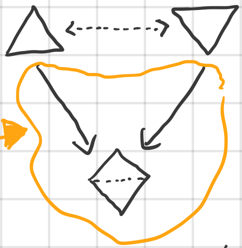
Pre Talk: (The Gist)

Objective: Discuss Summer Research Done w/ Dr. Ballard & Dr. McFaddin This Research involved studying the homology of Flops

So what's a flop #

The "categorical" picture (you may recall this from Ballard's presentation)



Top is a flop

← just focus on the word "equivalent"

words for the grown-ups "Common Blow-up!"

the morphism that defines this is just an equation that looks like $\frac{f(x)}{g(x)}$ ← polynomials

Flops which arise from variation of GIT → all of our flops will have

Conjecture: (Bondal & orlov) any two spaces related by a flop are Derived equivalent (i.e Homologically equivalent)

* Joke about how that's a loaded word "Derived" * ← maybe do the "acting on story?"

I've heard "quick" explanations like: "Just formally invert quasi-isomorphisms"

blah blah Minimal Model program

→ 3-fold flop

The Atiyah flop: (2-dimensional)

$$X = \mathbb{A}^2_k \times_k \mathbb{A}^2_k$$

$$\begin{tikzcd} X/\!/_+ \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X/\!/_- \arrow[dl] \\ & X/\!/_0 & \end{tikzcd}$$

← G.I.T Quotients ...

← leave on the board

The larger Concept here dealing w/ the words G.I.T and Derived is **ACTION**

G.I.T (The idea)

I learned the Ballard-Drinfeld quotient approach

Let's start w/ the building blocks: Complex Vector spaces ← write as schemes real quick

pick our favorite vector space {denote $V = \mathbb{C}^2$}

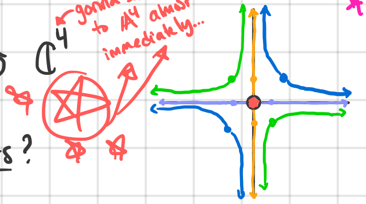
what I'm gonna draw → $\mathbb{R}^2 \subseteq \mathbb{C}^4$ (through both talks you can just think this)

$$\mathbb{C}^4 \cong \mathbb{C}^2 \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}) = \mathbb{C}^2 \oplus (\mathbb{C}^2)^*, \quad \mathbb{C} \setminus \{0\} = \mathbb{C}^\times \circlearrowleft \mathbb{C}^4$$

← gonna switch to $\mathbb{A}^4$ almost immediately...

$$t \cdot (\vec{a}, \vec{b}) = (t\vec{a}, \tfrac{1}{t}\vec{b})$$

what are the orbits?



* hyperbolic action *

Attracting locus: (positive locus)

$X^+ := \mathrm{Hom}^{\mathbb{G}_m}(\mathbb{A}^1_+, X) = \{x \in X \mid \lim_{\substack{t\to\infty \\ t \in \mathbb{C}^\times}} t \cdot x \text{ exists in } X\}$

is the attracting locus, and

in this example $\lim_{t\to\infty} t \cdot (\vec{a}, \vec{b}) = \lim_{t\to\infty} (t \cdot \vec{a}, \frac{1}{t} \cdot \vec{b})$

So $\vec{a} = \vec{0}$ $\vec{b} =$ anything!

i.e. $V \times \{0\} \subseteq V \times V^*$

repelling locus: (negative locus)

$X^- := \mathrm{Hom}^{\mathbb{G}_m}(\mathbb{A}^1_-, X) = \{x \in X \mid \lim_{\substack{t\to\infty \\ t \in \mathbb{C}^\times}} t^{-1} \cdot x \text{ exists in } X\}$

is the repelling locus.

in this example $\lim_{t\to\infty} \frac{1}{t} \cdot (\vec{a}, \vec{b}) = \lim_{t\to\infty} (\frac{1}{t}\vec{a}, t \cdot \vec{b})$

So $\vec{b} = \vec{0}$ $\vec{a} =$ anything

i.e. $\{0\} \times V^* \subseteq V \times V^*$

Semi-stable loci:

$$X^{ss}_+ := (V \times V^*) \setminus \underbrace{(V \times V^*)^+}_{V \times \{0\}} \cong V \times (V^* \setminus \{0\})$$

$$X^{ss}_- := (V \times V^*) \setminus \underbrace{(V \times V^*)^-}_{\{0\} \times V^*} \cong (V \setminus \{0\}) \times V^*$$

Title of Section: Interesting Loci

* The representation theory lingo for these loci are the polynomial subrep & rational but non-polynomial

(where I lose 1st years)

Now we take the quotient

$X/\!/_+ := \mathbb{C}^\times \backslash X_+^{ss}$

how we move from modules to bundles

$\cong \mathrm{Tot}\left(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}(V^*)}\right)$

again Sorry algebraic!

and

$X/\!/_- := \mathbb{C}^\times \backslash X_-^{ss}$

kinda like take $\frac{V - \{0\}}{\mathbb{C}^\times} = \mathbb{P}(V)$ and @ each point you have 2-lines $V^* \cong \mathbb{C} \times \mathbb{C}$ i.e. 2-copies of the tautological line bundle!

$\cong \mathrm{Tot}\left(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}(V)}\right)$

Sorry only have algebraic description

$k[a,c,e]$ $\quad \frac{1}{f} = e \quad$ $k[b,d,f]$

$\cong$

$k[x_1, x_2, y_1, y_2]^{\mathbb{G}_m}_{y_1} \cup k[x_1, x_2, y_1, y_2]^{\mathbb{G}_m}_{y_2}$

a $x_1 y_1$ $\quad$ $x_2 y_1$ c $\quad$ $y_2 \frac{1}{y_1}$ e

b $x_1 y_2$ $\quad$ $x_2 y_2$ d

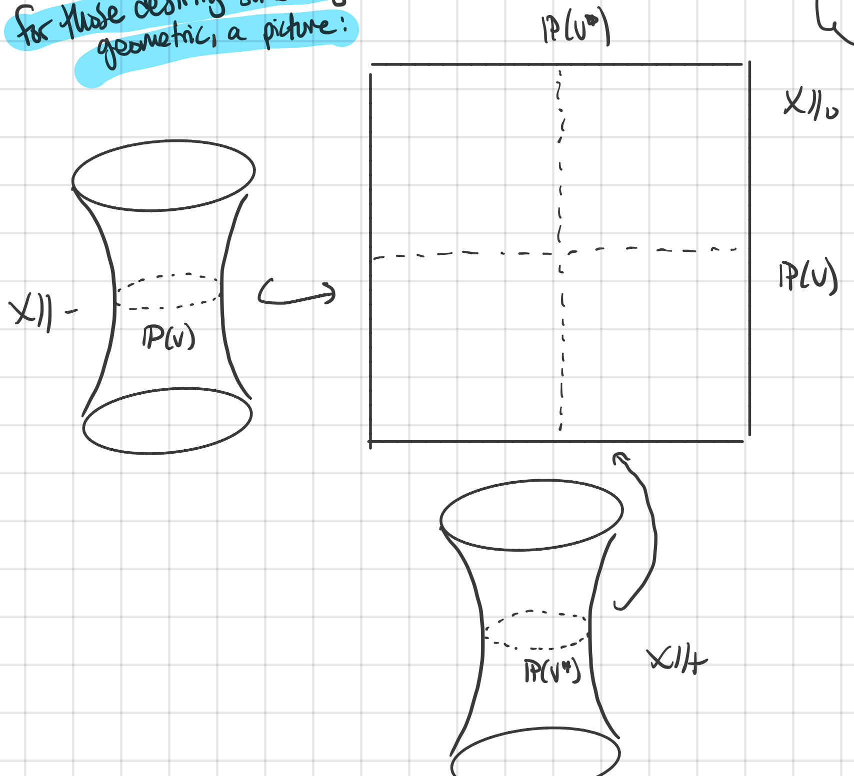
$ae - b$

$ce - d$

finally $X/\!/_0 = \mathrm{Spec}\left(k[a,b,c,d]/(ad-bc)\right)$

"Singular quadric"

for those desiring something more geometric, a picture:



highly singular at atleast 4-points!

Moral of the Story: Quotients get messy!

So what do we do? Try #1 notice $X/\!/_0$ is singular to avoid messyness like that we consider quotient stacks... for anyone that has done anything with projective space knows it's easier to consider the charts (affine patches) can we do that? yes since the spaces are less important to us, we care more about the stuff it "acts" on...

Big Ideas:

$$\begin{tikzcd}
\text{Sheaf of functions} \arrow[r, leftrightarrow] \arrow[d, equal] & \text{Geometry of the space} \\
\text{"Ring"} \arrow[d, phantom, "\circlearrowright"] & \\
\text{Modules} \arrow[r, leftrightarrow] & \text{Bundles}
\end{tikzcd}$$

for the "first" step we can say two spaces are the same if the have the same modules i.e act on same things

This a Morita "like" equivalence

Big Ideas (cont.): but the universe is "squishy" and Morita is too rigid So....

we will say two spaces are "the same" if they act on the same stuff up to homotopy (which we gauge by homology)

donut is the same as a coffee cup

But every 20 years there is a new homology ... sometimes they are the same So we want to consider a place where all the present and future ones could live ... This is the Derived Category

Crude cutting...

Slightly more formal:

- consider chains of Modules (present & future homology)
- consider maps of these chains up to homotopy
- formally invert quasi-isomorphisms (induce isomorphisms of homology)

(localization)

Notation: let R be a ring

only consider bounded chains...

$$D(\mathrm{Mod}(R)), \quad D^b(\mathrm{Mod}(R))$$

More A.G.'y:

variety

$$D^b(\mathrm{Qcoh}(X))$$

Quasi-coherent

The Big ~~Result~~ Trick

$$\begin{tikzcd}
G \times X \arrow[r, "1 \times f"] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

only consider $\mathbb{C}^\times$-equivariant Morphisms i.e. ones that preserves $\mathbb{C}^\times$-structure

$$D_{\mathbb{C}^\times}(\mathrm{Qcoh}(X^{ss}_{\pm})) \cong D(\mathrm{Qcoh}(X^{\pm}))$$

More general:

quotient stack

$$D_G(\mathrm{Qcoh}(X)) \cong D(\mathrm{Qcoh}([X/G]))$$

algebraic group

Integral Transforms

with this new concept of sameness we will need maps

$$D(\mathrm{Qcoh}(X)) \longrightarrow D(\mathrm{Qcoh}(Y))$$

"~~Stretch~~ Spread it out across Y"

$$\begin{tikzcd}
 & \pi_X^*\mathcal{F} \arrow[rr, mapsto] & & S(\pi_X^*\mathcal{F}) \arrow[dr, mapsto] & \\
\mathcal{F} \arrow[ur, mapsto] & X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y & \pi_{Y*}(S(\pi_X^*\mathcal{F}))
\end{tikzcd}$$

How do we stretch it out?

The kernel

let $K \in D(QCoh(X \times Y))$

then define for $\mathcal{F} \in D(QCoh(X))$

$$\Phi_K(\mathcal{F}) := \pi_{Y*}\left(K \otimes \pi_X^*(\mathcal{F})\right)$$

(These functors need to be "derived")

But wait... what about that "group action"

Now say $G \circlearrowleft X$ and $H \circlearrowleft Y$ Then $G \times H \circlearrowleft X \times Y$

$$\begin{tikzcd}
 & \pi_Y^*\mathcal{F} \arrow[rr, mapsto] & & K \otimes \pi_X^*\mathcal{F} \arrow[ddr, mapsto, "\pi_{Y*}"] & \\
 & & \overset{G\times H}{\circlearrowleft} X \times Y \arrow[dl] \arrow[dr] & & \\
\mathcal{F} \arrow[uur, mapsto, "\pi_X^*"] & \underset{G\circlearrowleft}{X} & & \underset{H\circlearrowleft}{Y} & \left[\pi_{Y*}(K \otimes \pi_X^*\mathcal{F})\right]^G
\end{tikzcd}$$

gives trivial action by H

Take G invariants

//

recall

The BDF kernel:

$X = V \oplus V^\vee = \mathbb{A}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} \cong \text{Spec}(\mathbb{C}[x_1, x_2, y_1, y_2])$

$\mathbb{C}^2 \times (\mathbb{C}^2)^*$ w/ the $\mathbb{C}^\times$-action is equivalent to $\deg(x_i) = +1$ $\deg(y_i) = -1$

$$Q := \frac{\mathbb{C}[x_1, x_2, x_1', x_2', y_1, y_2, y_1', y_2', u]}{\left\langle \begin{matrix} x_1' - x_1 u & y_1 - u y_1' \\ x_2' - x_2 u & y_2 - u y_2' \end{matrix} \right\rangle} \cong \mathbb{C}[x_1, x_2, y_1', y_2', u] \overset{\text{or}}{=} \mathbb{C}[x_1, x_2, y_1 u^{-1}, y_2 u^{-1}, u]$$

$\mathbb{C}[x_1, x_2, y_1, y_2]$-bimodule via the maps

$$\overset{p}{\begin{matrix} x_i \longmapsto x_i \\ y_i \longmapsto y_i' u \end{matrix}} \qquad \overset{s}{\begin{matrix} x_i \longmapsto u x_i \\ y_i' \longmapsto y_i' \end{matrix}}$$

and hence $\tilde{Q}$ is an $\mathcal{O}_{X \times X}$-sheaf of modules then we embedd into the derived category to get our kernel!

let $i : X_+^{ss} \hookrightarrow X$ and $j : X_-^{ss} \hookrightarrow X$ then

$$\Phi_{(i \times j)^* \tilde{Q}} : D^b(X_+^{ss}) \longrightarrow D^b(X_-^{ss}) \text{ is an equivalence!}$$

Thus by "Big Trick" get our result

Cool Geometric properties of The BDF kernel

* $\mathbb{G}_m$ is the scheme for $\mathbb{C}^\times$ *

w/ previous maps p & s induce maps of schemes $\hat{p}$, & $\hat{s}$

$$\begin{tikzcd}
 & \mathrm{Spec}(Q) \arrow[d, shift right, "\hat{p}"'] \arrow[d, shift left, "\hat{s}"] \\
\mathbb{G}_m \times X \arrow[ur, "\tau"] \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\sigma"'] & X
\end{tikzcd}$$

This is known as a partial compactification of the action of $\mathbb{G}_m$ on X

Boundary:

$$\partial_Q := \mathrm{Spec}(Q) - \tau(\mathbb{G}_m \times X)$$

un-stable loci:

$$X^{us}_{\hat{s}} := \hat{s}(\partial_Q) \qquad X^{us}_{\hat{p}} := \hat{p}(\partial_Q)$$

Semi-stable loci:

$$X^{ss}_{\hat{s}} := X - \hat{s}(\partial_Q) \qquad X^{ss}_{\hat{p}} := X - \hat{p}(\partial_Q)$$

Awesome thing:

$$X^{ss}_{\hat{s}} = X^{ss}_{+} \quad \text{and} \quad X^{ss}_{\hat{p}} = X^{ss}_{-}$$

Talk: (outline)

- Grassmannian Flop
    - rank 1 gives Atiyah
- windows (the idea)
- The BDF – kernel
- (partial) Compactification
    - loci
        - Boundary
        - unstable
        - Semi-stable
- Localization
- $\mathrm{Im}\,\Phi_{\tilde{Q}^+} \cong \mathrm{Im}\,\Phi_{\tilde{Q}^-}$
    - Kapranov window
    - lascoux
    - $\Phi_{Q^+}(V) = (V)_{\geq 0}$
    - $\Phi_{Q^+}\underset{i^*}{(\mathcal{K})} = \mathcal{K}$
- Future directions...

$\not\leftarrow \quad \mathrm{Hom}(N_{p}, N_{p_1}) = 0$

$\rightarrow$

only points forward

The Talk:

(Joint work w/ Dr. Ballard & Dr. McFaddin)

The Grassmannian Flop:

Let V and W be complex V.S. s/t $*\ \dim V = n \leq \dim W = m\ *$

$*$ first write it's affine

and denote:

$$X = \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V) = \mathrm{spec}\big(\underbrace{k[\overset{A}{\overset{\shortparallel}{x_{ij}}}, \overset{B}{\overset{\shortparallel}{y_{ij}}}]}_{R}\big)$$

and $$X_-^{ss} = \{\phi \mid \mathrm{rank}\,\phi \leq n-1\} \oplus \mathrm{Hom}(W,V)$$

$$X_+^{ss} = \mathrm{Hom}(V,W) \oplus \{\psi \mid \mathrm{rank}\,\psi \leq n-1\}$$

$*$ again note

$\{\phi \mid \mathrm{rank}\,\phi \leq n-1\} / GL = Gr(n,W)$

hence the name...

The Grassmannian Flop is the following

$$\begin{tikzcd}
\left[X_+^{ss}/GL(V)\right] \arrow[rr] \arrow[dr, "\text{open}"'] & & \left[X_-^{ss}/GL(V)\right] \arrow[dl, "\text{open}"] \\
& \left[X/GL(V)\right] &
\end{tikzcd}$$

$[X_+^{ss}/GL]$ is also a bundle over the grassmannian

$*$ note for $\dim V = 1$ this recovers the Atiyah flop $*$

Donovan & Segal (2012) showed that there was a derived equivalence between $[X_+^{ss}/GL(V)]$ and $[X_-^{ss}/GL(V)]$ using 'windows'

(~10min)

The Idea of window(s):

$$\begin{tikzcd}
& D^b\left(\left[X_+^{ss}/GL\right]\right) \\
D^b\left(\left[X/GL\right]\right) \arrow[ur, "\text{restriction}"'] \arrow[dr, "\text{restriction}"] & \\
& D^b\left(\left[X_-^{ss}/GL\right]\right)
\end{tikzcd}$$

(i.e. every "module" of bigger one can be restricted to module of the smaller one)

when we restrict our restrictions to this window it's an equivalence to both

we almost have proven that a generalized BDF-kernel also realizes this...
(and it's essential image is a window!)

Just as in the pre-talk

The Big ~~Result~~ Trick

$$\begin{tikzcd} G \times X \arrow[r, "1 \times f"] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

only consider G-equivariant morphisms i.e. ones that preserves G-structure

$$D_G(QCoh(X)) \cong D(QCoh([X/G]))$$

quotient stack

algebraic group

So we only need to consider the GL-equivariant derived categories

$$D^b_{GL}(X^{ss}_+) \text{ and } D^b_{GL}(X^{ss}_-)$$

and provide a kernel for this equivalence

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The BDF kernel

$$\begin{tikzcd} GL \times X \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\pi"'] & X \end{tikzcd}$$

action

projection

A Ring

$k[\mathrm{End}(W)]$

$$Q = \left(\sigma^{\#}(R),\ p^{\#}(R),\ \underset{\overset{\shortparallel}{C}}{c_{ij}} \right)$$

$$\cong k[A \cdot C^{-1}, B, C]$$

$$= k[\tilde{A}, B, C]$$

$$\cong k[A, B] \otimes k[\mathrm{End}(W)]$$

$$\begin{tikzcd} R \arrow[r, shift left, "s^{\#}"] \arrow[r, shift right, "p^{\#}"'] & Q \end{tikzcd}$$

$A \overset{s^{\#}}{\longmapsto} \tilde{A}$ $\qquad$ $A \overset{p^{*}}{\longmapsto} \tilde{A}C$

$B \longmapsto CB$ $\qquad$ $B \longmapsto B$

again Q has a R bi-module structure and hence $\tilde{Q}$ is an $\mathcal{D}_{X \times X}$-module

we can consider the functor

$$\Phi_{\tilde{Q}} : D^b_{GL}(QCoh(X)) \to D^b_{GL}(QCoh(X))$$

<u>another useful functor:</u>

$$Id : D^b_{GL}(QCoh(X)) \longrightarrow D^b_{GL}(QCoh(X))$$

$$Id = \Phi_{\Delta} \quad \text{where } \Delta = GL \times X$$

Furthermore w/ this bi-module structure we have the following isomorphism

$$\frac{\mathbb{C}[A_1, A_2, B_1, B_2, C]}{(A_1 - A_2C,\ B_2 - CB_1)} \cong Q$$

$A_1, B_1 \leftarrow p^{\#}$ structure

$A_2, B_2 \leftarrow s^{\#}$ structure

//

Similar to the pre-talk this is also

a partial compactification

$$\begin{tikzcd} & & \mathrm{Spec}(Q) \arrow[d, shift right, "p"'] \arrow[d, shift left, "s"] \\ GL\times X \arrow[urr, "\mu"] \arrow[rr, shift left, "\pi"] \arrow[rr, shift right, "\sigma"'] & & X \end{tikzcd}$$

if don't get that far in pretalk first define

and Recall $\partial_Q = \mathrm{Spec}(Q) - \tau(GL\times X)$ $X^{us}_s = s(\partial_Q)$ $X^{us}_p = p(\partial_Q)$

~20min

again! again!

$X^{ss}_s = X - s(\partial_Q) = X^{ss}_+$ $X^{ss}_p = X - p(\partial_Q) = X^{ss}_-$

So not only will this provide a kernel for this equivalence but it also encodes all of the information of the loci "internally"

The "first half" (of the equivalence)

$$\begin{tikzcd} & X^{ss}_+\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X^{ss}_+ & & X \end{tikzcd}$$

denote $i: X^{ss}_+ \hookrightarrow X$ the inclusion of the open set

and $\tilde{Q}_+ := (i\times 1)^* \tilde{Q}$

Then $\Phi_{\tilde{Q}_+} : D^b_{GL}(QCoh(X^{ss}_+)) \longrightarrow D^b_{GL}(QCoh(X^{ss}))$

This functor is faithful

Quick why:

it has a left inverse $i^*: D^b_{GL}(QCoh(X)) \to D^b_{GL}(QCoh(X^{ss}_+))$, to see this note that X^{ss}_+ has an affine chart consisting of the localizations of the maximal minors of B, denote one of those by t_I then w/ $I \subseteq \{1,\dots,m\}$, $|I|=n$ indexing the rows and note $R_{t_I}\otimes_S Q \cong K[GL]\otimes R$ as if you invert a minor of "B_2" from above you invert the determinant of C, and hence as $K[GL]\otimes R$ is the kernel to the identity and $i^*\circ\Phi_{\tilde{Q}^+}$

$$\begin{tikzcd} & X^{ss}_+\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X^{ss}_+ & & X \\ & & X^{ss}_+ \arrow[u, "i"'] \end{tikzcd} \rightsquigarrow \quad i^*\circ\pi_{2*} \cong \pi_* \qquad \begin{tikzcd} & X^{ss}_+\times X^{ss}_+ \arrow[dl, "\pi_1"'] \arrow[dr, "\pi"] & \\ X^{ss}_+ & & X^{ss}_+ \end{tikzcd}$$

Next full

Bousfield localizations

Definition 3.3.1. Let $\mathcal{T}$ be a triangulated category. A **Bousfield localization** is an exact endofunctor $L : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation

$$\delta : \mathrm{Id}_{\mathcal{T}} \to L$$

such that:

a) $L\delta = \delta L$ and
b) $L\delta : L \to L^2$ is invertible.

If instead we have an endofunctor $C : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\epsilon : C \to 1$ such that

a) $C\epsilon = \epsilon C$ and
b) $C\epsilon : C^2 \to C$ is invertible,

then one calls C a **Bousfield colocalization**.

★ $\Phi_{\tilde{Q}}$ is a Bousfield co-localization! → $\tilde{Q} * \tilde{Q} = \tilde{Q}$ ↑ convolution (calculus of kernels)

Definition 3.3.3. Suppose we have maps of endofunctors

$$C \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{T}} \xrightarrow{\delta} L$$

of a triangulated category $\mathcal{T}$ such that

$$Cx \xrightarrow{\epsilon_{Cx}} x \xrightarrow{\delta_x} Lx$$

is an exact triangle for any object x. Then $C \to \mathrm{Id}_{\mathcal{T}} \to L$ is called a a **Bousfield triangle** for $\mathcal{T}$ if any of the following equivalent conditions are satisfied:

(i) L is Bousfield localization and $C(\epsilon_x) = \epsilon_{Cx}$,
(ii) C is a Bousfield colocalization and $L(\delta_x) = \delta_{Lx}$,
(iii) L is Bousfield localization and C is a Bousfield colocalization.

Lemma 3.3.4. *Let $C \to \mathrm{Id}_{\mathcal{T}} \to L$ be a Bousfield triangle for a triangulated category $\mathcal{T}$. Then there is a weak semi-orthogonal decomposition*

$$\mathcal{T} = \langle \mathrm{Im}\, L \overset{\curvearrowright}{\underset{\not\leftarrow}{,}} \mathrm{Im}\, C \rangle. \tag{3.11}$$

Here Im *denotes the essential image.*

The following is an exact Triangle

$$Q_X \xrightarrow{\eta^\sharp} \Delta_X \to \mathrm{cone}_{\eta^\sharp} \to Q_X[1]$$

Thus by calculus of kernels the following is a Bousfield Triangle

$$\Phi_{\tilde{Q}_X} \xrightarrow{\hat{\eta}} 1 \to \Phi_{\mathrm{cone}\,\eta}$$

Lemma 3.3.5. *Let* $C_1 \xrightarrow{\epsilon_1} 1 \xrightarrow{\delta_1} L_1$ *and* $C_2 \xrightarrow{\epsilon_2} 1 \xrightarrow{\delta_2} L_2$ *be Bousfield triangles for a triangulated category* $\mathcal{T}$ *such that* $L_1 C_2 \xrightarrow{L_1(\epsilon_2)} L_1$ *is an isomorphism. Then there is a weak semi-orthogonal decomposition*

$$\mathcal{T} = \langle \operatorname{Im} C_2 \circ L_1, \operatorname{Im} C_2 \circ C_1, \operatorname{Im} L_2 \rangle.$$

This induces a fully[redacted] *functor*

$$F : \mathcal{T} / \operatorname{Im} C_1 \to \mathcal{T}.$$ ← follows from a Result of Bondal

We now see that from Lemma 3.25 it follows, since the map $Q(\eta)$ is just ρ_X, which is an isomorphism by the previous Lemma. With this proof and by denoting $J_+ := j_* \circ j^*$ where $j : X_s^{\mathrm{ss}} \to X$, and Γ_+ as the local cohomology, we are now ready to prove that $\Phi_{\tilde{Q}+}$ is full.

Proposition 3.28. *Let* X *be an object of* $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$. *There is a semi-orthogonal decomposition*

$$\mathsf{D}(\mathrm{Qcoh}^{\mathrm{GL}(V)} X) = \langle \operatorname{Im} \Phi_{\mathrm{cone}\,\eta}, \operatorname{Im} \Phi_Q \circ \Gamma_+, \operatorname{Im} \Phi_{\tilde{Q}+} \rangle,$$

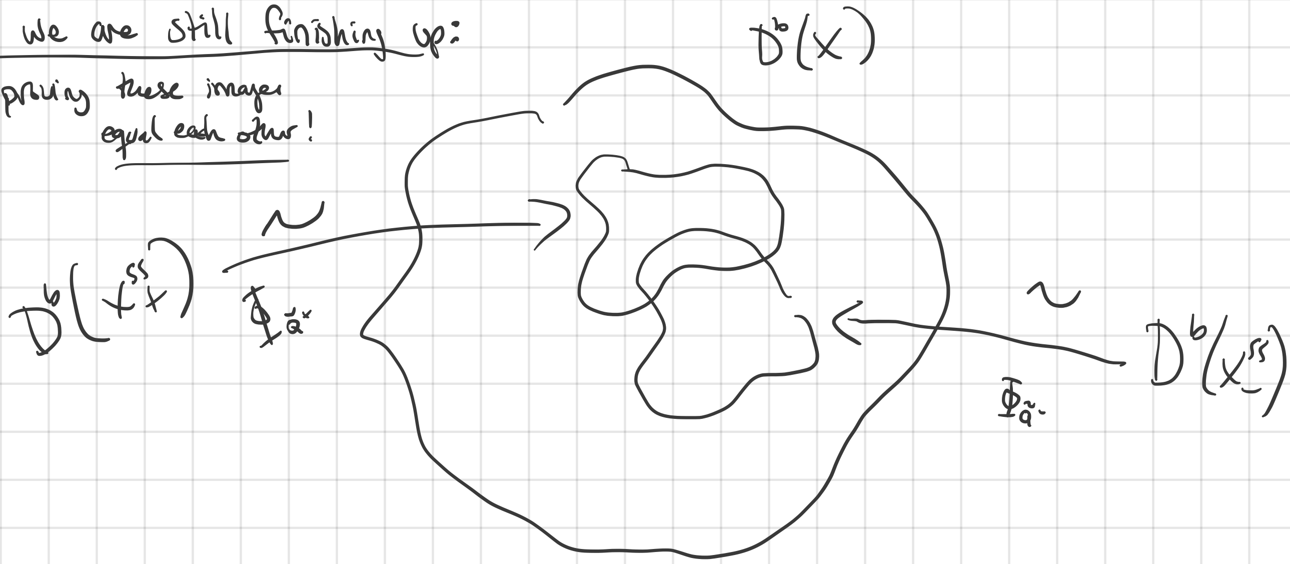
where Im *denotes the essential image. Furthermore,* $\Phi_{\tilde{Q}+}$ *is fully-faithful.*

here since these are Calabi-Yau $\Phi_{\tilde{Q}^+}$ is an equivalence!

Similar for $\Phi_{\tilde{Q}_-}$

what we are still finishing up:

proving these images equal each other!



$\mathrm{Im}\,\Phi_{\tilde{Q}^+} \cong \mathrm{Im}\,\Phi_{\tilde{Q}^-}$ (or why we think it is)

we consider the exceptional collection on the grassmannian's studied by Kapparanov.
consider $Gr(n, W)$ denote V as the canonical v.bund.

$$\delta_{m,n} := \{\text{Young diagrams } \gamma \text{ with height} \leq m - n \text{ and width} \leq n\}$$

and Kapranov's collection as

$$\mathcal{K} := \left\{ L_\alpha V^* \,\middle|\, \alpha \in \delta_{m,n} \right\}.$$

* of course the way I have written it as a module of $\mathbb{C}$ we can pullback to get a module over X ! *

Don & Segal showed that Both $D^b(X^{ss}_+)$ and $D^b(X^{ss}_-)$ are equivalent to this "window"

* check the time *

if too little

Jump to future

directions !

to show $\operatorname{Im} \Phi_{\tilde{Q}^+} \subseteq \langle \mathcal{K} \rangle$

we show that $F_\bullet \xrightarrow{\text{qis}} \tilde{Q}_+$ (using the Lascoux Resolution)

$$\begin{tikzcd} & X^{ss}_+ \times X \arrow[dl, "\pi_+"'] \arrow[dr, "\pi"] & \\ X^{ss}_+ \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\ & \operatorname{Spec}(\mathbb{C}) & \end{tikzcd}$$

$F_i = \bigoplus \quad p^* E \otimes_{\mathbb{C}} q^* E'$ $\longrightarrow$ denote as $E \boxtimes E'$

where $E \in \mathcal{K}$

as $\Phi_{E \boxtimes E'}(\mathcal{O}_x) = E \otimes \underset{\substack{\uparrow \\ \text{a v.s.}}}{V} \cong \bigoplus E$

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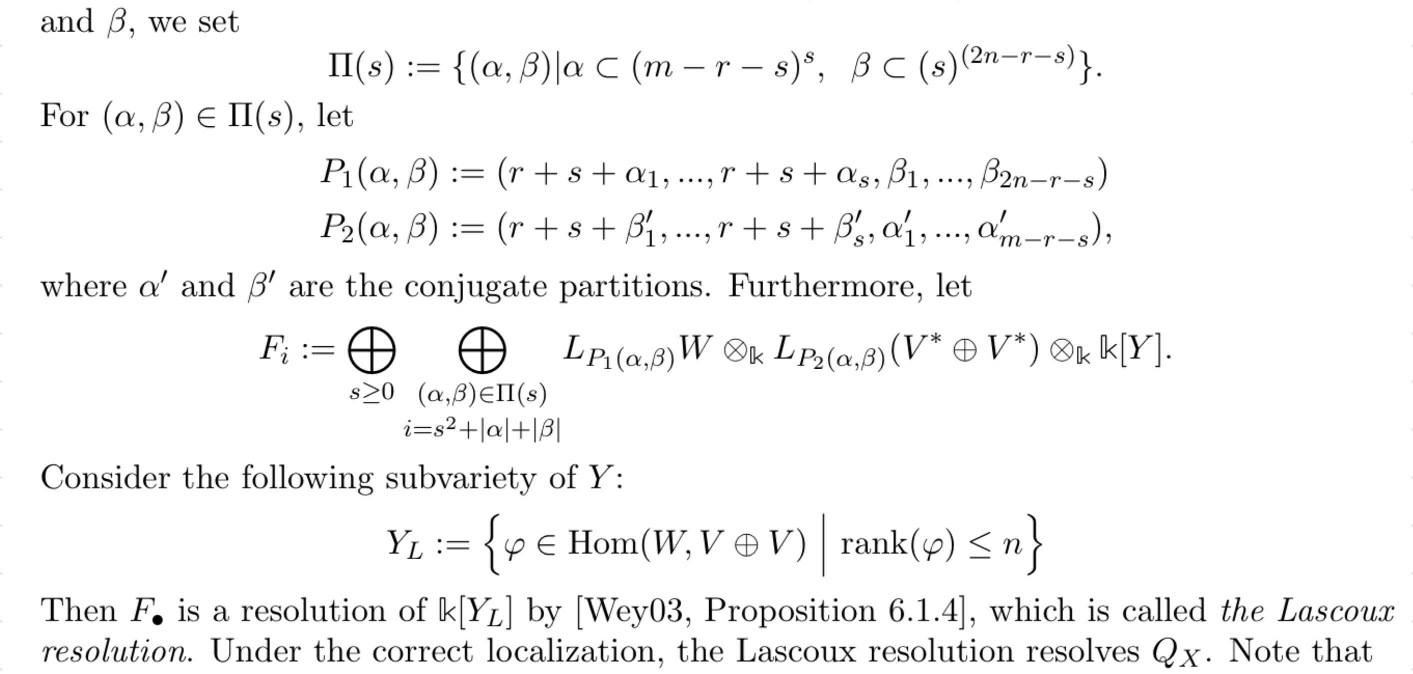

5.1. **The Lascoux Construction.** Let $Y = \operatorname{Hom}(W, V \oplus V)$. For $s \in \mathbb{N}$ and partitions α and β, we set

$$\Pi(s) := \{(\alpha, \beta) | \alpha \subset (m-r-s)^s, \quad \beta \subset (s)^{(2n-r-s)}\}.$$

For $(\alpha, \beta) \in \Pi(s)$, let

$$P_1(\alpha, \beta) := (r+s+\alpha_1, ..., r+s+\alpha_s, \beta_1, ..., \beta_{2n-r-s})$$
$$P_2(\alpha, \beta) := (r+s+\beta'_1, ..., r+s+\beta'_s, \alpha'_1, ..., \alpha'_{m-r-s}),$$

where α' and β' are the conjugate partitions. Furthermore, let

$$F_i := \bigoplus_{s \geq 0} \bigoplus_{\substack{(\alpha,\beta) \in \Pi(s) \\ i = s^2 + |\alpha| + |\beta|}} L_{P_1(\alpha,\beta)} W \otimes_{\Bbbk} L_{P_2(\alpha,\beta)}(V^* \oplus V^*) \otimes_{\Bbbk} \Bbbk[Y].$$

Consider the following subvariety of Y:

$$Y_L := \left\{ \varphi \in \operatorname{Hom}(W, V \oplus V) \;\middle|\; \operatorname{rank}(\varphi) \leq n \right\}$$

Then $F_\bullet$ is a resolution of $\Bbbk[Y_L]$ by [Wey03, Proposition 6.1.4], which is called *the Lascoux resolution*. Under the correct localization, the Lascoux resolution resolves Q_X. Note that

using the Lascoux we can write $\tilde{Q}^+$ as a resolution of elements of $\mathcal{K}$ and hence $\Phi_{\tilde{Q}^+}(E) \in \langle \mathcal{K} \rangle$

☆ probably don't need to write all this junk! ☆

To show $\langle \mathcal{E} \rangle \subseteq \mathrm{Im}\,\Phi_{\tilde{Q}*}$

want to show $\Phi_{\tilde{Q}}(i^* E) \overset{qis}{\cong} E$ for $E \in \langle \mathcal{E} \rangle$

To do this we are using the fact for flat V

$$\Phi_{\tilde{Q}}(V) = V_{\geq 0} \quad \text{i.e. the "polynomial rep part"}$$

and as $\mathcal{E}$ is only made of polynomial reps

$\Phi_{\tilde{Q}}(E) \cong E$ so just need no higher cohomology

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Future directions (generalization to the singular case)

HORI-MOLOGICAL PROJECTIVE DUALITY

JØRGEN VOLD RENNEMO AND ED SEGAL

(2017)

Let V be a vector space of odd dimension v. For any even number $0 \leq 2q < v$, we have a Pfaffian variety

$$\mathrm{Pf}_q \subset \mathbb{P}(\wedge^2 V^\vee)$$

consisting of all 2-forms on V whose rank is at most $2q$. This variety is not a complete intersection, and is usually highly singular – the singularities occur where the rank drops below $2q$. We only get smooth varieties in the cases $q = 1$, which gives the Grassmannian $\mathrm{Gr}(V, 2)$, and $q = \frac{1}{2}(v-1)$, which gives the whole of $\mathbb{P}(\wedge^2 V^\vee)$.

The projective dual of Pf_s is another Pfaffian variety; it's the locus

$$\mathrm{Pf}_s \subset \mathbb{P}(\wedge^2 V)$$

consisting of bivectors of rank at most $2s$, where $2s = v - 1 - 2q$.

These are derived equivalent

stops Q from being a co-localization

but this time highly singular so to use the BLT we will need to resolve our Q (simplicially) for any hope that it works

and each piece in this resolution will be a co-localization