1. Let I denote the 2×2 identity matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and let

$$M=\begin{pmatrix}I & A\\ B & C\end{pmatrix}, \quad N=\begin{pmatrix}I & B\\ A & C\end{pmatrix}.$$

where A,B,C are arbitrary 2×2 matrices which entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

2. A sequence of integers $\{f(n)\}$ for $n = 0,1,2,\ldots$ is defined as follows: $f(0) = 0$ and for $n > 0$,

$$f(n) = \begin{cases} f(n-1)+3, & \text{if } n = 0 \text{ or } 1 \pmod 6,\\ f(n-1)+1, & \text{if } n = 2 \text{ or } 5 \pmod 6,\\ f(n-1)+2, & \text{if } n = 3 \text{ or } 4 \pmod 6.\end{cases}$$

Derive an explicit formula for $f(n)$ when $n = 0 \pmod 6$, showing all necessary details in your derivation.

3. A computer is programmed to randomly generate a string of six symbols using only the letters A,B,C. What is the probability that the string will not contain three consecutive A's?

4. A 9×9 chess board has two squares from opposite corners and its central square removed (so 3 squares on the same diagonal are removed, leaving 78 squares). Is it possible to cover the remaining squares using dominoes, where each domino covers two adjacent squares? Justify your answer.

5. Let $f(x) = \int_0^x \sin(t^2 - t + x)\,dt$. Compute $f''(x) + f(x)$ and deduce that $f^{(12)}(0) + f^{(10)}(0) = 0$ ($f^{(10)}$ indicates 10 th derivative).

6. An enormous party has an infinite number of people. Each two people either know or don't know each other. Given a positive integer n, prove there are n people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the n people, then A knows B, whereas the second possibility means that if A and B are any two of the n people, then A does not know B).

7. Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty}\left|1 - \frac{a_{n+1}}{a_n}\right|$ is divergent.

PROBLEMS FROM 2004