukai kernel.

In the case $d = 1$, the Grassmann flop reduces to a twisted form of a standard Atiyah flop. If the flop is honestly twisted, then even the existence of a derived equivalence is new for any honestly twisted form of a Grassmann, or even Atiyah, flop. One should note that the results of [DS14] do not translate directly to non-algebraically closed \mathbb{k} , see Remark 5.3.3.

In the untwisted $d = 1$ case, Theorem 1 recovers the well-known result of Bondal and Orlov [BO95]. However, the proof is genuinely new. In general, the results for Grassmann flops rely only on cohomology computations of vector bundles on Grassmannians via the Borel-Bott-Weil Theorem.

Theorem 1 is proven by

- (1) showing that Q induces an equivalence, Theorem 5.3.1, and
- (2) checking that Q base changes to the fiber product of the flop, Theorem 3.1.22.

The next result demonstrates that Q provides a canonical window in the derived category of the quotient stack $[X/\text{GL}(V)]$, similar to situation studied in [BDF17].

Theorem 2. Let $Q_+ := Q|_{X^+ \times X}$. Then

- The functor

$$\Phi_{Q_+} : D^b(X^+) \rightarrow D^b([Z/\text{GL}(V)])$$

is fully-faithful.

- The restriction map $j^* : D^b([Z/\text{GL}(V)]) \rightarrow D^b(X^+)$ is a left inverse to Φ_{Q_+} .

- Kapranov's representations form a set of generators for the essential image of Φ_{Q_+} .

Theorem 2, in particular, provides a completely geometric explanation for the appearance of Kapranov's representations. Our method of monoid compactification can therefore be seen as part of a program to produce canonical windows for quotients via linearly reductive groups.

All results naturally flow from the fact that Q is a idempotent functor on $\mathbf{D}^b[X/\mathrm{GL}(V)]$. This can be summarized as the following consequence of Lemmas 3.1.16 and 3.1.23.

Theorem 3. There exists a morphism of kernels $Q \rightarrow \Delta$ inducing an isomorphism of $Q \circ Q \rightarrow Q$, where \circ denotes convolution of kernels and Δ is the kernel of the identity.

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Notation. Throughout, \mathbb{k} denotes a field of characteristic zero. So that no confusion arises we consider $0 \in \mathbb{N}$. Further, for $\ell \in \mathbb{N}$, $\ell \neq 0$, we let $[\ell] := \{1, \dots, \ell\}$. We let $\mathrm{Vec}_{\mathbb{k}}$ denote the category of finite-dimensional \mathbb{k} -vector spaces and \mathbb{k} -linear transformations, and for a \mathbb{k} -algebra S we denote Aff_S the category of affine $\mathrm{Spec}(S)$ -schemes and morphisms of schemes. We utilize standard results in Geometric Invariant Theory and attempt to align our notation with that of [Mum65]. All schemes considered here are \mathbb{k} -schemes, and we denote the global sections of the structure sheaf of a \mathbb{k} -scheme Z as $\mathbb{k}[Z]$. The word *point* will always mean \mathbb{k} -point.

2. BACKGROUND

Throughout, fix a d -dimensional \mathbb{k} -vector space V .

Definition 2.0.1. For any \mathbb{k} -vector space W we can naturally associate a scheme over $\mathrm{Spec}(\mathbb{k})$ defined as the spectrum of the symmetric algebra of the dual space W^\vee , that is

$$\mathrm{Spec}(\mathrm{Sym}(W^\vee))$$

we will refer to this scheme as the **geometric bundle** of W over $\mathrm{Spec}(\mathbb{k})$. In general for a \mathbb{k} -scheme X and a locally free \mathcal{O}_X -module M we will refer to the relative spectrum of the symmetric sheaf of algebras of M^\vee as the corresponding geometric bundle over X .

With the above definition in mind we will denote by $\mathrm{GL}(V)$ the linear algebraic group

$$\mathrm{GL}(V) := \mathrm{Spec}\left(\mathbb{k}\left[C, (\det(C))^{-1}\right]\right),$$

where $C = (c_{ij})_{i,j \in [d]}$ is a collection of indeterminates. We use $\det(C)$ to denote the polynomial

$$\det(C) := \sum_{\delta \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}(\delta)} \left(\prod_{i \in [d]} c_{i\delta(i)} \right).$$

Next, recall that for a \mathbb{k} -scheme Z , an action of $\mathrm{GL}(V)$ on Z is defined by a morphism of schemes

$$\sigma_Z : \mathrm{GL}(V) \times_{\mathbb{k}} Z \rightarrow Z.$$

If Z and Y are \mathbb{k} -schemes with $\mathrm{GL}(V)$ -action given by σ_Z and σ_Y , we say that a morphism $f : Z \rightarrow Y$ is $\mathrm{GL}(V)$ -equivariant whenever the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}(V) \times_{\mathbb{k}} Z & \xrightarrow{1_{\mathrm{GL}(V)} \times_{\mathbb{k}} f} & \mathrm{GL}(V) \times_{\mathbb{k}} Y \\ \sigma_Z \downarrow & & \downarrow \sigma_Y \\ Z & \xrightarrow{f} & Y \end{array}$$

Most of this work deals with categories whose objects carry a $\mathrm{GL}(V)$ -action and $\mathrm{GL}(V)$ -equivariant morphisms. To denote such a restriction, we simply use the superscript $\mathrm{GL}(V)$. For example, let R be a commutative ring with a $\mathrm{GL}(V)$ -action. We denote the category of modules with an inherited $\mathrm{GL}(V)$ -action and $\mathrm{GL}(V)$ -equivariant morphisms as $\mathrm{Mod}^{\mathrm{GL}(V)}(R)$.

Next we will establish further notational conventions. Given a collection $A = \{a_{ij}\}_{i \in [m], j \in [d]}$ of indeterminates, for any subsets $J \subseteq [d]$, $I \subseteq [m]$, we let $A_{I,J}$ denote the collection of variables

$$A_{I,J} := (a_{ij})_{i \in I, j \in J}.$$

For $I \subseteq [m]$ with $|I| = d$, we may list this set in increasing order, and denote the corresponding ordered set by $I := \{\ell_1, \dots, \ell_d\}$. We then write

$$\det(A_{I,[d]}) := \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) \left(\prod_{1 \leq i \leq d} A_{\ell_i, \sigma(i)} \right).$$

Given two collections $A = \{a_{ij}\}_{i \in [m], j \in [d]}$ and $B = \{b_{ij}\}_{i \in [d], j \in [m]}$, we use BA to denote the collection of polynomials

$$\left\{ \sum_{\ell=1}^m b_{i\ell} a_{\ell j} \right\}_{i \in [d], j \in [d]}.$$

Lastly, given two \mathbb{k} -algebras R and S and elements $r \in R$ and $s \in S$, we let r^L and s^R denote the elements $r \otimes 1$ and $1 \otimes s$ in $R \otimes_{\mathbb{k}} S$.

3. THE KERNEL

Recall that in [BDF17] the authors exhibit a kernel using a partial compactification of a certain \mathbb{G}_m -action. We follow a similar line of reasoning, and begin by defining our main categories of interest. Let W and W' be arbitrary finite dimensional \mathbb{k} -vector spaces. Next, consider the vector spaces

$$\mathrm{Hom}_{\mathrm{Vec}_{\mathbb{k}}}(V, W) \oplus \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{k}}}(W', V),$$

which carry a natural action, for $\varsigma \in \mathrm{GL}(V)$ (a point), $\varphi \in \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{k}}}(V, W)$, and $\vartheta \in \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{k}}}(W', V)$ as $\varsigma \cdot (\varphi \circ \varsigma, \varsigma^{-1} \circ \vartheta)$. The category of all such geometric bundles of the above type will be denoted as $\mathrm{AM}_{\mathbb{k}}^{\mathrm{GL}(V)}$, whose morphisms are morphisms of schemes which are $\mathrm{GL}(V)$ -equivariant relative to the above action. Let us provide a few more specifics concerning the action of $\mathrm{GL}(V)$ on arbitrary objects of $\mathrm{AM}_{\mathbb{k}}^{\mathrm{GL}(V)}$. For such an object Z , the induced action

$$\sigma_Z : \mathrm{GL}(V) \times_{\mathbb{k}} Z \rightarrow Z$$

is equivalent to the co-action as Hopf algebra modules. Choosing bases for V , W and W' , we may write

$$Z = \text{Spec}(\mathbb{k}[\{a_{ij}\}_{i \in [d], j \in [m_0]}, \{b_{ij}\}_{i \in [m_1], j \in [d]}]),$$

where $m_1 = \dim W$ and $m_0 = \dim W'$. Letting $B := (b_{ij})$ and $A := (a_{ij})$, we have

$$(3.1) \quad Z = \text{Spec}(\mathbb{k}[A, B]).$$

The co-action on the global sections

$$\sigma_Z^\sharp : \mathbb{k}[Z] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[Z],$$

is defined on the generators as

$$b_{ij} \mapsto (\det(C))^{-1} \sum_{r=1}^d \text{Adj}(C)_{rj} \otimes_{\mathbb{k}} b_{ir}$$

$$a_{ij} \mapsto \sum_{r=1}^d c_{ir} \otimes_{\mathbb{k}} a_{rj},$$

where $\text{Adj}(C)$ is the adjoint matrix of C .

To build the readers intuition further let $\varphi \in \text{Hom}(V, W)$, and choose a basis of V and W such that $\varphi = (\varphi_{ij})$ with $i \in [m_0]$ and $j \in [d]$, and $\varsigma \in \text{GL}(V)$ such that under the chose of basis for V then $\varsigma = (\varsigma_{s\ell})$ for $s, \ell \in [d]$. Therefore the action of $\text{GL}(V)$ on $\text{Hom}(V, W)$ can be defined as

$$(\varsigma, \varphi) \mapsto \varphi \circ \varsigma$$

$$\varphi_{ij} \mapsto \sum_{\ell=1}^d \varphi_{i\ell} \varsigma_{\ell j}$$

Thus we can define a co-action on the dual $(\text{Hom}(V, W))^\vee$ by:

$$\varphi_{ji} = (\varphi_{ij})^\vee \mapsto \left(\sum_{\ell=1}^d \varphi_{i\ell} \varsigma_{\ell j} \right)^\vee = \sum_{\ell=1}^d \varsigma_{j\ell} \varphi_{\ell i}$$

Therefore if we identify $(\text{Hom}(V, W))^\vee$ with $\text{Sym}^1((\text{Hom}(V, W))^\vee)$ we get an action on $\text{Spec}(\text{Sym}((\text{Hom}(V, W))^\vee))$, i.e. the geometric bundle of $\text{Hom}(V, W)$ over $\text{Spec}(\mathbb{k})$. With a similar calculation on the regular functions of $\text{Hom}(W', V)$ we derive part of the action on Z described above.

Furthermore, the projection

$$\pi_Z : \text{GL}(V) \times_{\mathbb{k}} Z \rightarrow Z$$

induces the map

$$\pi_Z^\sharp : \mathbb{k}[Z] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[Z].$$

Further, let $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$ denote the full subcategory of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$ consisting of objects of the form

$$\text{Hom}(V, W) \oplus \text{Hom}(W', V)$$

such that $\dim(W), \dim(W') \geq \dim(V)$.

3.1. The functor. Before turning attention to our functor \mathbf{Q} , we introduce the functor Δ , which gives the kernel of the identity functor.

Notation 3.1.1. If $Z = \text{Spec}(R)$ is an element of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, we define the following scheme

$$\Delta_Z := Z \times_{\mathbb{k}} \text{GL}(V).$$

The assignment $Z \mapsto \Delta_Z$ defines a functor $\Delta : \text{AM}_{\mathbb{k}}^{\text{GL}(V)} \rightarrow \text{Aff}_{\mathbb{k}}^{\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V)}$.

We aim to use Δ_Z to produce the Fourier-Mukai kernel for the identity functor on the bounded $\text{GL}(V)$ -equivariant derived category $\text{D}^b(\text{Qcoh}^{\text{GL}(V)} Z)$ by associating to it an object of $\text{D}^b(\text{Qcoh}^{\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V)} Z \times_{\mathbb{k}} Z)$. To achieve this, consider the morphism

$$\pi_Z \times_{\mathbb{k}} \sigma_Z : Z \times_{\mathbb{k}} \text{GL}(V) \rightarrow Z \times_{\mathbb{k}} Z.$$

We define a sheaf of modules over $Z \times_{\mathbb{k}} Z$ associated to Δ_Z as

$$\tilde{\Delta}_Z := (\pi_Z \times_{\mathbb{k}} \sigma_Z)_* \mathcal{O}_{\Delta_Z},$$

where \mathcal{O}_{Δ_Z} denotes the structure sheaf of the affine scheme Δ_Z . It remains to define an action that realizes this sheaf as a $(\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V))$ -equivariant sheaf over $Z \times_{\mathbb{k}} Z$, which is provided in the next lemma.

Lemma 3.1.2. *For $Z = \text{Spec}(R)$ an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, the scheme $\text{GL}(V) \times_{\mathbb{k}} Z = \Delta_Z$ has a natural $(\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V))$ -action*

$$\sigma_{\Delta_Z} : \text{GL}(V) \times_{\mathbb{k}} \text{GL}(V) \times_{\mathbb{k}} \Delta_Z \rightarrow \Delta_Z$$

uniquely determined by the co-action

$$\sigma_{\Delta_Z}^{\sharp} : \mathbb{k}[\Delta_Z] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\Delta_Z]$$

defined by

$$\begin{aligned} \sigma_{\Delta_Z}^{\sharp}(1 \otimes r) &= (\iota_1 \otimes 1_{\Delta_Z}) \circ \sigma_Z^{\sharp}(r) \\ \sigma_{\Delta_Z}^{\sharp}(t \otimes 1) &= \left((1 \otimes \mu^{\sharp}) \circ (\beta^{\sharp} \otimes 1) \circ s^{\sharp} \circ \mu^{\sharp}(t) \right) \otimes 1_R. \end{aligned}$$

Here $r \in R$, $t \in \mathbb{k}[\text{GL}(V)]$ and $\iota_1 : \mathbb{k}[\text{GL}(V)] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\text{GL}(V)]$ is the natural inclusion into the first component; $\beta^{\sharp} : \mathbb{k}[\text{GL}(V)] \rightarrow \mathbb{k}[\text{GL}(V)]$ is the co-inverse, $\mu^{\sharp} : \mathbb{k}[\text{GL}(V)] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\text{GL}(V)]$ is the group co-multiplication and $s^{\sharp} : \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\text{GL}(V)] \rightarrow \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} \mathbb{k}[\text{GL}(V)]$ switches the factors in the tensor product.

Moreover, the map $\pi_Z \times_{\mathbb{k}} \sigma_Z : \text{GL}(V) \times_{\mathbb{k}} Z \rightarrow Z \times_{\mathbb{k}} Z$ is equivariant with respect to this $\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V)$ action.

The proof is a straight-forward diagram chase and is left to the reader.

Lemma 3.1.3. *Let $Z = \text{Spec}(R)$ be an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$ and M an object of $\text{Mod}^{\text{GL}(V)}(R)$. Then there is a $\mathbb{k}[\text{GL}(V)]$ -co-module isomorphism*

$$\left(\mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} M \right)^{\text{GL}(V)} \cong M,$$

where $\mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} M$ is given the left $\mathbb{k}[\text{GL}(V)]$ -co-action as a $\mathbb{k}[\Delta_Z]$ -module.

Proof. Note that there is a natural morphism

$$M \rightarrow \left(\mathbb{k}[\mathrm{GL}(V)] \otimes_{\mathbb{k}} M \right)^{\mathrm{GL}(V)}$$

given by the equivariant structure of M . Since the extension $\bar{\mathbb{k}}/\mathbb{k}$ is faithfully-flat, it suffices to show that this map is an isomorphism over $\bar{\mathbb{k}}$. Assume that $\bar{\mathbb{k}} = \mathbb{k}$.

By the Peter-Weyl Theorem, there is a decomposition $\mathbb{k}[\mathrm{GL}(V)] = \bigoplus S_i \otimes S_i^\vee$, where S_i runs over every irreducible representation of $\mathrm{GL}(V)$. Furthermore, since $\mathrm{GL}(V)$ is linearly reductive, we have a decomposition $M = \bigoplus M_i$ into irreducible components. Thus, we have

$$\left(\mathbb{k}[\mathrm{GL}(V)] \otimes_{\mathbb{k}} M \right)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes (S_i^\vee \otimes M_i)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes \mathrm{Hom}_{\mathbb{k}}^{\mathrm{GL}(V)}(S_i, M_i),$$

and our result follows from Schur's Lemma. \square

Lemma 3.1.4. *For any object Z of $\mathrm{AM}_{\mathbb{k}}^{\mathrm{GL}(V)}$, the object*

$$\tilde{\Delta}_Z \in \mathrm{D}^b(\mathrm{Qcoh}^{\mathrm{GL}(V) \times_{\mathbb{k}} \mathrm{GL}(V)}(Z \times_{\mathbb{k}} Z))$$

is the Fourier-Mukai kernel of the identity functor on $\mathrm{D}^b(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z)$.

Proof. First note that since Δ_Z is flat via either module structure and the Reynolds operator is flat, it is sufficient to prove this on the level of R -modules. For an R -module M , the integral transform associated to $\tilde{\Delta}_Z$ is given by

$$\Phi_{\tilde{\Delta}_Z}(\tilde{M}) := \left[\mathbf{R}\pi_{2*} \left(\tilde{\Delta}_Z \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* \tilde{M} \right) \right]^{\mathrm{GL}(V)},$$

where π_i are the natural $\mathrm{GL}(V)$ -equivariant projections $Z \times_{\mathbb{k}} Z \rightarrow Z$ (see [BFK14, Section 2] for background). Our desired result is a consequence of the following calculation:

$$\begin{aligned} \Phi_{\tilde{\Delta}_Z}(\tilde{M}) &= \left[\mathbf{R}\pi_{2*} \left[((\pi_Z \times_{\mathbb{k}} \sigma_Z)_* \mathcal{O}_{\Delta_Z}) \otimes_{\mathcal{O}_{Z \times_{\mathbb{k}} Z}}^{\mathbf{L}} (\mathbf{L}\pi_1^* \tilde{M}) \right] \right]^{\mathrm{GL}(V)} \\ &\cong \left[\pi_{2*}(\pi_Z \times_{\mathbb{k}} \sigma_Z)_* \left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_{\Delta_Z}} \left((\pi \times_{\mathbb{k}} \sigma_Z)^* \pi_1^* \tilde{M} \right) \right] \right]^{\mathrm{GL}(V)} \\ &\cong \left[\sigma_{Z*} \pi_Z^* \tilde{M} \right]^{\mathrm{GL}(V)} \\ &\cong \left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_Z} \tilde{M} \right]^{\mathrm{GL}(V)} \\ &\cong \left[(\mathcal{O}_{\mathrm{GL}(V)} \otimes_{\mathbb{k}} \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \tilde{M} \right]^{\mathrm{GL}(V)} \\ &\cong \left[\mathcal{O}_{\mathrm{GL}(V)} \otimes_{\mathbb{k}} \tilde{M} \right]^{\mathrm{GL}(V)} \\ &\cong \tilde{M}, \end{aligned}$$

where the first isomorphism follows from the projection formula, and the last follows from Lemma 3.1.3. Furthermore, on the second isomorphism we may forego the process of deriving these functors as they are either exact or remain an adapted class (as discussed above). \square

We now define the natural generalization of the functor Q from [BDF17, Defn 2.1.6].

Definition 3.1.5. Given an object $Z = \text{Spec}(R)$ of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, define

$$Q_Z := \left(\pi_Y^\sharp(R), \sigma_Y^\sharp(R), C \right) \subseteq \mathbb{k}[\text{GL}(V) \times_{\mathbb{k}} Z].$$

that is the \mathbb{k} -subalgebra of $\mathbb{k}[\text{GL}(V) \times Z]$ generated by the images of $\sigma_Z^\sharp, \pi_Z^\sharp$ and the image of the inclusion $\mathbb{k}[\text{End}(V)] \hookrightarrow \mathbb{k}[\text{GL}(V) \times_{\mathbb{k}} Z]$. For ease of notation we denote $\mathbf{Q}_Z := \text{Spec}(Q_Z)$.

Remark 3.1.6. Similar to the functor Q in [BDF17, Def 2.1.6] our definition provides a partial compactification of the action of $\text{GL}(V)$ on Z . For ease of reference we recall the definition of a partial compactification next.

Definition 3.1.7. Let G be an algebraic group and Z a \mathbb{k} -scheme with G -action. Let \tilde{Z} be a \mathbb{k} -scheme together an action of $G \times_{\mathbb{k}} G$ which is equipped with a $(G \times_{\mathbb{k}} G)$ -equivariant open immersion

$$i : G \times_{\mathbb{k}} Z \hookrightarrow \tilde{Z},$$

as well as a $(G \times_{\mathbb{k}} G)$ -equivariant morphism

$$(p, s) : \tilde{Z} \rightarrow Z \times_{\mathbb{k}} Z$$

such that the following diagram commutes

$$\begin{array}{ccc} & & \tilde{Z} \\ & \nearrow i & \downarrow p \\ G \times_{\mathbb{k}} Z & \xrightarrow{\pi} & Z \\ & \xrightarrow{\sigma} & \downarrow s \\ & & Z \end{array}$$

where σ is the action of G on Z and π is the projection to Z . In this case, we refer to \tilde{Z} , with the maps p, s, i , as a *partial compactification of the action of G on Z* .

Example 3.1.8. If $\dim V = 1$, the category $\text{AM}_{\mathbb{k}}^{\text{GL}(V)} = \text{AM}_{\mathbb{k}}^{\text{G}_m}$ is a subcategory of $\text{CR}_{\mathbb{k}}^{\text{G}_m}$ as studied in [BDF17]. In this case, the definition of Q given here recovers that found in loc. cit.

Lemma 3.1.9. *Let $Z = \text{Spec}(R)$ be an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$. Then there are morphisms*

$$\mathbf{Q}_Z \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{s} \end{array} Z.$$

which precompose with the open immersion $\Delta_Z \rightarrow \mathbf{Q}_Z$ to give the morphisms π_Z and σ_Z .

Proof. By definition, the maps π_Z^\sharp and σ_Z^\sharp both have images which lie in Q_Z . \square

Lemma 3.1.10. *For any object Z of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, we have an isomorphism*

$$(3.2) \quad Q_Z \cong \mathbb{k}[A^L, B^L, A^R, B^R, C] / (B^L - B^R C, A^R - C A^L) \cong \mathbb{k}[A^L, B^R, C].$$

Proof. We provide the reader with an easily verifiable isomorphism defined on the generators by

$$\begin{aligned} c_{ij} &\mapsto c_{ij}, & \pi_Z^\sharp(a_{ij}) &\mapsto a_{ij}^L, & \pi_Z^\sharp(b_{ij}) &\mapsto b_{ij}^L, \\ \sigma_Z^\sharp(a_{ij}) &\mapsto a_{ij}^R, & \sigma_Z^\sharp(b_{ij}) &\mapsto b_{ij}^R. \end{aligned}$$

\square

Remark 3.1.11. It follows from Equation (3.2) that \mathbf{Q}_Z is equivalent to the closed subvariety of $(Z \times_{\mathbb{k}} Z) \times_{\mathbb{k}} \text{End}(V)$, consisting of the following points

$$\{(\psi_1, \psi_2, \psi_3, \psi_4, \varphi) \mid \psi_1 = \psi_3 \circ \varphi, \psi_4 = \varphi \circ \psi_2\}$$

Lemma 3.1.12. For any object Z of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, the scheme \mathbf{Q}_Z admits a $(\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V))$ -action, denoted $\sigma_{\mathbf{Q}_Z}$, which is uniquely defined by the co-action

$$\sigma_{\mathbf{Q}_Z}^{\#} : \mathbb{k}[A^L, B^R, C] \rightarrow \mathbb{k}[D^L, (\det D^L)^{-1}] \otimes \mathbb{k}[D^R, (\det D^R)^{-1}] \otimes \mathbb{k}[A^L, B^R, C],$$

which maps the generators

$$\begin{aligned} b_{ij}^R &\mapsto (\det(D^R))^{-1} \sum_{r=1}^n \text{Adj}(D^R)_{rj} \otimes_{\mathbb{k}} b_{ir}^R, \\ a_{ij}^L &\mapsto \sum_{r=1}^n d_{ir}^L \otimes_{\mathbb{k}} a_{rj}^L, \\ c_{ij} &\mapsto (\det(D^L))^{-1} \sum_{r=1}^n \text{Adj}(D^L)_{sj} \otimes_{\mathbb{k}} d_{ir}^R \otimes_{\mathbb{k}} c_{rs}, \end{aligned}$$

where $\text{Adj}(D)$ is the adjoint of the matrix D .

Proof. This follows by restricting the action of $(\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V))$ on Δ_Z that was defined in Lemma 3.1.2. \square

The next lemma gives explicit descriptions of the two module structures that Q_Z possesses.

Lemma 3.1.13. For $Z = \text{Spec}(\mathbb{k}[A, B])$ an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$, we have the following two $\mathbb{k}[A, B]$ -module structures on Q_Z given by $p^{\#}$ and $s^{\#}$, respectively:

$$\begin{aligned} p^{\#} : \mathbb{k}[A, B] &\rightarrow \mathbb{k}[A^L, B^R, C] \\ B &\mapsto B^R C \\ A &\mapsto A^L \\ s^{\#} : \mathbb{k}[A, B] &\rightarrow \mathbb{k}[A^L, B^R, C] \\ B &\mapsto B^R \\ A &\mapsto C A^L \end{aligned}$$

Proof. These are just the maps induced by the description of Q_Z from Lemma 3.1.10 under the identification

$$Q_Z = \mathbb{k}[A^L, B^R, C].$$

\square

Proposition 3.1.14. For any object Z of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$ the assignment $Z \mapsto \mathbf{Q}_Z$ defines a functor $\mathbf{Q} : \text{AM}_{\mathbb{k}}^{\text{GL}(V)} \rightarrow \text{Aff}_{\mathbb{k}[\text{End}(V)]}^{\text{GL}(V) \times \text{GL}(V)}$.

Proof. Let $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$ be objects of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$ with $f \in \text{Hom}_{\text{AM}_{\mathbb{k}}^{\text{GL}(V)}}(X, Y)$. Note that $\mathbf{Q}f : \mathbf{Q}_X \rightarrow \mathbf{Q}_Y$ is defined as the restriction of $f \otimes 1 : X \times_{\mathbb{k}} \text{GL}(V) \rightarrow Y \times_{\mathbb{k}} \text{GL}(V)$,

which is well defined since f is assumed to be $\mathrm{GL}(V)$ -equivariant. With this description it is readily verified that indeed \mathbf{Q} is functorial. \square

Remark 3.1.15. It follows immediately from the definition that Δ is a subfunctor of \mathbf{Q} . Furthermore, this definition easily extends to any affine variety with a $\mathrm{GL}(V)$ -action; yet this level of generalization is outside the scope of this paper. We note that our choice of subcategory $\mathrm{AM}_{\mathbb{k}}^{\mathrm{GL}(V)} \subset \mathrm{Aff}_{\mathbb{k}}^{\mathrm{GL}(V)}$, is intended to give an appropriate generalization of the varieties considered in [DS14] while not having to encounter any unnecessary technical difficulties in the statements of this preliminary section.

Now, we prove some properties of Q that will be used in Section 3.3 to prove the fullness of a Fourier-Mukai transform constructed using Q .

Lemma 3.1.16. *For an object $Z = \mathrm{Spec}(R)$ of $\mathrm{AM}_{\mathbb{k}}^{\mathrm{GL}(V)}$, we have*

$$\mathrm{Tor}_i({}_p Q_Z, {}_s Q_Z) = 0$$

for all $i > 0$, where the subscripts preceding Q_Z denote the R -module structures given by p^\sharp or s^\sharp , respectively.

Proof. Let $R := \mathbb{k}[A, B]$ as in Equation (3.1). By Lemma 3.1.13, we have

$$\begin{aligned} {}_p Q_Z &\cong \mathbb{k}[A, B, B', C] / (B - B'C) \\ {}_s Q_Z &\cong \mathbb{k}[A, B, A', C] / (A - CA') \end{aligned}$$

Let us compute $Q_Z \otimes_p^{\mathbf{L}} Q_Z$ using the above expressions:

$$\begin{aligned} Q_Z \otimes_p^{\mathbf{L}} Q_Z &= \mathbb{k}[A, B, A', C] / (A - CA') \otimes_{\mathbb{k}[A, B]}^{\mathbf{L}} \mathbb{k}[A, B, B', C] / (B - B'C) \\ &\cong \mathbb{k}[A, B, A', C] / (A - CA') \otimes_{\mathbb{k}[A, B]} \mathcal{K}_{\mathbb{k}[A, B, B', C]}(B - B'C) \\ &\cong \mathcal{K}_{\mathbb{k}[A, B, A', B', C_1, C_2] / (A - C_2 A')} (B - B' C_1) \\ &\cong \mathcal{K}_{\mathbb{k}[B, A', B', C_1, C_2]} (B - B' C_1), \end{aligned}$$

where we resolved the regular sequence $(B - B'C)$ by the Koszul complex, denoted by \mathcal{K} , on the second line.

Finally, we see that the sequence $(B - B' C_1)$ is still regular in the ring $\mathbb{k}[B, A', B', C_1, C_2]$ and hence all the higher homologies vanish. \square

Notation 3.1.17. Similar to an observation of $\mathbb{k}[\Delta_Z]$, the ring Q_Z is naturally associated to a sheaf of modules over Z , with its module structure defined via s or p . We may thus realize Q_Z as a sheaf of modules over $Z \times_{\mathbb{k}} Z$, and we denote this module by

$$\widehat{Q}_Z := (p \times_{\mathbb{k}} s)_* \mathcal{O}_{Q_Z}.$$

We will use the same notation in the derived setting (see Section 3.2, particularly Remark 3.2.4). Furthermore, as Δ_Z is an open subset of \mathbf{Q}_Z we will denote the natural open immersion as

$$\eta : \mathrm{GL}(V) \times_{\mathbb{k}} Z \rightarrow \mathbf{Q}_Z.$$

We now specialize to the case where $Z = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$ is an arbitrary object of $\mathrm{HP}_{\mathbb{k}}^{\mathrm{GL}(V)}$, so that $\dim W, \dim W' \geq \dim V = d$. We also recall that we denote

$\dim W = m_0$ and $\dim W' = m_1$. Now consider the two open sets

$$U^+ := \left(\text{Hom}(V, W) \setminus \{\varphi : \text{rank}(\varphi) \leq (d-1)\} \right) \oplus \text{Hom}(W', V)$$

$$U^- := \text{Hom}(V, W) \oplus \left(\text{Hom}(W', V) \setminus \{\vartheta : \text{rank}(\vartheta) \leq (d-1)\} \right).$$

It will be useful to denote the following open covers of these quasi-affine sets. Let

$$(3.3) \quad U^+ = \bigcup_{J \subseteq [m_0], |J|=d} U_J^+,$$

$$(3.4) \quad U^- = \bigcup_{I \subseteq [m_1], |I|=d} U_I^-,$$

where

$$U_I^+ := \text{Spec} \left(\mathbb{k} \left[A, B, (\det(A_{[d], I})^{-1}) \right] \right)$$

$$U_J^- := \text{Spec} \left(\mathbb{k} \left[A, B, (\det(B_{J, [d]})^{-1}) \right] \right)$$

and (for example) $\det(A_{[d], I})$ denotes the $(d \times d)$ minor of A consisting of the rows indexed by I . Therefore, we have the following affine open covers:

$$(3.5) \quad U^+ \times_{Z//0} U^- = \bigcup_{I \subseteq [m_0], J \subseteq [m_1], |I|=|J|=d} U_I^+ \times_{Z//0} U_J^-,$$

$$(3.6) \quad U^+ \times_{\mathbb{k}} U^- = \bigcup_{I \subseteq [m_0], J \subseteq [m_1], |I|=|J|=d} U_I^+ \times_{\mathbb{k}} U_J^-,$$

where $Z//0 := \text{Spec}(\mathbb{k}[A, B]^{\text{GL}(V)})$ denotes the invariant theoretic quotient of Z .

Lemma 3.1.18. *Let Z be an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$. There is an isomorphism*

$$\mathbb{k}[Z \times_{Z//0} Z] \cong \mathbb{k}[A^L, B^L, A^R, B^R] / (B^L A^L - B^R A^R),$$

where the generators and relations are as in Definition 3.1.5.

Proof. From Weyl's fundamental theorems for the action of $\text{GL}(V)$ (for example see [KP96, Chapter 2.1] or the original text [Wey46]) we have

$$Z//0 = \{D \in \text{Hom}_{\mathbb{k}}(W, W) \mid \text{rank } D \leq \dim V\}.$$

The map $Z \rightarrow Z//0$ is thus given by the homomorphism

$$\mathbb{k}[Z//0] \rightarrow \mathbb{k}[A, B]$$

$$D \mapsto BA.$$

Hence,

$$\mathbb{k}[Z \times_{Z//0} Z] = \mathbb{k}[A^L, B^L] \otimes_{\mathbb{k}[Z//0]} \mathbb{k}[A^R, B^R] \cong \mathbb{k}[A^L, B^L, A^R, B^R] / (B^L A^L - B^R A^R).$$

□

Lemma 3.1.19. *There exists a morphism*

$$\kappa := p^\# \otimes s^\# : \mathbb{k}[A^L, B^L] \otimes_{\mathbb{k}[Z//0]} \mathbb{k}[A^R, B^R] \rightarrow Q_Z.$$

Proof. This follows since $p^\#$ and $s^\#$ are equal on $\mathbb{k}[Z//0]$, by definition. \square

Lemma 3.1.20. *With the conventions above we have the following containment of ideals in the ring $\mathbb{k}[A^L, B^L, A^R, B^R, C]$:*

$$(B^L A^L - B^R A^R) \subset (B^L - B^R C, A^R - C A^L).$$

Proof. This follows from

$$(B^L - B^R C)A^L + B^R(CA^L - A^R) = B^L A^L - B^R A^R.$$

\square

Proposition 3.1.21. *Let $Z = \text{Spec}(\mathbb{k}[A, B])$ be an object of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$ and Q_Z as in Equation (3.2). Let $\widehat{Q}_Z|_{U^+ \times_{\mathbb{k}} U^-}$ be the restriction of \widehat{Q}_Z to the open subset $U^+ \times_{\mathbb{k}} U^- \subset Z \times_{\mathbb{k}} Z$. Then κ restricts to an isomorphism*

$$\kappa|_{U^+ \times_{\mathbb{k}} U^-} : \widehat{Q}_Z|_{U^+ \times_{\mathbb{k}} U^-} \xrightarrow{\sim} \mathcal{O}_{U^+ \times_{Z//0} U^-}.$$

Proof. We look affine-locally using the covers of Equations 3.5 and 3.6. We need only show that under the above localization the map $\kappa : \mathbb{k}[Z \times_{Z//0} Z] \rightarrow Q_Z$ becomes an isomorphism. For surjectivity, it suffices to show that there is an element (we find two such) which map to C . Indeed, we have

$$\begin{aligned} ((B^R)_{J[d]})^{-1} A^L &\mapsto C \\ B^R ((A^L)_{[d]I})^{-1} &\mapsto C, \end{aligned}$$

easily verified by the relations $B^L - B^R C$ and $A^R - C A^L$ in $Q(\mathbb{k}[A, B])$ given in Definition 3.1.5.

For injectivity, it suffices to check that under this localization we have the containment

$$(B^L - B^R C, A^R - C A^L) \subset (B^L A^L - B^R A^R),$$

since the opposite containment is Lemma 3.1.21. To see this, simply note that by multiplying by the appropriate elements in the above identification, we have

$$(B^L)_{J[d]} = (B^R)_{J[d]} C \quad \text{and} \quad (A^R)_{[d]I} = C (A^L)_{[d]I}.$$

Hence, multiplying by the appropriate units in our localization, we have

$$(B^L - B^R C, A^R - C A^L) = ((B^R)_{J[d]} (A^R - C A^L), (B^L - B^R C) (A^L)_{[d]I}).$$

For example, by Equation (3.1), we have

$$(B^R)_{J[d]} (A^R - C A^L) = ((B^R)_{J[d]} A^R - (B^L)_{J[d]} A^L) \in (B^L A^L - B^R A^R),$$

while the other relation follows similarly. This gives our desired isomorphism. \square

Consider the restriction $Q|_{U^+ \times U^-}$. By descent, we have a corresponding object P on the quotient $Z^+ \times Z^-$.

Theorem 3.1.22. *For an object Z of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$ we have an isomorphism*

$$P \cong \mathcal{O}_{Z^+ \times_{Z_0} Z^-}$$

Proof. This follows immediately by passing to the quotient in Proposition 3.1.21. \square

We now examine a useful invariant when studying kernels in the next subsection. Note that for Z , an object of $\mathbf{AM}_{\mathbb{k}}^{\mathrm{GL}(V)}$, the tensor product $Q_Z \otimes_p Q_Z$ is equipped with a natural $\mathrm{GL}(V)^{\times 4}$ -action. This induces a $\mathrm{GL}(V)^{\times 3}$ -action, which we denote

$$\sigma_3 : \mathrm{GL}(V)^{\times 3} \times_{\mathbb{k}} \mathbf{Q}_Z^{\times 2} \rightarrow \mathbf{Q}_Z^{\times 2}$$

and is defined as the product of the following compositions

$$\begin{array}{ccc} & \mathrm{GL}(V)^{\times 3} \times_{\mathbb{k}} \mathbf{Q}_Z^{\times 2} & \\ \pi_{1,2,4} \swarrow & & \searrow \pi_{2,3,5} \\ \mathrm{GL}(V)^{\times 2} \times_{\mathbb{k}} \mathbf{Q}_Z & & \mathrm{GL}(V)^{\times 2} \times_{\mathbb{k}} \mathbf{Q}_Z \\ \sigma_{\mathbf{Q}_Z} \downarrow & & \downarrow \sigma_{\mathbf{Q}_Z} \\ \mathbf{Q}_Z & & \mathbf{Q}_Z \end{array}$$

Here $\pi_{i,j,k} : \mathrm{GL}(V)^{\times 3} \times_{\mathbb{k}} \mathbf{Q}_Z^{\times 2} \rightarrow \mathrm{GL}(V)^{\times 2} \times_{\mathbb{k}} \mathbf{Q}_Z$ is the projection onto the i^{th} , j^{th} and k^{th} components. For any ring T with $\mathrm{GL}(V)^{\times 3}$ -action, we will denote the invariant subring associated to the action corresponding to the middle component of $\mathrm{GL}(V)^{\times 3}$ by T^{\boxtimes} . The notation $(-)^{\boxtimes}$ is suggestive of pinching a module in the middle. Since taking invariants is functorial for equivariant morphisms, we obtain the following:

Lemma 3.1.23. *The following diagram commutes*

$$\begin{array}{ccc} & (Q_Z \otimes_p \otimes_{\sigma} \Delta_Z)^{\boxtimes} & \\ (1 \otimes \eta)^{\boxtimes} \nearrow & & \searrow \sim \\ (Q_Z \otimes_p \otimes_s Q_Z)^{\boxtimes} & & Q_Z \\ (\eta \otimes 1)^{\boxtimes} \searrow & & \nearrow \sim \\ & (\Delta_Z \otimes_{\pi} \otimes_s Q_Z)^{\boxtimes} & \end{array}$$

Furthermore, the morphism $\rho_Z : (Q_Z \otimes_p \otimes_s Q_Z)^{\boxtimes} \rightarrow Q_Z$ is an isomorphism.

Proof. First recall that we have a presentation from Lemma 3.1.13 of ${}_s Q_Z$ and ${}_p Q_Z$, which for ease of calculation we set the following simplified notation, with the hope that no confusion arises:

$$\begin{aligned} {}_p Q_Z &\cong \mathbb{k}[A, B^L, B^R, C] / (B^L - B^R C) \cong \mathbb{k}[A, B^R, C] \\ &:= \mathbb{k}[A, B, C] \\ {}_s Q_Z &\cong \mathbb{k}[A^L, A^R, B, C] / (A^R - C A^L) \cong \mathbb{k}[A^L, B, C] \\ &:= \mathbb{k}[A, B, C] \end{aligned}$$

Further we recall the notational preference that for \mathbb{k} -algebras R, S and $r \in R, s \in S$ that the following pure tensors will be denoted: $r \otimes 1 := r^L$ and $1 \otimes s := s^R$. With these

conventions we have the following presentations of rings:

$$\begin{aligned} Q_Z \underset{p}{\otimes} Q_Z &\cong \mathbb{k}[A^L, A^R, B^L, B^R, C^L, C^R] / (B^L C^L - B^R, A^L - C^R A^R) \\ &\cong \mathbb{k}[A^R, B^L, C^L, C^R] \\ \mathbb{k}[\Delta_Z] \underset{\pi}{\otimes} Q_Z &\cong \mathbb{k}[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^L)^{-1}] / (B^L - B^R, A^L - C^R A^R) \\ &\cong \mathbb{k}[A^R, B^L, C^L, C^R, \det(C^L)^{-1}] \\ Q_Z \underset{p}{\otimes} \mathbb{k}[\Delta_Z] &\cong \mathbb{k}[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^R)^{-1}] / (B^L - B^R (C^R)^{-1}, A^L - C^R A^R) \\ &\cong \mathbb{k}[A^R, B^L, C^L, C^R, \det(C^R)^{-1}] \end{aligned}$$

Hence, commutativity of the above diagram is clear. Furthermore, one verifies that we have an isomorphism $\mathbb{k}[\Delta_Z] \underset{\pi}{\otimes} Q_Z \cong Q_Z \underset{p}{\otimes} \mathbb{k}[\Delta_Z]$, and thus

$$(\mathbb{k}[\Delta_Z] \underset{\pi}{\otimes} Q_Z)^{\boxtimes} \cong (Q_Z \underset{p}{\otimes} \mathbb{k}[\Delta_Z])^{\boxtimes}.$$

It is clear that the maps on the right-hand side of the diagram are isomorphisms since $\mathbb{k}[\Delta_Z]$ is the kernel of the identity by Lemma 3.1.4. We claim that

$$\begin{aligned} (Q_Z \underset{p}{\otimes} Q_Z)^{\boxtimes} &= \mathbb{k}[A^R, B^L, C^L \cdot C^R] \\ (Q_Z \underset{p}{\otimes} \mathbb{k}[\Delta_Z])^{\boxtimes} &= \mathbb{k}[A^R, B^L, C^L \cdot C^R] \end{aligned}$$

from which it follows that these rings are isomorphic. This claim is simply Weyl's Theorem for the invariants of $\mathbb{k}[V \otimes V^\vee]$. □

3.2. The integral kernel. We now use \mathbf{Q} to construct Fourier-Mukai kernels. We begin by recalling the following from [BDF17, Definition 3.1.4].

Definition 3.2.1. Let \tilde{Z} be a partial compactification of an action $\sigma : G \times_{\mathbb{k}} Z \rightarrow Z$, with maps p, s , and i as above. We define the *boundary* of \tilde{Z} to be

$$\partial_{\tilde{Z}}^s := \tilde{Z} \setminus i(G \times_{\mathbb{k}} Z),$$

the *s-unstable locus* to be

$$Z_s^{\text{us}} := s(\partial_{\tilde{Z}}^s),$$

and the *s-semistable locus* to be

$$Z_s^{\text{ss}} := Z \setminus Z_s^{\text{us}}.$$

One similarly defines the *p-unstable* and *p-semistable* loci.

Remark 3.2.2. It follows from [BDF17, Example 3.1.10] that for an object Z of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$, the *s-semistable locus* Z_s^{ss} coincides with U^+ from Equation (3.3). Similarly, the *p-semistable locus* Z_p^{ss} coincides with U^- from Equation (3.4).

Definition 3.2.3. For an object Z of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$, we let

$$\widehat{Q}_Z := (p \times s)_* \mathcal{O}_{Q_Z} \in \text{D}^{\text{b}} \left(\text{Qcoh}^{\text{GL}(V) \times_{\mathbb{k}} \text{GL}(V)} Z \times_{\mathbb{k}} Z \right),$$

where the pushforward is understood to be derived. We denote by \widehat{Q}_Z^+ the quasi-coherent sheaf on $Z_s^{\text{ss}} \times_{\mathbb{k}} Z$ realized by restricting \widehat{Q}_Z from $Z \times_{\mathbb{k}} Z$. That is,

$$\widehat{Q}_Z^+ = (j \times 1_Z)^* \widehat{Q}_Z,$$

where $j : Z_s^{\text{ss}} \rightarrow Z$ is the inclusion. Finally, taking \widehat{Q}_Z^+ as the Fourier-Mukai kernel, we have the functor

$$(3.7) \quad \Phi_{\widehat{Q}_Z^+} : \mathbb{D}^b(\text{Qcoh}^{\text{GL}(V)} Z_s^{\text{ss}}) \rightarrow \mathbb{D}^b(\text{Qcoh}^{\text{GL}(V)} Z).$$

Remark 3.2.4. Since the functor $(p \times s)_*$ is exact, \widehat{Q}_Z is just the $\text{GL}(V)$ -linearized sheaf associated to Q_Z with its (p, s) -bimodule structure given in Lemma 3.1.2. This justifies our use of \widehat{Q}_Z in Notation 3.1.17.

Lemma 3.2.5. *Let Z be an object of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$. Then $\Phi_{\widehat{Q}_Z^+}$ is faithful.*

Proof. Our proof follows from the fact that the functor

$$i^* : \mathbb{D}^b(\text{Qcoh}_{\text{GL}(V)}(Z_s^{\text{ss}})) \rightarrow \mathbb{D}^b(\text{Qcoh}_{\text{GL}(V)}(Z))$$

is the left inverse of $\Phi_{\widehat{Q}_Z^+}$. To see this, note that for any maximal minor m of B , we have $R_m \otimes_s Q_Z \cong \mathbb{k}[\text{GL}(V)] \otimes_{\mathbb{k}} R = \mathbb{k}[\Delta_Z]$. Indeed, inverting a minor on the left amounts to inverting the determinant of C . Since Δ_Z is the kernel of the identity, we obtain the desired result. \square

The fullness of this functor depends on certain localization properties, which are the focus of the next section.

3.3. Bousfield localizations. This section recalls Bousfield (co)-localizations which will be used to establish fullness of the functor $\Phi_{\widehat{Q}_Z^+}$ from Equation (3.7). We recall that the existence of a Bousfield triangle produces a semi-orthogonal decomposition, and we show that the essential image of our functor is an inclusion into one of these pieces. We refer the reader to [Kra10] for a more detailed treatment of these concepts. While the proofs of the statements refer to [BDF17] we recall all of the statements here for ease of reference.

Definition 3.3.1. Let \mathcal{T} be a triangulated category. A *Bousfield localization* is an exact endofunctor $L : \mathcal{T} \rightarrow \mathcal{T}$ equipped with a natural transformation $\delta : 1_{\mathcal{T}} \rightarrow L$ such that:

- a) $L\delta = \delta L$ and
- b) $L\delta : L \rightarrow L^2$ is invertible.

A *Bousfield co-localization* is given by an endofunctor $C : \mathcal{T} \rightarrow \mathcal{T}$ equipped with a natural transformation $\epsilon : C \rightarrow 1_{\mathcal{T}}$ such that:

- a) $C\epsilon = \epsilon C$ and
- b) $C\epsilon : C^2 \rightarrow C$ is invertible.

Definition 3.3.2. Assume there are natural transformations of endofunctors

$$C \xrightarrow{\epsilon} 1_{\mathcal{T}} \xrightarrow{\delta} L$$

of a triangulated category \mathcal{T} such that

$$Cx \xrightarrow{\epsilon_{Cx}} x \xrightarrow{\delta_x} Lx$$

is an exact triangle for any object x of \mathcal{T} . Then we refer to $C \rightarrow 1_{\mathcal{T}} \rightarrow L$ as a *Bousfield triangle* for \mathcal{T} when any of the following equivalent conditions are satisfied:

- 1) L is a Bousfield localization and $C(\epsilon_x) = \epsilon_{Cx}$
- 2) C is a Bousfield co-localization and $L(\delta_x) = \delta_{Lx}$
- 3) L is a Bousfield localization and C is a Bousfield co-localization.

For a proof that the above properties are indeed equivalent, we refer the reader to [BDF17, Definition 3.33]. Denoting $S := \mathbb{k}[\Delta_Z]/Q_Z$, we have morphisms

$$Q_Z \xrightarrow{\eta^\sharp} \mathbb{k}[\Delta_Z] \rightarrow S \rightarrow Q_Z[1]$$

in $D^b(\text{Mod}^{\text{GL}(V)} \mathbb{k}[Z])$, where η^\sharp is the morphism induced by η as in Equation 3.1.17. This yields an exact triangle. Furthermore, if we let $\hat{\eta} : \Phi_{\hat{Q}_Z} \rightarrow 1$ denote the morphism induced by η^\sharp , we see that for any x in $D^b(\text{Qcoh}^{\text{GL}(V)} Z)$ the following is also exact:

$$\Phi_{\hat{Q}_Z}(x) \rightarrow x \rightarrow \Phi_{\tilde{S}}(x)$$

With these observations in mind, we present one of the main results of this section.

Proposition 3.3.3. *Let Z be an object of $\text{AM}_{\mathbb{k}}^{\text{GL}(V)}$. Then the triangle of functors*

$$\Phi_{\hat{Q}_Z} \xrightarrow{\hat{\eta}} 1 \rightarrow \Phi_{\tilde{S}}.$$

is a Bousfield triangle.

Proof. This follows identically as in [BDF17, Lemma 3.3.6], by Lemma 3.1.16 and Lemma 3.1.23. \square

We are now ready to prove that $\Phi_{\hat{Q}_+}$ is full. Let $J_+ := j_* \circ j^*$, where $j : Z_s^{\text{ss}} \rightarrow Z$ is the natural inclusion, and let Γ_+ be the local cohomology.

Proposition 3.3.4. *Let Z be an object of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$. There is a semi-orthogonal decomposition*

$$D(\text{Qcoh}^{\text{GL}(V)} Z) = \langle \text{Im } \Phi_{\tilde{S}}, \text{Im } \Phi_Q \circ \Gamma_+, \text{Im } \Phi_{\hat{Q}_+} \rangle,$$

where Im denotes the essential image. Furthermore, $\Phi_{\hat{Q}_+}$ is fully-faithful.

Proof. This follows identically to the proof of Proposition 3.3.9 in [BDF17] \square

Letting $j' : Z_p^{\text{ss}} \rightarrow Z$ be the inclusion and Γ_+ its local cohomology, we have the following dual statement.

Proposition 3.3.5. *Let Z be an object of $\text{HP}_{\mathbb{k}}^{\text{GL}(V)}$. There is a semi-orthogonal decomposition*

$$D(\text{Qcoh}^{\text{GL}(V)} Z) = \langle \text{Im } \Phi_{\tilde{S}}, \text{Im } \Phi_Q \circ \Gamma_-, \text{Im } \Phi_{\hat{Q}_-} \rangle,$$

where Im denotes the essential image. Furthermore, $\Phi_{\hat{Q}_-}$ is fully-faithful.

4. A GEOMETRIC RESOLUTION

For this section, we will denote Z as the scheme

$$\text{Hom}(V, W) \oplus \text{Hom}(W', V).$$

Having established that $\Phi_{\hat{Q}_+}$ is fully faithful, the remaining objective of this work is to examine the essential image of the functor $\Phi_{\hat{Q}_+}$. We will show that this image is generated by an exceptional collection first discovered by Kapranov in [Kap88]. The method which we use is based on the underlying techniques of the well known ‘geometric technique’ of Kempf (see e.g. [Wey03]).

4.1. A sketch of Kempf. The objective of the method of Kempf is to provide a free resolution of special modules by pulling back to a trivial geometric bundle over a projective variety.

Consider an algebraic variety Y . The total space of the sheaf $\mathcal{O}_Y^{\oplus n}$ is the scheme $Y \times \mathbb{A}^n$. Now let X be the total space of a locally free sheaf $\mathcal{F} \subset \mathcal{O}_Y^{\oplus n}$ on Y . Let π denote the projection $Y \times \mathbb{A}^n \rightarrow Y$.

We have the exact sequence of locally free sheaves on $Y \times \mathbb{A}^n$

$$0 \longrightarrow \pi^* \mathcal{F} \longrightarrow \pi^* \mathcal{O}_Y^{\oplus n} \xrightarrow{f} \pi^* \mathcal{T} \longrightarrow 0,$$

where \mathcal{T} is the quotient sheaf.

Consider the section $s := f \circ \text{taut} : \mathcal{O}_{Y \times \mathbb{A}^n} \rightarrow \pi^* \mathcal{T}$, where taut denotes the tautological section of $\pi^* \mathcal{O}_Y^{\oplus n}$ on $Y \times \mathbb{A}^n$. Then, we have the following statement.

Proposition 4.1.1. *With the above notation, a locally free resolution of the sheaf \mathcal{O}_X as a $\mathcal{O}_{Y \times \mathbb{A}^n}$ -module is given by the Koszul complex*

$$\mathcal{K}(s)_\bullet : 0 \rightarrow \bigwedge^{\text{rk}(\mathcal{T})} (\pi^* \mathcal{T}^\vee) \rightarrow \dots \rightarrow \bigwedge^2 (\pi^* \mathcal{T}^\vee) \rightarrow \pi^* \mathcal{T}^\vee \rightarrow \mathcal{O}_{Y \times \mathbb{A}^n}$$

Proof. On the vanishing locus $Z(s)$, the tautological section taut factors through $\pi^* \mathcal{F}$. Hence, the vanishing locus is the total space of the sheaf \mathcal{F} , which is X . We see that the section is regular as the codimension of $Z(s)$ equals the rank of the sheaf $\pi^* \mathcal{T}$; and the Koszul complex resolves \mathcal{O}_X . For more details, see [Wey03, Proposition 3.3.2]. \square

4.2. The resolution. Now we are ready to present a resolution which will open a window to view $\text{Im}(\Phi_{\widehat{Q}^+})$. First recall that we set $\dim(V) := d$. We define \mathbf{Q}_Z^+ as the base change:

$$\begin{array}{ccc} \mathbf{Q}_Z^+ & \longrightarrow & \mathbf{Q}_Z \\ \downarrow & & \downarrow^{p \times s} \\ Z_s^{\text{ss}} \times Z & \hookrightarrow & Z \times Z \end{array}$$

Let \mathcal{S} be the tautological bundle on $\text{Gr}(d, W)$ i.e. the locally free sheaf on $\text{Gr}(d, W) = [\text{Hom}(V, W)_s^{\text{ss}} / \text{GL}(V)^L]$ corresponding to the $\text{GL}(V)^L$ -representation V . Then, we have the Euler sequence for the Grassmannian $\text{Gr}(d, W)$:

$$0 \rightarrow \mathcal{S} \rightarrow W \rightarrow \mathcal{Q} \rightarrow 0.$$

Consider the pullback of the above sequence to $\text{Gr}(d, W) \times \text{Hom}(W', V)$ along q and apply $\mathcal{H}om(t^*V, -)$, where $q : \text{Gr}(d, W) \times \text{Hom}(W', V) \rightarrow \text{Gr}(d, W)$ and $t : \text{Gr}(d, W) \times \text{Hom}(W', V) \rightarrow \text{Hom}(W', V)$ are projections

$$(4.1) \quad 0 \rightarrow \mathcal{H}om(t^*V, q^* \mathcal{S}) \xrightarrow{p} \mathcal{H}om(t^*V, q^* W) \xrightarrow{\Xi} \mathcal{H}om(t^*V, q^* \mathcal{Q}) \rightarrow 0.$$

Let us denote $\mathcal{T} := \mathcal{H}om(t^*V, q^* \mathcal{Q})$. We denote the total space of the locally free sheaf $\mathcal{H}om(A, B)$ as $\mathbf{Hom}(A, B)$. From the discussion in the previous subsection, we get the following result:

Lemma 4.2.1. *The following Koszul complex is a free resolution for $\mathcal{O}_{\mathbf{Hom}(t^*V, q^*\mathcal{S})}$ as an $\mathcal{O}_{\mathrm{Gr}(d, W) \times Z}$ -module.*

$$(4.2) \quad \mathcal{K}(s)_\bullet : \bigwedge^{d(m-d)} \pi^* \mathcal{T}^\vee \rightarrow \dots \rightarrow \bigwedge^2 \pi^* \mathcal{T}^\vee \rightarrow \pi^* \mathcal{T}^\vee \rightarrow \mathcal{O}_{\mathrm{Gr}(d, W) \times Z}$$

where $\pi : \mathrm{Gr}(d, W) \times \mathrm{Hom}(V, W) \times \mathrm{Hom}(W', V) \rightarrow \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ is the projection morphism.

Proof. We choose $Y = \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$, and $\mathcal{F} = \mathcal{H}om(t^*V, q^*\mathcal{S})$, and apply Proposition 4.1.1. Notice that the total space of $\mathcal{H}om(t^*V, q^*W)$ on $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ is $\mathrm{Gr}(d, W) \times Z$. \square

Now, we can identify $[\mathbf{Q}_Z^+ / \mathrm{GL}(V)^L]$ as the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$.

Lemma 4.2.2. *The quotient space $[\mathbf{Q}_Z^+ / \mathrm{GL}(V)^L]$ is $\mathrm{GL}(V)^R$ -equivariantly isomorphic to the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$ as schemes over $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$.*

Proof. Recall from Equation (3.2), that \mathbf{Q}_Z is associated to the module

$$\mathbb{k}[A^L, B^R, C]$$

Geometrically, we may view \mathbf{Q}_Z as the total space of the locally free sheaf $\mathrm{End}(V)$ over $\mathrm{Spec} \mathbb{k}[A^L, B^R]$. Once we base change to the semistable locus and take the quotient with respect to the $\mathrm{GL}(V)^L$ action, we get that $[\mathbf{Q}_Z^+ / \mathrm{GL}(V)^L]$ is isomorphic to the total space

$$\mathbf{Hom}(t^*V, q^*\mathcal{S}) \rightarrow \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V).$$

Moreover, the inclusion, $\mathcal{H}om(t^*V, q^*\mathcal{S}) \rightarrow \mathcal{H}om(t^*V, q^*W)$ realizes it as a subspace of the total space $\mathbf{Hom}(t^*V, q^*W)$ over $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ which is $Z \times \mathrm{Gr}(d, W)$.

This inclusion $\mathcal{H}om(t^*V, q^*\mathcal{S}) \rightarrow \mathcal{H}om(t^*V, q^*W)$ is induced by the ring homomorphism

$$\begin{aligned} k[A^L, A^R, B^R] &\rightarrow k[A^L, B^R, C] \\ A^L &\mapsto A^L \\ A^R &\mapsto CA^L \\ B^R &\mapsto B^R. \end{aligned}$$

which is equivariant with respect to the remaining $\mathrm{GL}(V)^R$ -action. \square

We denote $\pi_1 : [Z_s^{\mathrm{ss}} / \mathrm{GL}(V)^L] \rightarrow [\mathrm{Hom}(V, W)_s^{\mathrm{ss}} / \mathrm{GL}(V)^L]$ as the projection. Putting Lemma 4.2.1 and Lemma 4.2.2 together, we get a resolution of the sheaf $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$.

Corollary 4.2.3. *The Koszul complex (4.2) is a locally free resolution of the sheaf $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$ of $\mathcal{O}_{\mathrm{Gr}(d, W) \times Z}$ -modules.*

Remark 4.2.4. We note that we could also have constructed a locally free resolution of \widehat{Q}_Z^+ on $Z^{\mathrm{ss}} \times Z$ by the same method, and this will also lead to a similar proof as in the remainder of this paper.

5. ANALYZING THE INTEGRAL TRANSFORM

In this section, we show that the kernel \widehat{Q}_Z^+ induces a derived equivalence for a Grassmann flop. We begin by showing that the essential image of this functor coincides with the ‘window’ description studied by Donovan and Segal in [DS14, Section 3.1]. Recall that Z is scheme

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$$

with $\dim(W) =: m, \dim(W') =: m' \geq d := \dim(V)$. Specifically, we will show that the image of $\Phi_{\widehat{Q}_Z^+}$ is generated by a collection of vector bundles corresponding to representations identified by Kapranov [Kap88].

Let us recall Kapranov’s collection. Consider the standard $\mathrm{GL}(V)$ representation V , where $\mathrm{GL}(V)$ acts by left multiplication. Consider the Schur modules of V associated to a Young diagram (or equivalently, partition) α , and denote them by $L_\alpha V$. Kapranov’s collection is defined by

$$\mathfrak{K}_{d,m} := \left\{ L_\alpha V \mid \alpha \in \text{Young diagrams of height } \leq m - d \text{ and width } \leq d \right\}.$$

We also consider pull backs of these representations to $\mathrm{Gr}(d, W)$ along the structure morphism. As V pulls back to the tautological bundle \mathcal{S} , the Schur functors $L_\alpha V$ pull back to $L_\alpha \mathcal{S}$ and these are the locally free sheaves considered by Kapranov. By abuse of notation, we will consider $\mathfrak{K}_{d,m}$ as a collection of locally free sheaves on $\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$ or $\mathrm{Hom}(W', V)$ (again, by pulling back along the structure morphism). Note that when $\mathbb{k} = \mathbb{C}$, this is exactly the dual of the zeroth window W_0 from [DS14, Section 3.1].

It is the objective of this section to show that the thick triangulated subcategory generated by elements of $\mathfrak{K}_{d,m}$ is equivalent to $\mathrm{Im}(\Phi_{\widehat{Q}_Z^+})$. We show one containment in Proposition 5.1.1, which relies on the work of Section 4.

5.1. Windows from a resolution. Consider the projection $\pi_1 : Z_s^{\mathrm{ss}} \rightarrow \mathrm{Hom}(V, W)_s^{\mathrm{ss}}$. To demonstrate that the image of $\Phi_{\widehat{Q}_Z^+}$ is contained in $\langle \mathfrak{K}_{d,m} \rangle$, we exhibit a particular $\mathrm{GL}(V)^L \times \mathrm{GL}(V)^R$ -equivariant resolution \mathcal{K}_\bullet of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Hom}(V, W)_s^{\mathrm{ss}} \times Z$. Equivalently, this is a $\mathrm{GL}(V)^R$ -equivariant resolution of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Gr}(d, W) \times Z$. The resolution obtained in equation (4.2.1) in Section 4.2 is the one we are looking for and resolves the functor $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*$.

In this subsection, we will show that the components \mathcal{K}^i of the resolution have a filtration whose associated graded pieces are of the form $J \boxtimes K$ with $K \in \mathfrak{K}_{d,m}$. This decomposition of the Fourier-Mukai transform $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*$ yields a functorial way to describe $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*(M)$ using objects of $\mathfrak{K}_{d,m}$ for all objects $\pi_1^*(M) \in \mathrm{D}^b([Z_s^{\mathrm{ss}}/\mathrm{GL}(V)])$. As such objects generate $\mathrm{D}^b([Z_s^{\mathrm{ss}}/\mathrm{GL}(V)])$ this is enough to conclude the goal of this section, $\mathrm{Im}(\Phi_{\widehat{Q}_Z^+}) \subseteq \mathfrak{K}_{d,m}$.

Proposition 5.1.1. *With notation as above, we have*

$$\mathrm{Im}(\Phi_{\widehat{Q}_Z^+}) \subseteq \langle \mathfrak{K}_{d,m} \rangle,$$

where $\langle \mathfrak{K}_{d,m} \rangle$ is the thick triangulated subcategory generated by elements in $\mathfrak{K}_{d,m}$.

Proof. By Corollary 4.2.3, we have a quasi-isomorphism with the Koszul complex

$$\mathcal{K}_\bullet \cong (\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$$

The components of the Koszul complex are $\bigwedge^l \pi^* \mathcal{H}om(t^*V, q^*Q)^\vee$ for $0 \leq l \leq d$. We can appeal to the Cauchy Formula, e.g. [Wey03, Theorem 2.3.2(a)], to get a filtration on $\bigwedge^i t^* \pi^* \mathcal{H}om(t^*V, q^*Q)^\vee$ whose associated graded pieces are

$$\pi^* \left(\bigoplus_{|\lambda|=i} L_\lambda V \boxtimes L_{\lambda'} Q^\vee \right).$$

Thus, each term in the Koszul complex can be generated using iterated exact sequences from the locally free sheaves

$$\pi^* (L_\lambda V \boxtimes L_{\lambda'} Q^\vee).$$

These components, in turn, generate \widehat{Q}_Z^+ . Hence, for all M , $\Phi_{\widehat{Q}_Z^+}(\pi_1^* M)$ is generated by objects of the form

$$\Phi_{\pi^*(L_\lambda V \boxtimes L_{\lambda'} Q^\vee)}(\pi_1^* M) = \mathbf{R}\Gamma(M \otimes L_\lambda Q^\vee) \otimes_{\mathbb{k}} L_\lambda V$$

all of which lie in $\mathfrak{K}_{d,m}$. Now, since π_1 is an affine map, $\mathrm{D}^b([Z^{\mathrm{ss}}/\mathrm{GL}(V)])$ is generated by the essential image of π_1^* . The result follows. \square

5.2. Truncation operator. In this section we will see that $\Phi_{\widehat{Q}_Z^+}$ has a useful description on $\mathrm{GL}(V)$ -representations. Yet before we go deeper into the representation theory we define a truncation operator over our field \mathbb{k} of characteristic zero.

Definition 5.2.1. Let $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(\mathbb{k}[\mathrm{Hom}(V, W)])$, we define the *truncation operator* as follows

$$M_{\geq 0} := \left(M \otimes \mathbb{k}[\mathrm{End}(V)] \right)^{\mathrm{GL}(V)}$$

Recall, further that there is a $\mathrm{GL}(V) \times_{\mathbb{k}} \mathrm{GL}(V)$ -module decomposition

$$(5.1) \quad \mathbb{k}[\mathrm{End}(V)] \cong \bigoplus N_i^\vee \otimes_{\mathbb{k}} N_i,$$

where we sum over all irreducible representations of $\mathrm{GL}(V)$ with all positive weights [Pro07], these representations are also referred to as *polynomial representations*. Since $\mathrm{GL}(V)$ is linearly reductive over a field of characteristic zero, we may decompose any $\mathrm{GL}(V)$ -module M as $M \cong \bigoplus M_i$, where M_i is irreducible and we have the following description of the truncation operator 5.2.1:

Lemma 5.2.2. *Let $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(\mathbb{k}[\mathrm{Hom}(V, W)])$; then decompose M over \mathbb{k} into irreducibles as*

$$(5.2) \quad M = \bigoplus_{M_i \text{ irreducible}} M_i.$$

Then the truncation operator may be described as follows

$$M_{\geq 0} = \bigoplus_{\substack{M_i \text{ irreducible} \\ \text{and polynomial}}} M_i$$

\square

Lemma 5.2.3. *For any $M \in \text{Mod}^{\text{GL}(V)}(\mathbb{k}[\text{Hom}(V, W)])$, $M_{\geq 0}$ is a $\mathbb{k}[\text{Hom}(V, W)]$ -submodule of M and $(-)_{\geq 0}$ is exact.*

Proof. The exactness of the functor follows since $\text{GL}(V)$ is linearly reductive and thus our operator is just a projection. That $M_{\geq 0}$ is a $\mathbb{k}[\text{Hom}(V, W)]$ -submodule follows since $\mathbb{k}[\text{Hom}(V, W)]_{\geq 0} = \mathbb{k}[\text{Hom}(V, W)]$ since $\mathbb{k}[\text{Hom}(V, W)]$ is a polynomial representation. \square

To deliver a cleaner picture we define some more notation $Y' := \text{Hom}(V, W)$. For the remainder of this subsection we will exploit the commutativity of the following diagram.

$$\begin{array}{ccc} U_Z^+ & \xleftarrow{j} & \text{Hom}(V, W) \oplus \text{Hom}(W', V) \\ q_1|_{U_Z^+} \downarrow & & \downarrow q_1 \\ U_{Y'}^+ & \xleftarrow{i} & \text{Hom}(V, W) \end{array}$$

Lemma 5.2.4. *Let $M \in \text{Mod}^{\text{GL}(V)}(\mathbb{k}[\text{Hom}(V, W)])$ then*

$$\Phi_{Q_{Y'}}(M) = M_{\geq 0}$$

Proof. The coaction map defines a morphism

$$M_{\geq 0} \rightarrow (\mathbb{k}[\text{End}(V)] \otimes M_{\geq 0})^{\text{GL}(V)} \hookrightarrow (\mathbb{k}[\text{End}(V)] \otimes M)^{\text{GL}(V)},$$

which we claim is an isomorphism. Notice that the coaction map lands in $\mathbb{k}[\text{End}(V)] \subset \mathbb{k}[\text{GL}(V)]$ as $M_{\geq 0}$ is a polynomial representation. To check that this map is an isomorphism, we may base change to $\bar{\mathbb{k}}$ (which is faithfully flat over \mathbb{k}). Hence, assume that $\mathbb{k} = \bar{\mathbb{k}}$.

Using equation (5.1) and Lemma 5.2.2, we get

$$\begin{aligned} (\mathbb{k}[\text{End}(V)] \otimes M)^{\text{GL}(V)} &\cong \bigoplus N_j \otimes (N_j^\vee \otimes M_i)^{\text{GL}(V)} \\ &\cong M_{\geq 0} \end{aligned}$$

where we are considering the left $\text{GL}(V)$ invariant submodule and the second line follows from Schur's Lemma.

Finally, by Lemma 3.1.10 we have $Q_{Y'} \cong \mathbb{k}[A] \otimes \mathbb{k}[\text{End}(V)]$, and we get

$$\begin{aligned} (Q_{Y'} \otimes M)^{\text{GL}(V)} &\cong \mathbb{k}[A] \otimes (\mathbb{k}[\text{End}(V)] \otimes M)^{\text{GL}(V)} \\ &\cong M_{\geq 0}. \end{aligned}$$

\square

Lemma 5.2.5. *We have an isomorphism*

$$(q_1 \times \text{Id})_* {}_s(Q_Z)_p \cong (\text{Id} \times q_1)^* {}_s(Q_{Y'})_p$$

as objects of $\text{Mod}^{\text{GL}(V) \times \text{GL}(V)}(Y' \times Z)$.

Proof. This follows from the following calculation.

$$\begin{aligned} {}_sQ_Z &\cong \mathbb{k}[A^L, B^R, C] \\ &\cong \mathbb{k}[A^R, B^R] \otimes_{\mathbb{k}[A]} \mathbb{k}[A^L, C] \\ &\cong Z \otimes_{\mathbb{k}[Y']} Q_{Y'}, \end{aligned}$$

where the first isomorphism follows from Lemma 3.1.13 and in the second line, $k[A]$ acts on the left by going to A^R and on the right by going CA^L . \square

Corollary 5.2.6. *Let $M \in \text{Mod}^{\text{GL}(V)}(k[Y'])$, then*

$$\Phi_{Q_Z}(q_1^*M) \cong q_1^*\Phi_{Q_{Y'}}(M)$$

Proof. This follows from Lemma 5.2.5 which says that it is true at the level of the Fourier-Mukai kernels. \square

Lemma 5.2.7. *For $L_\alpha V \in \mathfrak{K}_{d,m}$ we have that*

$$(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)_{\geq 0} \cong L_\alpha V$$

Proof. To see this we will denote the irreducible components as $(-)_\beta$ where β is the highest weight corresponding to the isotypical piece, and by $\beta \geq 0$ we denote weights correspond to polynomial representations.

$$\begin{aligned}
 (\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)_{\geq 0} &= \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)_\beta \\
 &\cong \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^*L_\alpha V \otimes L_\beta V^\vee)^{\text{GL}(V)} \\
 &\cong \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^*(L_\alpha V \otimes L_\beta V^\vee))^{\text{GL}(V)} \\
 &\cong \bigoplus_{\beta \geq 0} \mathbf{R}\Gamma(\text{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) \\
 (5.3) \quad &\cong \bigoplus_{\beta \geq 0} \Gamma(\text{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) \\
 (5.4) \quad &\cong \bigoplus_{\beta \geq 0} \Gamma(\text{Hom}(V, W), L_\alpha V \otimes L_\beta V^\vee)^{\text{GL}(V)} \\
 &\cong \bigoplus_{\beta \geq 0} (\text{Sym}(\text{Hom}(W, V)) \otimes L_\alpha V \otimes L_\beta V^\vee)^{\text{GL}(V)} \\
 (5.5) \quad &\cong \text{Sym}(\text{Hom}(W, V)) \otimes L_\alpha V \\
 &\cong \mathcal{O}_{\text{Hom}(V, W)} \otimes L_\alpha V
 \end{aligned}$$

Equation (5.3) follows from [Kap88, Lemma 3.2.a] (this uses the assumption that $L_\alpha V \in \mathfrak{K}_{d,m}$ and the fact that the weights of the irreducible summands of $L_\alpha V \otimes L_\beta V^\vee$ are all strictly larger than $-(m-d)$.) Equation (5.4) follows as $\text{Gr}(d, W)$ has co-dimension greater than 2 in the global quotient stack $[\text{Hom}(V, W)/\text{GL}(V)]$. Equation (5.5) follows from Schur's Lemma and the fact that all representations in $\text{Sym}(\text{Hom}(W, V)) \otimes L_\alpha V$ are polynomial (this uses the fact that $L_\alpha V$ is polynomial). \square

Proposition 5.2.8. *If $L_\alpha V \in \mathfrak{K}_{d,m}$ then*

$$\Phi_{Q_Z^+}(L_\alpha V) \cong L_\alpha V$$

Proof. This result follows from another calculation,

$$\begin{aligned}\Phi_{Q_Z^+}(L_\alpha V) &\cong \Phi_Q(\mathbf{R}j_*\mathbf{L}j^*L_\alpha V) \\ &\cong \pi^*\Phi_{Q_{Y'}}(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V) \\ &\cong \pi^*\left(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V\right)_{\geq 0},\end{aligned}$$

where the second line follows from Corollary 5.2.6 and the last line by Lemma 5.2.4. Hence our result follows from Lemma 5.2.7. \square

Corollary 5.2.9. $\text{Im } \Phi_{\widehat{Q}_Z^+} = \langle \mathfrak{K}_{d,m} \rangle$.

Proof. This is an immediate consequence of Proposition 5.1.1 and Lemma 5.2.8. \square

Note that we have a similar equality for $\Phi_{\widehat{Q}_Z^-}$.

Corollary 5.2.10. $\text{Im } \Phi_{\widehat{Q}_Z^-} = \langle (\mathfrak{K}_{d,m'})^\vee \rangle = \langle \mathfrak{K}_{d,m'} \otimes \det(V^*)^{m'-d} \rangle$.

Proof. We can switch the roles of W and W' by taking transposes. This is anti-equivariant, ie equivariant up to inversion in $\text{GL}(V)$. Consequently, we replace all representations with their duals which gives the first equality. The second is a standard identity. \square

5.3. The equivalence. Finally, we combine things to provide Fourier-Mukai equivalences for (twisted) Grassmann flops. As usual, let \mathbb{k} be an (arbitrary) field of characteristic zero.

We recall that P is the object obtained by the restriction of Q_Z^+ to $Z^+ \times Z^-$.

Theorem 5.3.1. *Assume $\dim W' \geq \dim W$. The wall crossing functor*

$$\Phi_P : \text{D}^b(Z^+) \rightarrow \text{D}^b(Z^-)$$

is fully-faithful. If $\dim W' = \dim W$, it is an equivalence.

Proof. Proposition 3.3.4 tells us that $\Phi_{\widehat{Q}_Z^+}$ is fully-faithful. Thus, we reduce to checking that j_-^* is fully-faithful on the image of $\Phi_{\widehat{Q}_Z^+}$. Also, from Proposition 3.3.5, we know that j_-^* is fully-faithful on the image of $\Phi_{\widehat{Q}_Z^-}$.

From Corollaries 5.2.9 and 5.2.10, we see that

$$\text{Im } \Phi_{\widehat{Q}_Z^+} \subseteq \text{Im } \Phi_{\widehat{Q}_Z^-} \otimes \det(V^*)^{d-m'}.$$

Since restriction commutes with tensoring with a line bundle, if j_-^* is fully-faithful on a full subcategory \mathcal{C} then it is also on $\mathcal{C} \otimes \mathcal{L}$ for any line bundle \mathcal{L} . Now Corollaries 5.2.9 and 5.2.10 show j_-^* must be fully-faithful on the image of $\Phi_{\widehat{Q}_Z^+}$.

If $\dim W' = \dim W$, then both varieties are Calabi-Yau. As Calabi-Yau's can have no nontrivial admissible subcategories our fully-faithful functor must be an equivalence. \square

Remark 5.3.2. If $\overline{\mathbb{k}} = \mathbb{k}$, once one knows that

$$\text{Im } \Phi_{\widehat{Q}_Z^+} = \langle \mathfrak{K}_{d,m} \rangle$$

one can conclude Theorem 5.3.1 using [DS14, Proposition 3.6]. But, the technology presented here makes for a simple direct proof.

Remark 5.3.3. In general, if we have two smooth projective varieties X and Y over \mathbb{k} , then the existence of an equivalence

$$D^b(X_{\overline{\mathbb{k}}}) \cong D^b(Y_{\overline{\mathbb{k}}})$$

does not guarantee the existence of an equivalence

$$D^b(X) \cong D^b(Y).$$

A simple class of counter-examples is Severi-Brauer varieties.

One needs, at least, a kernel over \mathbb{k} which base changes to furnish the equivalence to appeal to [Orl02, Lemma 2.12]. Without providing a kernel for general \mathbb{k} for the equivalence in [DS14], the results in loc.cit. cannot be used to deduce equivalences over arbitrary fields of characteristic zero.

One can go even further. We give the following definition.

Definition 5.3.4. We say

$$\begin{array}{ccc} Y^+ & \overset{\longleftarrow}{\dashrightarrow} & Y^- \\ & \searrow & \swarrow \\ & Y_0 & \end{array}$$

is a *twisted Grassmann flop* if the base change to the separable closure of \mathbb{k}

$$\begin{array}{ccc} Y_{\mathbb{k}^{\text{sep}}}^+ & \overset{\longleftarrow}{\dashrightarrow} & Y_{\mathbb{k}^{\text{sep}}}^- \\ & \searrow & \swarrow \\ & (Y_0)_{\mathbb{k}^{\text{sep}}} & \end{array}$$

is isomorphic to a Grassmann flop.

Example 5.3.5. Let A be a central simple \mathbb{k} -algebra of degree n . For $0 < l < n$, the l -th generalized Severi-Brauer variety of A $\text{SB}_l(A)$ is the variety parameterizing right ideals of dimension ln in A . Such a variety is a twisted form of $\text{Gr}(l, n)$, ie

$$\text{SB}_l(A)_{\mathbb{k}^{\text{sep}}} \cong \text{Gr}(l, n)_{\mathbb{k}^{\text{sep}}}.$$

On $\text{SB}_l(A)$, the tautological vector bundle \mathcal{T} , whose fibers are the ideals, base changes to $\text{Hom}(W, \mathcal{S})$. Let T denote the associate geometric vector bundle. The map

$$\text{SB}_l(A) \rightarrow \text{Spec } \Gamma(T, \mathcal{O}_T)$$

contracts the zero section and base changes to $X^+ \rightarrow X_0$. One can then take two copies of $\text{Spec } \Gamma(T, \mathcal{O}_T)$ and identify them with the involution that base changes to transposition the linear maps. The resulting diagram is a (n honestly) twisted Grassmann flop.

We also have equivalences for twisted Grassmann flops in characteristic zero.

Corollary 5.3.6. *Assume $\text{char } \mathbb{k} = 0$. If we have a twisted Grassmann flop, then there is an equivalence*

$$D^b(Y^+) \rightarrow D^b(Y^-).$$

Proof. Theorem 3.1.22 says that the structure sheaf of the fiber product $Y_{\mathbb{k}^{\text{sep}}}^+ \times_{(Y_0)_{\mathbb{k}^{\text{sep}}}} Y_{\mathbb{k}^{\text{sep}}}^-$ is a Fourier-Mukai kernel. Applying [Orl02, Lemma 2.12] shows that the $Y^+ \times_{Y_0} Y^-$ is also a Fourier-Mukai kernel. \square

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