

## Limits Prep Worksheet 3

### Rational Expression/Functions

#### Rational Expression

A rational expression can be written as a fraction of polynomials.

**Example(s):**

$$\frac{x^2 - 2x + 7}{3x}$$

$$\frac{9 - x^2}{x^2 + 6x + 9}$$

Often we would like to write such an expression in its simplest form, i.e. we would like to cancel out common factors.

**Example(s):**

$$\frac{9x^2 - 4}{3x^2 + x - 14}$$

*Step 1:* Factor the numerator and the denominator separately.

$$\frac{9x^2 - 4}{3x^2 + x - 14} = \frac{(x - 2)(x + 2)}{(x - 2)(3x + 7)}$$

*Step 2:* Cancel out like terms from the top and bottom.

$$\frac{(x - 2)(x + 2)}{(x - 2)(3x + 7)} \sim \frac{x + 2}{3x + 7}$$

*Notice:* I did **not** write *equals* since this does not equal when  $x = 2$ .  
(This is a common procedure in limits that we will see shortly)

**Problem 1.** Write the following rational expressions in lowest terms. If there are any restrictions on the variables, indicate them.

(a)  $\frac{s^2 - s - 6}{s^2 + s - 12}$

(c)  $\frac{7 - b}{b - 7}$

(b)  $\frac{22r^2 - 11r}{6 - 12r}$

(d)  $\frac{9 - x^2}{x^2 + 6x + 9}$

**Problem 2.** Add, subtract, multiply, or divide as indicated. Write all answers in lowest terms. If there are any restrictions on the variables, indicate them.

$$(a) \frac{n}{m+3} - \frac{-3n+7}{m+3}$$

$$(g) \frac{1}{x^2-1} - \frac{1}{x^2+3x+2}$$

$$(b) \frac{r}{r^2-s^2} + \frac{s}{r^2-s^2}$$

$$(h) \frac{9k-18}{6k+12} \cdot \frac{3k+6}{15k-30}$$

$$(c) \frac{4x}{8x+4} \cdot \frac{14x+7}{6}$$

$$(i) \frac{3x+12}{6x-30} \cdot \frac{x^2-x-2}{x^2-16}$$

$$(j) \frac{27-3k^2}{3k^2+8k-3} \div \frac{k^2-6k+9}{6k^2-19k+3}$$

$$(d) (y^2-4) \div \frac{4y+8}{5}$$

$$(k) \frac{2}{3-m} - \frac{2}{m-3} + \frac{3}{m^2-9}$$

$$(e) \frac{9}{2t} + \frac{4}{t^2}$$

$$(l) \frac{4z}{z^2+6z+8} - \frac{2z-1}{z^2-z-6}$$

$$(f) \frac{12}{p^2+6p+9} - \frac{2}{p+3}$$

$$(m) \frac{x^2+10x+21}{x^2+14x+49} \cdot \frac{x^2+12x+35}{x^2-6x-27}$$

## Rational Functions and Holes

Rational functions often have a restricted domain. One way to have a restricted domain which will appear in the graph as a Hole. A hole happens when you can “cancel” the “bad point(s)”

### Hole Example

Consider the rational function

$$f(x) = \frac{x^2 + 6x + 9}{x + 3}$$

Since division by zero is not well defined (mathy lingo for doesn't make sense) we see that we can't plug in  $-3$ , that is the domain is all real numbers  $x$  such that  $x \neq -3$ . Yet notice we can “cancel out” the  $x + 3$  in the bottom (denominator)

$$\begin{aligned} \frac{x^2 + 6x + 9}{x + 3} &= \frac{(x + 3)(x + 3)}{x + 3} \\ &\sim \frac{x + 3}{1} = x + 3 \end{aligned}$$

The graph of a rational function with a hole looks like the “canceled out” graph, that is the graph of the function associated to the equation obtained after canceling out, yet at the “bad points” we draw an open circle.

### Hole Graphing Example

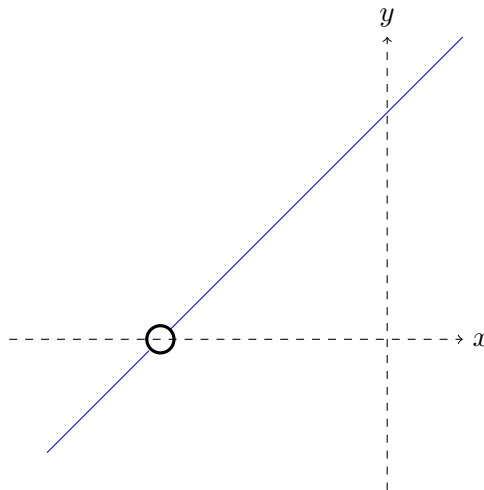
Consider the rational function from the previous example:

$$f(x) = \frac{x^2 + 6x + 9}{x + 3}$$

Recall this has a hole at  $x = -3$ . Once we did the canceling we saw for any value of  $x$  so that  $x \neq -3$  we had:

$$f(x) = x + 3$$

Which is just a line but we cannot plug in  $-3$  so it's graph looks like:



### Calculus Connection

So in your calculus class you would write the observation of this page as

$$\lim_{x \rightarrow -3^-} \frac{x^2 + 6x + 9}{x + 3} = \lim_{x \rightarrow -3^+} \frac{x^2 + 6x + 9}{x + 3} = \lim_{x \rightarrow -3} \frac{x^2 + 6x + 9}{x + 3} = 0$$

## Rational Functions and Vertical Asymptotes

Another way a rational function may have a restricted domain will appear in the graph as a Vertical Asymptote. A vertical asymptote happens where a value of  $x$  makes the denominator zero (but you cannot cancel out)

### Vertical Asymptotes

The graph of a rational function will have a vertical asymptote at  $x = a$  if the denominator is zero at  $x = a$  but the numerator is **NOT** zero at  $x = a$

*Note:* The part about the numerator not being zero assures there is no canceling out.

### Vertical Asymptote Example

Consider the following rational function:

$$f(x) = \frac{2x^2 + x + 3}{3x^2 - 5x - 7}$$

This function has zeros in the denominator i.e.  $0 = 3x^2 - 5x - 7$  when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - (4)(3)(-7)}}{2(3)} = \frac{5 \pm \sqrt{109}}{6}$$

and this function has no (*real*) zeros in the numerator i.e.  $0 \neq 2x^2 + x + 3$  since:

$$b^2 - 4ac = 1 - 4(2)(3) = -23$$

so it has a vertical asymptote(s) at

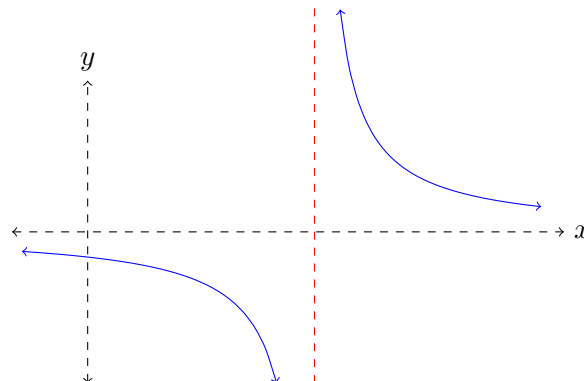
$$x = \frac{5 + \sqrt{109}}{6}, \frac{5 - \sqrt{109}}{6}$$

### Graphing Vertical Asymptotes

When graphing a vertical asymptote at  $x = a$  it is like the graph is driving towards a wall at the vertical line  $x = a$  and must veer to one side... Consider

$$\frac{1}{x - 3}$$

It has a vertical asymptote at  $x = 3$  since 3 makes the denominator zero but not the numerator zero, so the graph looks like:



How can you tell if it goes up or down???

### Calculus Connection

So in your calculus class you would write the observation of the previous page as

$$\lim_{x \rightarrow 3^-} \frac{1}{x+3} = -\infty$$

$$\lim_{x \rightarrow 3^+} \frac{1}{x+3} = \infty$$

$$\lim_{x \rightarrow 3} \frac{1}{x+3} = \text{DNE}$$

### Rational Functions and End Behavior

The End Behavior for rational functions is tricky... they can sometimes be described with horizontal asymptotes, when they cannot be these they go off to intinity just like polynomials.

#### Horizontal Asymptotes

For a rational function: ( $a$  and  $b$  are non-zero real numbers)

$$f(x) = \frac{ax^n + \dots}{bx^m + \dots}$$

1. (smaller top) If  $n < m$  then the  $x$ -axis ( $y = 0$ ) is the horizontal asymptote.
2. (equal top and bottom) If  $n = m$  then the line  $y = \frac{a}{b}$  is the horizontal asymptote.
3. (bigger top) If  $n > m$  there are no horizontal asymptotes.

#### Calculus Connection

So in your calculus class you would write the observation of the previous page as

$$\lim_{x \rightarrow \pm\infty} \frac{ax^n + \dots}{bx^m + \dots} = \begin{cases} 0 & n < m \\ \frac{a}{b} & n = m \\ \pm\infty & \text{otherwise} \end{cases}$$

**Problem 3.** Determine whether the following functions are rational functions. If so, write them in the form  $p(x)/q(x)$ , where  $p$  and  $q$  are polynomials.

a)  $f(x) = 2x^3 + 7$

c)  $f(x) = \frac{x^2}{2-x} - \frac{1}{x-2}$

b)  $g(x) = \frac{2^x + 1}{2^x - 2}$

d)  $g(x) = \frac{\sqrt{x} + 3}{\sqrt{x}}$

**Problem 4.** Compare and discuss the end behaviors of the following functions. Identify any horizontal asymptotes.

(a)  $f(x) = \frac{2x^2 + 1}{x^2 + 2}$

(b)  $g(x) = \frac{4x^3 + 2x}{2x^2 + x + 1}$

(c)  $h(x) = \frac{x}{x^2 + 5}$

**Problem 5.** Evaluate the following limits of rational functions:

(a)  $\lim_{x \rightarrow \infty} (10 + x^{-2})$

(b)  $\lim_{x \rightarrow -\infty} \frac{3x^4 + 10x^5}{2x^7 + 4}$

(c)  $\lim_{x \rightarrow \infty} \left( \frac{5x^3}{x^3 + 2x^2} + 1 \right)$

**Problem 6.** Determine the end behavior of the following rational functions. Then, determine their  $x$ -intercepts,  $y$ -intercepts, and horizontal/vertical asymptotes, if there are any.

(a)  $y = \frac{x + 3}{x + 5}$

(c)  $h(x) = \frac{x^2 - 4}{x^3 + 4x^2}$

(b)  $y = \frac{x + 2}{(x - 1)^2}$

(d)  $k(x) = \frac{x^2 - 9}{x^2 - 3x + 2}$