Chapter 4
Sections 4.2 - 4.4

Main Topic # 1: [Log is "like" square root]

The main concept for Logs is the concept of the **opposite** in the sense of the inverse of a function. Recall how one calculates the positive branch of the square root:

$$\sqrt{4} = \underline{\ 2\ }$$

That is because

$$(\underline{\ 2\ })^2 = \underline{\ 4\ }$$

In slight more generality:

$$\sqrt{b} = c$$

means

$$c^2 = b$$

It is the same idea for the Log and an exponential. First the technical definition:

The Log

For $a > 0$ we call the inverse of the function $f(x) = a^x$ **Log base** a and write it as

$$\log_a(x) = f^{-1}(x)$$

Now the definition which mimics the idea of square root above.

The Opposite of an Exponential

For $a > 0$

$$\log_a(b) = c$$

means

$$a^c = b$$

or

$$\log_a(b)$$

is the power of $\underline{\ a\ }$ that is $\underline{\ b\ }$

That is the Log is the function that says "Gimme that exponent"

Finally, there is a special Log that we call **The Natural Log**:

$$\log_e(x) = \ln(x)$$

Learning Outcome # 1: [Using the meaning of Log to calculate]

Problem 1. Complete the following statements.

(a) If $y = \log_{10}(100)$, then $\underline{\ 10\ }^y = 100$.

(b) $\log_{10}(5.5)$ is the power of $\underline{\ 10\ }$ that gives $\underline{\ 5.5\ }$.

(c) $\log_2(\underline{\ 500\ })$ is the power of $\underline{\ 2\ }$ that gives 500.

(d) If $4^m = n$ then $\log_4(n) = \underline{\ m\ }$.

(e) $\log_e(556)$ is the power of $\underline{\ e\ }$ that gives $\underline{\ 556\ }$.

Problem 2. Rewrite the following using exponents instead of logs.

(a) $\log_e(5) \approx 1.609$

$e^{1.609} = 5$

(b) $\log_2(1) = 0$

$2^0 = 1$

(c) $\log_{100}(A) = B$

$100^B = A$

Problem 3. Rewrite the following using logs instead of exponents.

(a) $e^{15} \approx 3269017.373$

$\ln(3269017.373) \approx 15$

(b) $10^{-2} = \frac{1}{100}$

$\log_{10}\left(\frac{1}{100}\right) = -2$

(c) $7^t = H$

$\log_7(H) = t$

Problem 4. Evaluate the following without using a calculator:

(a) $3^{\log_3(7)} = 7$

(b) $\log_{11}(11^4) = 4$

(c) $\log_b(\underbrace{\sqrt{b^3}}_{(b^3)^{1/2}}) = 3/2$

Problem 5. Evaluate the following without using a calculator.

(a) $\cancel{e^{\ln}}(17) = 17$

(b) $\cancel{\ln(e}^3) = 3$

(c) $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

$\downarrow$

$\frac{1}{e^{1/2}} = e^{-1/2}$

Main Topic # 2: [Solving Equations with Log]

In the last section we talked about exponential functions. Today we want to talk about how to solve equations like:

$$3^x = 7$$

So we need a way to:

"un-do" raising to the x

Just like we did in equations before...

Addition and Subtraction: To solve the equation $x + 7 = 8$ we need to "un-do adding 7" by subtracting 7 from both sides and get:

opposite

$$x + \cancel{7} = 8$$
$$\cancel{-7} \quad -7$$

$$x = 1$$

Multiplication and Division: To solve the equation $3x = 9$ we need to "un-do the multiplcation by 3" by dividing by 3 on both sides of the equation to get :

$$\frac{\cancel{(3)}x}{\cancel{3}} = \frac{9}{3}$$

$$x = 3$$

This is exactly what **the inverse** is for functions. To be more specific when considering the function $f(x) = 3^x$ the inverse has the following property:

$$f^{-1}(f(x)) = x$$

To use this property in the first equation we see:

$$\cancel{\log_3}(3^x) = \log_3(9)$$

$$\parallel$$

$$x = \log_3(9) = \underline{2}$$

The Take-Away

For $a > 0$ we have:

Concelation property!

$$a^{\log_a(x)} = x$$

and

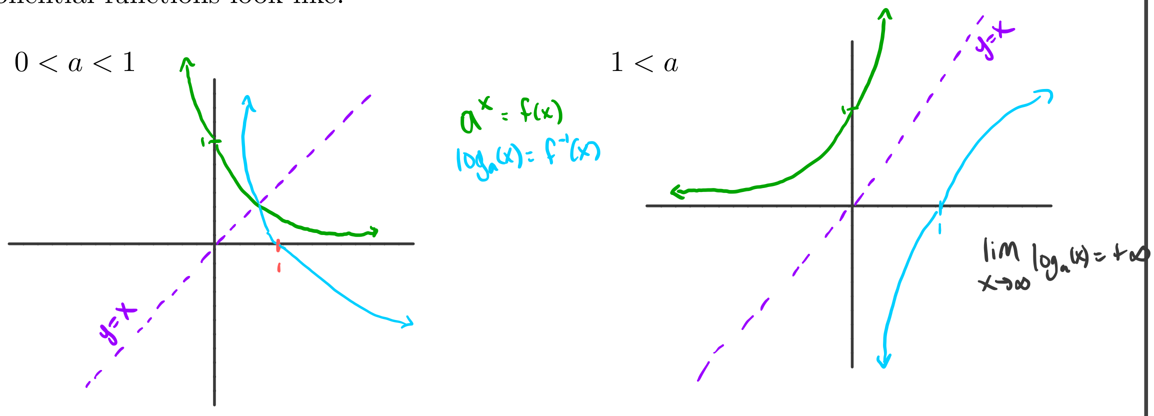
$$\log_a(a^x) = x$$

BECAUSE THEY ARE "OPPOSITES"!!!!

Finally, let's look at the graph of the Log, using what we learned about inverses in the previous section:

Sketching Logs

Exponential functions look like:



So we see that the **Domain** of $\log_a(x)$ is $(0, \infty)$

and the **Range** of $\log_a(x)$ is $(-\infty, \infty)$

Learning Outcome # 2: [Solving Basic Equations with Log]

Problem 6. Solve for x in the equations below.

(a) $\log_3(3^x) = \log_3(29)$

$x = \log_3(29)$

(b) $6 = 2(1.03)^x$

$\frac{6}{2} = \frac{2(1.03)^x}{2}$

$3 = (1.03)^x$

$\log_{1.03}(3) = x$

(c) $e^{-x} = \frac{1}{2}$

$\ln(e^{-x}) = \ln(\frac{1}{2})$

$-x = \ln(\frac{1}{2})$

$x = -\ln(\frac{1}{2})$

(d) $\boxed{3^x} - 7 = 12$

$+7 \quad +7$

$3^x = 19$

$x = \log_3(19)$

(e) $3e^{5x} + 2 = 8$

$x = \frac{\ln(2)}{5}$

(f) $4^{-7x} - 2 = 10$

$+2 \quad +2$

$4^{-7x} = 12$

$-7x = \log_4(12)$

$x = \frac{\log_4(12)}{-7}$

(g) $6 \cdot 3^{-2x} = 3^{4x}$

$\frac{6 \cdot 3^{-2x}}{3^{-2x}} = \frac{3^{4x}}{3^{-2x}}$

$6 = 3^{4x} \cdot 3^{2x} = 3^{6x}$

$6 = 3^{6x}$

$\log_3(6) = 6x \rightarrow x = \frac{\log_3(6)}{6}$

(h) $(e^x)^4 + 3 = 7$

$x = \frac{\ln(4)}{4}$

(i) $2(7^x)^2 + 3 = 15$

$-3 \quad -3$

$\frac{2(7^x)^2}{2} = \frac{12}{2}$

$(7^x)^2 = 6$

$7^{2x} = 6$

$2x = \log_7(6)$

$x = \frac{\log_7(6)}{2}$

Main Topic # 3: [The Laws of Logs]

Recall the Laws of Exponents

The Laws of Exponents

$$a^n \cdot a^m = a^{m+n}$$
$$(a^n)^m = a^{m \cdot n}$$
$$\frac{a^n}{a^m} = a^{n-m}$$
$$a^0 = 1$$

To understand the Laws of Logs we will first investigate what happens to all of the exponential laws when we apply $\log_a(\_\_)$ to both sides:

$$\log_a\left(a^n \cdot a^m\right) = \log_a\left(a^{m+n}\right) = m + n = \underline{\log_a(a^m)} + \underline{\log_a(a^n)}$$

$$\log_a\left((a^n)^m\right) = \log_a\left(a^{m \cdot n}\right) = m \cdot n = \underline{m} \cdot \underline{\log_a(a^n)}$$

$$\log_a\left(\frac{a^n}{a^m}\right) = \log_a\left(a^{n-m}\right) = n - m = \underline{\log_a(a^n)} - \underline{\log_a(a^m)}$$

$$\underline{0} = \log_a\left(a^0\right) = \log_a(1)$$

The awesome thing is the far right hand side has nothing to do with the base, which gives us our Laws of Logs, or as I like to call it:

Rob's Log Laws

The Laws of Logs

$$\log_a(A \cdot B) = \log_a(A) + \log_a(B)$$
$$\log_a(A^n) = n \cdot \log_a(A)$$
$$\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$$
$$\log_a(1) = 0$$

Learning Outcome # 3: [Identifying and Applying the Laws of Logs]

Problem 7. Match each expression of the left with its equivalent expression on the right for $A, B > 0$.

★ $\ln(AB)$	$\log_a(A^{3t})$
$\log_a\left(\frac{A}{B}\right)$	$\ln(A) + \ln(B)$
$\log_a(A^2) - \log_a(B)$	1
$t\log_a(A^3)$	$\frac{\ln(A)}{2}$
$\log_a(1)$	$2\log_a\left(\frac{A}{\sqrt{B}}\right)$
$\ln(e)$	0
$\ln(\sqrt{A})$	$\log_a(A) - \log_a(B)$

Problem 8. Rewrite each of the following as the sum/difference of simple logarithms.

(a) $\ln\left(\frac{3x^2}{yz}\right)$

$= \ln(3x^2) - \ln(yz)$

$= \ln(3) + \ln(x^2) - (\ln(y) + \ln(z))$

$= \ln(3) + 2\ln(x) - \ln(y) - \ln(z)$

(b) $\log_{10}\left(\frac{a^2b}{(cd)^3}\right)$

$= \log_{10}(a^2b) - \log_{10}((cd)^3)$

$= \log_{10}(a^2) + \log_{10}(b) - 3\log_{10}(cd)$

$= 2\log_{10}(a) + \log_{10}(b) - 3(\log_{10}(c) + \log_{10}(d))$

$= 2\log_{10}(a) + \log_{10}(b) - 3\log_{10}(c) - 3\log_{10}(d)$

(c) $\log_3\left(\frac{(z-1)^3}{z^{3/2}}\right)$

$= \log_3((z-1)^3) - \log_3(z^{3/2})$

$= 3\log_3(z-1) - 3/2\log_3(z)$

Problem 9. Rewrite each of the following as a single logarithm.

(a) $\ln(x) + \ln(3) - 2\ln(y)$

$= \ln(3x) - \ln(y^2)$

$= \ln\left(\frac{3x}{y^2}\right)$

(b) $\log_{10}(a) - 2\log_{10}(b) + 3\log_{10}(c) - 4\log_{10}(d)$

$= \log_{10}(a) - \log_{10}(b^2) + \log_{10}(c^3) - \log_{10}(d^4)$

$= \log_{10}\left(\frac{a}{b^2}\right) + \log_{10}\left(\frac{c^3}{d^4}\right) = \log_{10}\left(\frac{ac^3}{b^2d^4}\right)$

(c) $\frac{1}{2}\log_c(x) - \log_c(y) - \log_c(z-1) - \log_c(a)$

$= \log_c\left(\frac{\sqrt{x}}{y}\right) - \log_c\left(\frac{z-1}{a}\right) = \log_c\left(\frac{a\sqrt{x}}{y(z-1)}\right)$

Learning Outcome # 4: [Solving Equations using the Laws of Logs]

Problem 10. Solve the following equations:

(a) $3^{x+1} = 9^{2x}$

Done on next page

(b) $6^x = 7^{x-1}$

Done on page after next

(c) $3^{2x-1} = 5^x$

Done on page after next

Problem 11. Solve the following equations:

(a) $\log_{10}(x-3) = 4$

Done 3 pages later!

(b) $\log_2(x) + \log_2(x+2) = \log_2(6x+1)$

Done 3 pages later!

(c) $\log_3(x) - \log_3(x-1) = 2$

Done 4 pages later!

(d) $2\ln(x) = \ln(x+3) + \ln(x-1)$

Done 4 pages later!

* keep on writing the base when writting $\log_a$ sucks so w/ the log Laws I can solve every problem that deals w/ an exponential I can use ln *

10] (a) $3^{x+1} = 9^{2x}$

$\ln(3^{x+1}) = \ln(9^{2x})$ Take ln of both sides

$(x+1)\ln(3) = 2x\ln(9)$ use log law to pull down exponent

$x\ln(3) + \ln(3) = 2x\ln(9)$ distribute $\ln(3)\,(x+1)$

$-x\ln(3)$ $\quad -x\ln(3)$

$\ln(3) = 2x\ln(9) - x\ln(3)$ collect x's on same side of equation

$\ln(3) = x\left(2\ln(9) - \ln(3)\right)$ factor out common x

$\ln(3) = x\left(\ln\left(\frac{9^2}{3}\right)\right)$ use log laws to combine

$\ln(3) = x\ln(27)$ Reduce the fraction

$\overline{\ln(27)}$ $\quad \overline{\ln(27)}$

$\frac{\ln(3)}{\ln(27)} = x$ isolate the x by dividing both sides by $\ln(27)$

10] (b) $6^x = 7^{x-1}$

$\ln(6^x) = \ln(7^{x-1})$ Take ln of both sides

$x\ln(6) = (x-1)\ln(7)$ use log law to pull down exponent

$x\ln(6) = x\ln(7) - \ln(7)$ distribute the $\ln(7)(x-1)$

$-x\ln(7) \quad -x\ln(7)$

$x\ln(6) - x\ln(7) = -\ln(7)$ collect x's on one side of equation by subtracting

$x(\ln(6) - \ln(7)) = -\ln(7)$ factor out common x

$\frac{x\ln(\frac{6}{7})}{\ln(\frac{6}{7})} = \frac{-\ln(7)}{\ln(\frac{6}{7})}$ use log laws to combine

$x = \frac{-\ln(7)}{\ln(\frac{6}{7})}$ isolate x by dividing

10] (c) $3^{2x-1} = 5^x$

$\ln(3^{2x-1}) = \ln(5^x)$ Take ln of both sides

$(2x-1)\ln(3) = x\ln(5)$ use log law to pull down the exponent

$2x\ln(3) - \ln(3) = x\ln(5)$ distribute the $\ln(3)(2x-1)$

$-2x\ln(3) \quad -2x\ln(3)$

$-\ln(3) = x\ln(5) - 2x\ln(3)$ collect x's on one side of equation by subtracting

$-\ln(3) = x(\ln(5) - 2\ln(3))$ Factor out common x

$\frac{-\ln(3)}{\ln(\frac{5}{9})} = \frac{x\ln\left(\frac{5}{9}\right)}{\ln(\frac{5}{9})}$ use log laws to combine

$\frac{-\ln(3)}{\ln(\frac{5}{9})} = x$ isolate the x by dividing

This uses the cancellative properties we discussed earlier

11] (a) $\log_{10}(x-3)=4$

cancels!

$\cancel{10}^{\cancel{\log_{10}}(x-3)} = 10^4$ make both sides the exponet of the base

$x-3=10^4$ use cancelative property of log & exponet

$+3 \quad +3$

$x = 10003$ isolate the x by subtraction

11] (b) $\log_2(x) + \log_2(x+2) = \log_2(6x+1)$

$\log_2(x(x+2)) = \log_2(6x+1)$ use log laws to combine

$-\log_2(6x+1) \quad -\log_2(6x+1)$

$\log_2(x(x+2)) - \log_2(6x+1) = 0$ Collect x's on same side

$\log_2\left(\frac{x(x+2)}{6x+1}\right) = 0$ use log laws to combine

$2^{\log_2\left(\frac{x(x+2)}{6x+1}\right)} = 2^0$ make both sides the exponet of the base

$\frac{x(x+2)}{6x+1} = 1$ use cancelization property

$x(x+2) = 6x+1$ Multiply both sides by 6x+1 to get rid of fraction

$x^2-4x-1=0$ make look like quadratic

$x = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2}$ use quadratic equation

11] (c) $\log_3(x) - \log_3(x-1) = 2$

$\log_3\left(\frac{x}{x-1}\right) = 2$ use log laws to combine

cancels

$\cancel{3}^{\log_3\left(\frac{x}{x-1}\right)} = 3^2$ make both sides the exponet of the base

$\frac{x}{x-1} = 9$ use cancellative property

$x = 9(x-1)$ multiply get rid of faction

$\vdots$ } old stuff

$x = \frac{9}{8}$

11] (d) $2\ln(x) = \ln(x+3) + \ln(x-1)$

$2\ln(x) = \ln\left(\frac{x+3}{x-1}\right)$ use log laws to combine

$-2\ln(x)$ $-2\ln(x)$

$0 = \ln\left(\frac{x+3}{x-1}\right) - 2\ln(x)$ Collect x on both sides

$0 = \ln\left(\frac{x+3}{x(x-1)}\right)$ use log laws to combine

$e^0 = e^{\ln\left(\frac{x+3}{x(x-1)}\right)}$ make both sides the exponet of the base

$1 = \frac{x+3}{x(x-1)}$ use cancellative property

$x(x-1) = x+3$

$\vdots$ } old stuff

$x = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2$