1. Find the largest positive integer n with the property that $n+6(p^3+1)$ is prime whenever p is a prime number such that $2 \le p < n$. Justify your answer.

2. Find, and write out explicitly, a permutation $(p(1), p(2), \ldots, p(20))$ of $(1,2,\ldots,20)$ such that $k+p(k)$ is a power of 2 for $k=1,2,\ldots,20$, and prove that only one such permutation exists. (To illustrate, a permutation of $(1,2,3,4,5)$ such that $k+p(k)$ is a power of 2 for $k=1,2,\ldots,5$ is clearly $(1,2,5,4,3)$, because $1+1=2$, $2+2=4$, $3+5=8$, $4+4=8$, and $5+3=8$.)

3. We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1)=2$ and $t(2)=8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$.

4. A cubical box with sides of length 7 has vertices at $(0,0,0)$, $(7,0,0)$, $(0,7,0)$, $(7,7,0)$, $(0,0,7)$, $(7,0,7)$, $(0,7,7)$, $(7,7,7)$. The inside of the box is lined with mirrors and from the point $(0,1,2)$, a beam of light is directed to the point $(1,3,4)$. The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at $(0,1,2)$.

5. Define $f(x,y) = \dfrac{xy}{x^2+(y\ln(x^2))^2}$ if $x \neq 0$, and $f(0,y)=0$ if $y \neq 0$. Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and what its value is if the limit does exist.

6. Compute $\displaystyle\int_0^1 \left((e-1)\sqrt{\ln(1+ex-x)} + e^{(x^2)}\right) dx.$

7. Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ijth entry of A' is the jith entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form $A\mathbf{u}$ where $\mathbf{u}$ is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'\mathbf{v}$ where $\mathbf{v}$ is a 5×1 matrix with real entries.

PROBLEMS FROM 2005