Beyond Infinity

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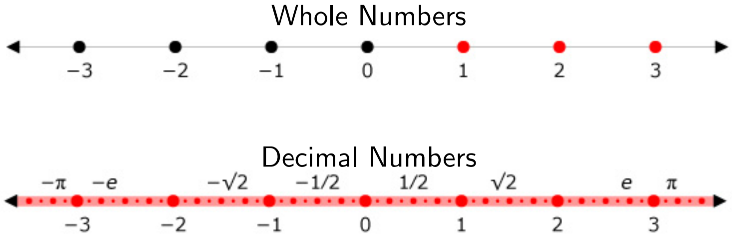
A Strange Thing

The collection of whole numbers and the collection of decimal numbers are both infinite, but are they really the same?



The whole numbers dot the landscape, while the decimal numbers fill it up.

Another Strange Thing

- What whole number comes after 1?
  - Easy, it's 2.
  - Next is 3.
  - Then 4.
  - And so on.
- After any given whole number, you can always specify the next one.
- In other words, the whole numbers can be *counted* or *listed* one after another.

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- What decimal number comes after 1?
  - Not 1.1, since 1.01 is closer.
  - 1.001 is even closer.
  - 1.0001 is closer still.
  - And so on.
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- Maybe there are too many decimal numbers to list (whatever that means).

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Comparing Without Counting

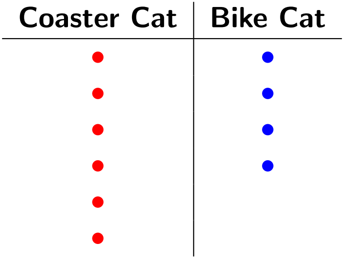
- We need a way to compare the sizes of things without counting.
- Infinity is hard, so let's try a finite version first.

Comparing Without Counting

- Both cats received lots of Christmas presents.
- They need to know who received more.
- Cats don't know how to count.

Comparing Without Counting

Strategy: Arrange the gifts in pairs with one from each cat.



- Unmatched gifts in Coaster cat's column
- Coaster cat got more gifts

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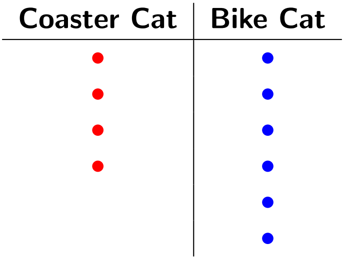

- Unmatched gifts in Bike cat's column
- Bike cat got more gifts

Comparing Without Counting

Strategy: Arrange the gifts in pairs with one from each cat.

Coaster Cat	**Bike Cat**
●	●
●	●
●	●
●	●
●	●
●	●

- No unmatched gifts in either column
- Each cat received the same number of gifts
- Gifts are said to be in *one-to-one correspondence*

Morals:

1. Always buy your cats an equal number of gifts.
2. Two collections are of the same size if there is a one-to-one correspondence between them.
3. If no such correspondence exists, one of the collections must be larger.

Comparing Without Counting

Let's apply the idea of a one-to-one correspondence to infinite collections.

- If there is a one-to-one correspondence between the whole numbers and the decimal numbers, then they must be the same size.
  - If a collection can be matched up with the whole numbers, we say that collection is *countable*.
- If there is no one-to-one correspondence between the whole numbers and the decimal numbers, then the collection of decimal numbers must be larger.
  - If a collection is too big to be matched up with the whole numbers, we say that collection is *uncountable*.

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Countable Collections: Even Numbers

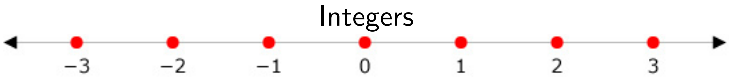
Row	Even
1	
2	
3	
$\vdots$	
100	
101	
$\vdots$	

Countable Collections: Even Numbers

Row	Even
1	2
2	4
3	6
$\vdots$	$\vdots$
100	200
101	202
$\vdots$	$\vdots$

We've demonstrated a one-to-one correspondence between the whole numbers and the even numbers, so the even numbers are countable.

Countable Collections: Integers



- Notice the integers are "doubly-infinite", extending infinitely both forward and backward.
- Perhaps this makes them uncountable.

Nope.

Countable Collections: Integers

Row	Integer
1	0
2	1
3	-1
4	2
5	-2
⋮	⋮
100	50
101	-50
⋮	⋮

We've demonstrated a one-to-one correspondence between the whole numbers and the integers, so the integers are countable.

Countable Collections: Fractions

We can play the same game with the collection of fractions as we did with the collection of decimals.

- What fraction comes after 0?
  - Not $\frac{1}{2}$, since $\frac{1}{4}$ is closer.
  - $\frac{1}{8}$ is even closer.
  - $\frac{1}{16}$ is closer still.
  - And so on.
- In fact, between *any* two fractions, there is another fraction (just take their average).
- There are decimal numbers that aren't fractions, however.
  - $\sqrt{3}$, π, e, $\ln(2)$, and many others
- The collection of fractions is quite large, but are these missing numbers enough to prevent it from being uncountable?

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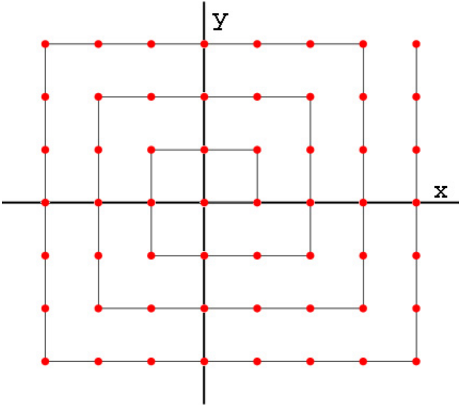
Countable Collections: Fractions

We have to order the fractions in a different way.

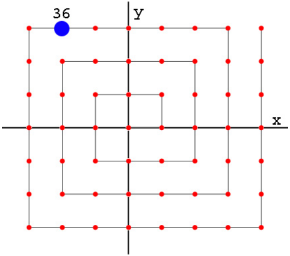


- Let x be the numerator and y be the denominator.
- Follow the spiral to visit all the fractions.

Countable Collections: Fractions

For example:

- Dot number 36 is at coordinate $(-2, 3)$.
- In our list of fractions, row 36 will contain $-\frac{2}{3}$.



This picture gives a one-to-one correspondence between the whole numbers and the fractions, so the fractions are countable.

An Uncountable Collection: Decimals

Finally, let's show that the collection of decimals really is uncountable.

- Actually, we're going to show that the collection of decimals between 0 and 1 (exclusive) is uncountable, but this is good enough.
- Start by assuming we've managed to list out all the decimals (like we've done with other countable sets).
- We'll show there's something horribly wrong with our list.
- In fact, any attempt to list the decimals will be doomed to failure, and so we'll have to conclude that they are uncountable.

An Uncountable Collection: Decimals

Maybe our list looks like this.

Row	Number
1	.394820...
2	.056733...
3	.870356...
4	.356734...
5	.745695...
6	.153455...
⋮	⋮

Highlight all the digits on the diagonal.

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Call this decimal number the "diagonal decimal".

An Uncountable Collection: Decimals

Form a new decimal from this "diagonal decimal" by adding 1 to each digit (wrapping 9's back to 0's).

$$\begin{aligned} \text{Diagonal Decimal:} \quad & .350795\ldots \\ \text{New Decimal:} \quad & .461806\ldots \end{aligned}$$

Notice the new decimal is between 0 and 1 (exclusive), and so is somewhere on the list.

Where does the new decimal appear on our list?

An Uncountable Collection: Decimals

New Decimal: $.461806\ldots$

- Is it in row 1?
  - No. The **first** digit of row **1**'s entry is a 3.
- Is it in row 2?
  - No. The **second** digit of row **2**'s entry is a 5.
- Is it in row 3?
  - No. The **third** digit of row **3**'s entry is a 0.
- And so on.

No matter which row you check, the new decimal differs in at least one place.

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An Uncountable Collection: Decimals

What does this mean?

- Our list, which supposedly listed all decimal numbers (between 0 and 1), is missing a number.
- The list can never be "patched up" by adding missing numbers, because we can always perform the diagonal operation again.
- Since no list will ever contain all the decimals, there must be more decimals than whole numbers.

The decimals constitute a larger kind of infinity than the whole numbers.

How Much Bigger?

How much bigger is uncountable than countable?

- Finite : Countable :: Countable : Uncountable

How Much Bigger?

Take a countable set (say, the whole numbers).

- Remove 100 numbers. You still have (countably) infinitely-many left.
- Remove 100 more. You still have (countably) infinitely-many left.
- And so on forever.

You can never empty a countable set by removing only finitely-many things at a time.

How Much Bigger?

Take an uncountable set (say, the decimal numbers).

- Remove countably-many numbers. You still have uncountably-many left.
- Remove countably-many more. You still have uncountably-many left.
- And so on forever.

You can never empty an uncountable set by removing only countably-many things at a time.

How Much Bigger?

In particular

- The fractions are countable.
- The decimals are uncountable.
- So, the decimals with all the fractions removed is still an uncountable collection.

What we have remaining are the *irrational* numbers.

- $\sqrt{3}$, π, e, $\ln(2)$, and many others

Essentially all the decimal numbers are irrational.

It's actually worse than that.

How Much Bigger?

- The collection of roots of numbers (square roots, cube roots, etc.) is countable.
- We can remove these as well, and we're left with the *transcendental numbers*.
  - π, e, $\ln(2)$, and many others.

Essentially all the decimal numbers are transcendental.

It's actually worse than that.

How Much Bigger?

- The *describable numbers* are all the numbers that can be described using finitely-many characters of any kind (linguistic, mathematical, etc.).
  - -1, $\frac{1}{2}$, $\sqrt{2}$, $\sum_{n=0}^{\infty} \frac{1}{n!}$, "the number of atoms in the universe one second after the Big Bang"
- The collection of describable numbers is countable.
- We can remove these from the decimal numbers, and we're left with the *indescribable numbers*.
- An indescribable number cannot be described in any fashion by someone with a finite lifespan.

Essentially all of the decimal numbers are indescribable.

How Much Bigger?

Constructing Even Larger Infinities

- So far, we've seen two kinds of infinities (countable and uncountable).
- There is a machine (called the *power set*) that takes one infinity and produces a larger one.
- Using the machine over and over gives you an endless stream of larger infinities.

Let's see how the power set works on a finite collection.

Constructing Even Larger Infinities

The *power set* takes a collection and gives you all possible combinations of the members (order doesn't matter).

Members of Original Set:	$1, 2, 3$
Members of Power Set:	$\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

- The original set contains numbers, while the power set contains *collections* of numbers.
- The original set has 3 numbers, while the power set contains 8 collections.

Constructing Even Larger Infinities

What about the power set of an infinite set like the whole numbers?

$$\begin{array}{rl} \text{Members of Original Set:} & 1, 2, 3, \ldots \\ \text{Members of Power Set:} & \{\}, \\ & \{1\}, \{2\}, \{3\}, \ldots, \\ & \{1,2\}, \{1,3\}, \{1,4\}, \ldots \\ & \{1,2,3\}, \{1,2,4\}, \{1,2,5\} \ldots \\ & \vdots \end{array}$$

Each row (except the first) has an infinitely-many things in it, and there are infinitely-many rows.

Constructing Even Larger Infinities

Why might we expect the power set to be larger than the original set?

Row	Power Set
1	$\{1\}$
2	$\{2\}$
3	$\{3\}$
4	$\{4\}$
5	$\{5\}$
$\vdots$	$\vdots$
100	$\{100\}$
101	$\{101\}$
$\vdots$	$\vdots$

We've matched up the whole numbers with a tiny, tiny portion of the power set.

Constructing Even Larger Infinities

Our endless stream of infinities might look like this.

Start with a countable collection.	Call its size $\aleph_0$.
Find the power set of the previous collection.	Call its size $\aleph_1$.
Find the power set of the previous collection.	Call its size $\aleph_2$.
And so on.	

- Each $\aleph$ denotes a larger kind of infinity than the one before it.
- All the $\aleph$ other than $\aleph_0$ are uncountable.
- You can never empty a collection of one $\aleph$ size by removing collections of a smaller $\aleph$ size (even if you get infinitely-many removals).
    - $\aleph_{100} - \aleph_{99} = \aleph_{100}$

Conclusion

1. Infinity is only the beginning.
2. The numbers we can describe are nothing compared to the numbers that exist.
3. Mathematics is a creative endeavor.

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Contact

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