B-spaces and their Characterization via Anisotropic Franklin Bases

G. Kyriazis, K. Park, and P. Petrushev *

B-spaces (generalized Besov spaces) generated by multilevel nested triangulations of compact polygonal domains in $\mathbb{R}^2$ are explored. Mild conditions are imposed on the triangulations which prevent them from deterioration and at the same time allow for a lot of flexibility and, in particular, arbitrarily sharp angles. It is shown that the B-spaces can be characterized by the corresponding anisotropic Franklin bases. This result is applied to nonlinear $n$-term approximation from anisotropic Franklin bases.

1. Introduction

We consider general B-spaces generated by sequences of multilevel nested triangulations of compact polygonal domains in $\mathbb{R}^2$. For a given polygonal domain $E$ in $\mathbb{R}^2$ we consider a sequence of nested triangulations $T_0, T_1, \ldots$ of general nature. Mild conditions are imposed on the triangulations which prevent them from deterioration. At the same time these conditions allow for a great deal of flexibility and, in particular, arbitrarily sharp angles. Generalized Besov spaces $B^{\alpha}_{pq}$ (called B-spaces) are naturally associated with every such sequence of triangulations and provide a specific nonstandard means of measuring the smoothness of the functions. A particular class of B-spaces needed in nonlinear approximation were introduced in [7] and further developed and used in [2, 3, 8].

In this article, we show that the general B-spaces can be characterized via Franklin bases obtained by applying the Gram-Schmidt orthogonalization process to the corresponding Courant elements. Similar results for Besov spaces in regular setups are obtained in [6] (see also the references in [6]). Further, we show how the B-spaces can be used to characterize the approximation spaces of nonlinear $n$-term approximation from anisotropic Franklin systems in $L_p$ ($1 < p < \infty$). This is a follow up paper of [9], where among other things it is

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proved that the anisotropic Franklin bases are Schauder bases for $C$ and $L_1$, and unconditional bases for $L_p$ ($1 < p < \infty$) and the corresponding Hardy space $H_1$.

The paper is organized as follows. In §2 we give all auxiliary results and introduce the anisotropic Franklin bases. In §3 we introduce the general B-spaces $B_{pq}^\alpha$ and show that the anisotropic Franklin bases characterize the B-spaces. In §4, we show that the approximation spaces of nonlinear $n$-term approximation from anisotropic Franklin bases can be characterized by certain B-spaces.

**Notation.** Throughout this article for a set $G \subset \mathbb{R}^2$, $|G|$ denotes the Lebesgue measure of $G$, while $\mathbf{1}_G$ denotes the characteristic function of $G$, and $\tilde{\mathbf{1}}_G := |G|^{-1/2}\mathbf{1}_G$. For a finite set $G$, $\#G$ denotes the cardinality of $G$. Positive constants are denoted by $c, c_1, \ldots$ (if not specified, they may vary at every occurrence), $A \approx B$ means $c_1 A \leq B \leq c_2 B$, and $A := B$ or $B =: A$ stands for “$A$ is by definition equal to $B$”. We set $\langle f, g \rangle := \int_E fg$.

2. Preliminaries

In this section we collect all prerequisites regarding triangulations, maximal operators, local approximation, etc., which will be needed in the development of the B-spaces and their characterization via Franklin bases.

2.1. Multilevel Triangulations

A set $E \subset \mathbb{R}^2$ is said to be a **bounded polygonal domain** if its interior $E^\circ$ is connected and $E$ is the union of a finite set $T_0$ of closed triangles with disjoint interiors: $E = \bigcup_{\triangle \in T_0} \triangle$. We call $T = \bigcup_{m=0}^\infty T_m$ a **locally regular triangulation** of $E$ or briefly an LR-triangulation with levels $(T_m)_{m \geq 0}$ if the following conditions are fulfilled:

(a) Every level $T_m$ is a partition of $E$, that is, $E = \bigcup_{\triangle \in T_m} \triangle$ and $T_m$ consists of closed triangles with disjoint interiors.

(b) The levels $(T_m)$ of $T$ are nested, i.e., $T_{m+1}$ is a refinement of $T_m$.

(c) Each triangle $\triangle \in T_m$ has at least two and at most $M_0$ children (subtriangles) in $T_{m+1}$, where $M_0 \geq 2$ is a constant.

(d) The valence $N_v$ of each vertex $v$ of any triangle $\triangle \in T_m$ (the number of the triangles from $T_m$ which share $v$ as a vertex) is at most $N_0$, where $N_0$ is a constant.

(e) **No hanging vertices condition**: No vertex of any triangle $\triangle \in T_m$ which belongs to the interior of $E$ lies in the interior of an edge of another triangle from $T_m$. 
(f) There exist constants $0 < r < \rho < 1$ ($r \leq \frac{1}{2}$) such that for each $\Delta \in \mathcal{T}_m$ ($m \geq 0$) and any child $\Delta' \in \mathcal{T}_{m+1}$ of $\Delta$,
\[ r|\Delta| \leq |\Delta'| \leq \rho|\Delta|. \tag{2.1} \]

(g) There exists a constant $0 < \delta \leq 1$ such that for $\Delta', \Delta'' \in \mathcal{T}_m$ ($m \geq 0$) with a common vertex,
\[ \delta \leq |\Delta'|/|\Delta| \leq \delta^{-1}. \tag{2.2} \]

We call $\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$ a regular triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$ if $\mathcal{T}$ satisfies conditions (a)-(e) of LR-triangulations and also the minimal angle condition, that is, $\min \text{angle}(\Delta) \geq \beta$ for every triangle $\Delta \in \mathcal{T}$, where $\beta > 0$ is a constant. Evidently, every regular triangulation is locally regular but not the other way around. For other types of triangulations, see [7].

We denote by $V_m$ the set of all vertices of triangles from $\mathcal{T}_m$ and by $E_m$ the set of all edges of triangles in $\mathcal{T}_m$. We also set $V := \cup_{m \geq 0} V_m$ and $E := \cup_{m \geq 0} E_m$.

We next give some basic facts concerning LR-triangulations. For more details about LR-triangulations, we refer the reader to [7] and [9].

The constants $M_0, N_0, r, \rho, \delta$, and $\#T_0$ associated with an LR-triangulation $\mathcal{T}$ are assumed fixed. We refer to them as parameters of $\mathcal{T}$.

The most important conditions (f)-(g) on LR-triangulations involve only areas of triangles but not angles. Consequently, if $\mathcal{T}$ is an LR-triangulation and $\Delta', \Delta'' \in \mathcal{T}_m$ have a common edge, then it may happen that $\Delta'$ is an equilateral triangle (or close to an equilateral triangle) but $\Delta''$ has an uncontrollably sharp angle.

In an LR-triangulation $\mathcal{T}$ there can be an equilateral (or close to such) triangle $\Delta_0$ at any level $\mathcal{T}_m$ with descendants $\Delta_1 \supset \Delta_2 \supset \ldots$ such that $\min \text{angle}(\Delta_j) \to 0$ as $j \to \infty$.

It is important to know how fast the area $|\Delta|$ of a triangle $\Delta \in \mathcal{T}_m$ may change when $\Delta$ moves away from a fixed triangle within the same level. Condition (f) suggests a geometric rate of change but in fact it is polynomial.

**Lemma 1.** If $\Delta, \Delta' \in \mathcal{T}_m$ can be connected by $n$ intermediate edges from $E_m$, then
\[ c_1^{-1}(n+1)^{-s} \leq |\Delta'|/|\Delta''| \leq c_1(n+1)^s, \tag{2.3} \]
where $s, c_1 > 0$ depend only on the parameters of $\mathcal{T}$.

**Graph distance.** We next introduce the $m$th level graph distance between vertices, which will play an important role in our further development: For any two vertices $v', v'' \in \mathcal{T}_m$, $m \geq 0$, we define the graph distance $\rho_m(v', v'')$ as the minimum number of edges from $E_m$ needed to connect $v'$ and $v''$.

The following lemma will be needed later on (see [9]).
Lemma 2. There exist constants $c > 0$ and $t > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $v^c \in \mathcal{V}_m$

$$\# \{ v \in \mathcal{V}_m : \rho_m(v, v^c) \leq n \} \leq cn^t, \quad n \geq 1.$$  \hfill (2.4)

Cells. For any vertex $v \in \mathcal{V}_m$ ($m \geq 0$), we denote by $\theta_v$ the union of all triangles from $\mathcal{T}_m$ which have $v$ as a common vertex. We denote by $\Theta_m$ the set of all such cells $\theta_v$ with $v \in \mathcal{V}_m$ and set $\Theta = \bigcup_{m \geq 0} \Theta_m$. For a given cell $\theta \in \Theta$, we shall denote by $v_\theta$ the “central” vertex of $\theta$.

For given $\theta', \theta'' \in \Theta_m$, we define the graph distance $\rho_m(\theta', \theta'')$ between $\theta'$ and $\theta''$ by $\rho_m(\theta', \theta'') := \rho_m(v_{\theta'}, v_{\theta''})$, where $v_{\theta'}, v_{\theta''} \in \mathcal{V}_m$ are the “central vertices” of $\theta', \theta''$.

Definition of $\theta^n_m$. We want to associate with each $x \in E$ a cell $\theta^n_m \in \Theta_m$, $m \geq 0$, which contains $x$. To this end we first associate with each triangle $\Delta \in \mathcal{T}_m$ a cell $\theta_{\Delta}^m \in \Theta_m$ such that $\Delta \subset \theta_{\Delta}^m$. Such a cell can be selected in three different ways. We choose one of them for each $\Delta \in \mathcal{T}_m$. Then for each $x \in E$ such that $x \in \Delta^0$ with $\Delta \in \mathcal{T}_m$, we define $\theta^n_m := \theta_{\Delta}^m$. If $x$ lies on the edge of a triangle from $\mathcal{T}_m$, we define $\theta^n_m$ as any cell from $\theta_m$ such that $x$ belongs to its interior, but if $x = v_\theta$ for some $\theta \in \Theta_m$, we set $\theta^n_m := \theta$.

Stars. In order to deal with graph distances and neighborhood relations it is convenient to employ the notion of the $n$th level star of a set: For any set $G \subset E$ and $m \geq 0$, we define the first $m$th level star of $G$ by

$$\text{Star}_m(G) := \text{Star}_m^1(G) := \cup \{ \theta \in \Theta_m : \theta \cap G \neq \emptyset \}$$ \hfill (2.5)

and inductively, $\text{Star}_m^k(G) := \text{Star}_m^1(\text{Star}_m^{k-1}(G))$, $k > 1$.

Maximal operator. Every LR-triangulation $\mathcal{T}$ of $E$ naturally generates a maximal operator $\mathcal{M}_2^\mathcal{T}$ defined by

$$\mathcal{M}_2^\mathcal{T} f(x) := \sup_{\theta \in \Theta} \left( \frac{1}{|\theta|} \int_{\theta} |f(y)|^s \, dy \right)^{1/s}$$ \hfill (2.6)

where the supremum is over all cells $\theta \in \Theta$ containing $x$ or $\theta = E$.

It is important that the Fefferman-Stein [5] vector valued maximal inequality holds for the maximal operator $\mathcal{M}_2^\mathcal{T}$ (for more details, see [9]):

Proposition 1. Let $\mathcal{T}$ be an LR-triangulation of $E \subset \mathbb{R}^2$. If $0 < p < \infty$, $0 < q \leq \infty$, and $0 < s < \min \{ p, q \}$, then for any sequence of functions $(f_j)_{j=1}^\infty$ on $E$,

$$\left\| \left( \sum_{j=1}^\infty |\mathcal{M}_2^\mathcal{T} f_j|^q \right)^{1/q} \right\|_p \leq c \left\| \left( \sum_{j=1}^\infty |f_j|^q \right)^{1/q} \right\|_p,$$ \hfill (2.7)

where $c$ depends only on $p$, $q$, $s$, and the parameters of $\mathcal{T}$.
2.2. Local Piecewise Linear Approximation and Courant Elements

The no-hanging-vertices condition (e) on LR-triangulations guarantees the existence of Courant elements, that is, for every cell $\theta \in \Theta_m$ there exists a unique continuous piecewise linear function $\varphi_\theta$ on $E$ which is supported on $\theta$ and satisfies $\varphi_\theta(v_\theta) = 1$. This is the so called Courant element associated to $\theta$. We denote $\Phi_m := \Phi_{m,T} := (\varphi_\theta)_{\theta \in \Theta_m}$ and $\Phi := \Phi_T := \bigcup_{m \geq 0} \Phi_m$.

We let $S_m$ denote the space of all continuous piecewise linear functions over $T_m$. Clearly, $S \in S_m$ if and only if $S = \sum_{v \in V_m} S(v) \varphi_\theta$. Evidently, $S_0 \subset S_1 \subset \ldots$ and it is easy to see [7] that $\bigcup_{m \geq 0} S_m = L^p(E)$, $0 < p \leq \infty$.

We shall frequently use the obvious fact that all norms of a polynomial on a triangle are equivalent, namely, if $P$ is a polynomial of degree $\leq k$ and $\triangle$ is a triangle in $\mathbb{R}^2$, then

$$\|P\|_{L^p(\triangle)} \approx |\triangle|^{1/p-1/q} \|P\|_{L^q(\triangle)}, \quad 0 < p, q \leq \infty, \quad (2.8)$$

with constants of equivalence depending only on $k$, $p$, and $q$.

The $L^p$-stability of $\Phi_m = (\varphi_\theta)_{\theta \in \Theta_m}$ is immediate from (2.8): If $(a_\theta)_{\theta \in \Theta_m}$ is an arbitrary sequence of real numbers and $S := \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta$, then

$$\|S\|_p \approx \left( \sum_{\theta \in \Theta_m} \|a_\theta \varphi_\theta\|_p^p \right)^{1/p} \approx \left( \sum_{\theta \in \Theta_m} |\theta| |a_\theta|^p \right)^{1/p}, \quad 0 < p \leq \infty. \quad (2.9)$$

The local piecewise linear approximation will play an important role in our further development. For $f \in L^q(E)$ and $\triangle \in T_m$, we denote the error of $L^q$-approximation to $f$ from $S_m$ on $\text{Star}_m(\triangle)$ by

$$S_\triangle(f)_q := S_\triangle(f, T)_q := \inf_{S \in S_m} \|f - S\|_{L^q(\text{Star}_m(\triangle))}. \quad (2.10)$$

The set $\Phi$ of all Courant elements is obviously redundant. A standard way of representing functions is by using the so called quasi-interpolant operators defined by

$$Q_m(f) := Q_{m,T}(f) = \sum_{\theta \in \Theta_m} \langle f, \tilde{\varphi}_\theta \rangle \varphi_\theta, \quad m \geq 0, \quad (2.11)$$

where the dual functions $\tilde{\varphi}_\theta$ are constructed so that they are supported in $\theta$ and $\langle \tilde{\varphi}_\theta, \varphi_{\theta'} \rangle = \delta_{\theta \theta'}$ for $\theta, \theta' \in \Theta_m$. In particular, the duals can be defined by

$$\tilde{\varphi}_\theta := \sum_{\triangle \in T_m, \triangle \subset \theta} 1_\triangle \cdot \lambda_{\triangle, \theta}, \quad (2.12)$$

where $\lambda_{\triangle, \theta}$ is the linear polynomial which is equal to $\frac{9}{N_v |\triangle|}$ at $v_\theta$ (the “central vertex” of $\theta$) and it takes the value $-\frac{3}{N_v |\triangle|}$ at the other two vertices of $\triangle$ (recall that $N_v$ is the valence of $v$).
Evidently, \( Q_m \) is a linear projector, i.e., \( Q_m(S) = S \) for \( S \in \mathcal{S}_m \). It is important that \( Q_m \) is locally bounded and provides good local approximation: If \( f \in L_q(E) \), \( 1 \leq q \leq \infty \), and \( \triangle \in \mathcal{T}_m \), then

\[
\| Q_m(f) \|_{L_q(\triangle)} \leq c \| f \|_{L_q(\text{Star}_m(\triangle))},
\]

and

\[
\| f - Q_m(f) \|_{L_q(\triangle)} \leq c S_\triangle(f)_q,
\]

where the constants depend only on \( q \) and the parameters of \( \mathcal{T} \).

We define

\[
q_m := Q_m - Q_{m-1} \quad \text{for} \quad m \geq 0, \quad \text{where} \quad Q_{-1} := 0.
\]

Clearly, \( q_m(f) \in \mathcal{S}_m \). For a given function \( f \) we define the coefficients \( (b_\theta(f))_{\theta \in \Theta_m} \) from the expression

\[
q_m(f) := \sum_{\theta \in \Theta_m} b_\theta(f) \varphi_\theta, \quad m \geq 0.
\]

It is readily seen that for \( f \in L_p(E) \), \( 1 \leq p \leq \infty \),

\[
f = \sum_{m=0}^{\infty} (Q_m(f) - Q_{m-1}(f)) = \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} b_\theta(f) \varphi_\theta,
\]

where \( Q_{-1}(f) := 0 \) and the convergence is in \( L_p(E) \). For the proofs of all of the above and more details, we refer the reader to [7] (see also [8]).

### 2.3. Anisotropic Franklin Bases

Here we define the Franklin system \( \mathcal{F}_T \) generated by Courant elements and present our main results on Franklin bases obtained in [9].

Let \( \mathcal{T} := \bigcup_{m \geq 0} \mathcal{T}_m \) be an LR-triangulation of \( E \) and recall that \( \mathcal{V}_m \) denotes the set of all vertices of triangles from \( \mathcal{T}_m \). We set \( \mathcal{V}_0 = \emptyset \) and \( \mathcal{V}_* = \mathcal{V}_m \setminus \mathcal{V}_{m-1} \) for \( m \geq 1 \) and write \( \mathcal{V}_* = \bigcup_{m=0}^{\infty} \mathcal{V}_m^* \).

Let \( \Theta_0 := E \). Choose \( \theta_{\max} \in \Theta_0 \) to be of maximum area and denote \( \Theta_0^* := \{ \theta_0 \} \cup \Theta_0 \setminus \{ \theta_{\max} \} \), i.e., we replace \( \theta_{\max} \) by \( \theta_0 = E \). Moreover, we associate \( \theta_0 \) with \( v_{\theta_{\max}} \) and set \( \varphi_{\theta_0} := 1_{\theta_0} \). For \( m \geq 1 \) denote by \( \Theta_m^* \) the set of all cells \( \theta \in \Theta_m \) with “central” vertices \( v_\theta \in \mathcal{V}_m^* \) and set \( \Theta^* := \bigcup_{m=0}^{\infty} \Theta_m^* \).

Note that for each \( m \), the set \( \{ \varphi_\theta : \theta \in \Theta_m^* \} \) is linearly independent. Also, \( \mathcal{S}_m = \text{span} \{ \varphi_\theta : \theta \in \Theta_m \} = \text{span} \{ \varphi_\theta : \theta \in \bigcup_{m=0}^{\infty} \Theta_m^* \} \).

We consider an arbitrary (but fixed) linear order \( \preceq \) on \( \Theta^* \) satisfying the following conditions:

(i) If \( \theta \in \Theta_m^* \) and \( \theta' \in \Theta_n^* \) with \( m < n \), then \( \theta \preceq \theta' \) and (ii) \( \theta_0 \preceq \theta \), \( \forall \theta \in \Theta^* \).

We now define the Franklin system \( \mathcal{F}_T \) by applying the Gram-Schmidt orthogonalization process to \( \{ \varphi_\theta \}_{\theta \in \Theta^*} \) in \( L_2(E) \) with respect to the order \( \preceq \). We obtain an orthonormal system \( \mathcal{F}_T := \{ f_\theta \}_{\theta \in \Theta^*} \) in \( L_2(E) \) consisting of continuous
piecewise linear functions. Each Franklin function \( f_\theta \) is uniquely determined (up to a multiple \( \pm 1 \)) by the conditions:

(a) \( f_\theta \in \text{span}\{ \varphi_{\theta'} : \theta' \preceq \theta \} \).

(b) \( \langle f_\theta, \varphi_{\theta'} \rangle = 0 \) for all \( \theta' \prec \theta \).

(c) \( \| f_\theta \|_2 = 1 \).

Note that \( f_{\theta_0} = \pm \mathbf{1}_{\theta_0} := \pm |E|^{-1/2} \mathbf{1}_E \).

Our main results on anisotropic Franklin systems from \cite{9} read as follows: The Franklin system \( F_T := \{ f_\theta \}_{\theta \in \Theta} \) is a Schauder basis for \( L^p(E) \), \( 1 \leq p \leq \infty \), with \( L_\infty(E) := C(E) \) and a unconditional basis for \( L^p(E), 1 < p < \infty \) and the corresponding Hardy space \( H_1(E,T) \). Also, \( H_1(E,T) \) is exactly the space of all functions in \( L_1 \) for which the Franklin system expansion converge unconditionally in \( L_1 \). Finally, the Franklin bases characterize the corresponding anisotropic BMO spaces.

For the purposes of this article, we need the localization properties of the Franklin functions \cite{9} (we use the notation from \S 2.1).

**Proposition 2.** There exist constants \( 0 < q_1 < 1 \) and \( c > 0 \) depending only on the parameters of \( T \) such that for any \( \theta \in \Theta^*_m \) (\( m \geq 0 \)),

\[
|f_\theta(x)| \leq c|\theta|^{-1/2}q_1^{\alpha_m(\theta,\theta_m)}, \quad x \in E,
\]

and for any \( s > 0 \) there exists a constant \( c_s \) such that

\[
|f_\theta(x)| \leq c_s|\theta|^{-1/2}(M_T^\alpha \mathbf{1}_\theta)(x), \quad x \in E,
\]

where \( M_T^\alpha \) is the maximal operator defined in (2.6). Furthermore,

\[
c_p^{-1}|\theta|^{1/p-1/2} \leq \|f_\theta\|_{L_\infty(\theta)} \leq \|f_\theta\|_p \leq c_p|\theta|^{1/p-1/2}, \quad 0 < p \leq \infty.
\]

3. **B-spaces**

In this section we define the general B-spaces \( B_{pq}^\alpha \) and show that they can be characterized by the corresponding Franklin bases.

3.1. **Definition of B-spaces and Basic Properties**

We begin by introducing the B-space \( B_{pq}^\alpha := B_{pq}^\alpha(T) \) induced by an arbitrary LR-triangulation \( T \) of a compact polygonal domain \( E \) in \( \mathbb{R}^2 \). Since our primary goal here is to relate them to the corresponding Franklin bases, we consider only B-spaces which are imbedded in \( L_1 \). We say that the indices \( \alpha, p, \) and \( q \) are **admissible** if one of the following holds:

(a) \( 0 < p, q \leq \infty \) and \( \alpha > (1/p - 1)_+ \) or

(b) \( 0 < p < 1, 0 < q \leq 1, \) and \( \alpha = 1/p - 1. \)
As will be shown these conditions guarantee the desired embedding.

For a given LR-triangulation of $E$, we define $B_{\alpha}^{pq}(T)$ as the set of all functions $f \in L^p(E)$ such that

$$
|f|_{B_{\alpha}^{pq}(T)} := \left( \sum_{m=0}^{\infty} \left( \sum_{\triangle \in T_m} (|\triangle|^{-\alpha} S_\Delta(f)_p) \right)^{q/p} \right)^{1/q} < \infty,
$$

(3.1)

where $S_\Delta(f)_p$ is the error of $L^p$-approximation to $f$ from $S_m$ on Star$_m(\triangle)$ (see (2.10)). We set

$$
\|f\|_{B_{\alpha}^{pq}(T)} := |E|^{-\alpha} \|f\|_p + |f|_{B_{\alpha}^{pq}(T)}.
$$

(3.2)

Evidently, $\|\cdot\|_{B_{\alpha}^{pq}(T)}$ is a norm if $p, q \geq 1$ and quasi-norm otherwise.

The B-space $B_{\alpha}^{pq}$ has an atomic decomposition. We define

$$
\|f\|_{B_{\alpha}^{pq}(T)}^A := \inf_{f = \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta} \left( \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m} (|\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p)^p \right)^{q/p} \right)^{1/q},
$$

(3.3)

where the infimum is taken over all representations $f = \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta$ in $L^p(E)$.

A third approach to the B-spaces $B_{\alpha}^{pq}$ is by using the decomposition via quasi-interpolants from (2.17). We define

$$
\|f\|_{B_{\alpha}^{pq}(T)}^Q := \left( \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m} (|\theta|^{-\alpha} \|b_\theta(f)\varphi_\theta\|_p)^p \right)^{q/p} \right)^{1/q}.
$$

(3.4)

The following lemma serves as a justification of our definition of admissible indices.

**Lemma 3.** If $\alpha, p, q$ are admissible indices and $\|f\|_{B_{\alpha}^{pq}(T)}^A < \infty$, then

$$
\|f\|_{L^1} \leq c \|f\|_{B_{\alpha}^{pq}(T)}^A.
$$

(3.5)

**Proof.** We consider only the case when $p < 1$, $q > 1$, and $\alpha > 1/p - 1$, since the other cases are similar. Let $f = \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta$ in $L^p$. Then using (2.8) and (2.9) we infer

$$
\|f\|_1 \leq c \|f\|_{B_{\alpha}^{pq}(T)}^A \leq c \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} \|a_\theta \varphi_\theta\|_1
$$

$$
\leq c \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} |\theta|^{1-1/p} \|a_\theta \varphi_\theta\|_p
$$

$$
= c |E| \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} (|\theta|/|E|)^{\epsilon} |\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p,
$$

where
where $\varepsilon := \alpha - (1/p - 1) > 0$. By (2.1)-(2.2) if $\theta \in \Theta_m$, then $|\theta|/|E| \leq c p^m$. We use this, the fact that $p < 1$, and Hölder’s inequality to obtain

$$
\|f\|_1 \leq c|E|^{\varepsilon} \sum_{m=0}^{\infty} \rho^{\varepsilon m} \left( \sum_{\theta \in \Theta_m} (|\theta|^{-\alpha}\|a_{\theta}\varphi_\theta\|_p)^p \right)^{1/p} \leq c|E|^{\varepsilon} \left( \sum_{m=0}^{\infty} \rho^{\varepsilon m q} \right)^{1/q},
$$

where $1/q + 1/q' = 1$. Since $0 < \rho < 1$, the above yields (3.5).

**Theorem 1.** For a given LR-triangulation $T$ of $E$ and admissible indices $\alpha$, $p$, and $q$, the norms $\|\cdot\|_{B_{pq}^\alpha(T)}$, $\|\cdot\|_{A_{B_{pq}^\alpha(T)}}$, and $\|\cdot\|_{Q_{B_{pq}^\alpha(T)}}$, defined in (3.2)-(3.4), are equivalent with constants of equivalence depending only on $\alpha$, $p$, $q$, and the parameters of $T$.

The proof of this theorem is fairly simple and will be omitted (see the proofs of the corresponding results in [2, 3, 7]; see also the more complicated proof of Theorem 2 below).

**Remark 1.** As was shown above the B-spaces $B_{pq}^\alpha$ are in essence sequence spaces and hence they can be interpolated by utilizing standard techniques. We do not present such result here. For some interpolation results on B-spaces, see [3].

**Remark 2.** In general the B-spaces are different from Besov spaces. However, if $T$ is a regular triangulation of a compact polygonal domain $E$ in $\mathbb{R}^2$, then the B-space $B_{pq}^\alpha(T)$ coincides with the Besov space $B_{q}^{2\alpha}(L_p(E))$ for sufficiently small $\alpha > 0$. For more details, see [7].

### 3.2. Franklin Basis Decomposition of B-spaces

Our main goal in this section is to show that the B-spaces $B_{pq}^\alpha(T)$ can be characterized via representations using Franklin bases. We define

$$
\|f\|_{F_{B_{pq}^\alpha(T)}} := \left( \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m^*} (|\theta|^{-\alpha}\|c_\theta(f)\varphi_\theta\|_p)^p \right)^{q/p} \right)^{1/q},
$$

where $c_\theta(f) := \langle f, \varphi_\theta \rangle$.

**Theorem 2.** Suppose $\alpha$, $p$, and $q$ are admissible indices and let $T$ be an LR-triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$. Then $f \in B_{pq}^\alpha(T)$ if and only if $\|f\|_{F_{B_{pq}^\alpha(T)}} < \infty$ and

$$
\|f\|_{F_{B_{pq}^\alpha(T)}} \approx \|f\|_{B_{pq}^\alpha(T)}.
$$

(3.7)
Proof. (a) We first show that if \( \|f\|_{B^p_{pq}} < \infty \), then
\[
\|f\|_{B^p_{pq}} \leq c \|f\|_{B^p_{pq}}^r.
\] (3.8)
We consider only the case when \( 1 < p < q < \infty \), since the other cases are the same or easier.

Let \( \triangle \in T_j \) \((j \geq 0)\) and denote briefly \( \triangle^* := \text{Star}_j(\triangle) \) (see (2.5)). Evidently, \( S_\triangle(g)_p = 0 \) if \( g \in S_{j-1} \). Therefore, using that \( f = \sum_{\theta \in \Theta_j} c_\theta f_\theta \) in \( L_p \) with \( c_\theta := c_\theta(f) := (f, f_\theta) \), we have
\[
S_\triangle(f)_p \leq \sum_{m=0}^\infty \| \sum_{\theta \in \Theta^*_m} c_\theta f_\theta \|_{L_p(\triangle^*)}. \quad (3.9)
\]

For \( \theta \in \Theta^*_m \), we denote \( g_\theta(x) := |\theta|^{-1/2} q_1^{\rho_m(\theta, \theta^m)/2} \). Then by (2.18), we have \( |f_\theta(x)| \leq c g_\theta(x) q_1^{\rho_m(\theta, \theta^m)/2}, \) \( x \in E \). Applying Hölder’s inequality, we obtain
\[
\left| \sum_{\theta \in \Theta^*_m} c_\theta f_\theta(x) \right|^p \leq c \left( \sum_{\theta \in \Theta^*_m} |c_\theta| g_\theta(x) q_1^{\rho_m(\theta, \theta^m)/2} \right)^p \leq c \left( \sum_{\theta \in \Theta^*_m} |g_\theta(x)| q_1^{\rho_m(\theta, \theta^m)/2} \right)^p \left( \sum_{\theta \in \Theta^*_m} q_1^{\rho_m(\theta, \theta^m)p'/2} \right)^{p'/p}, \quad (3.10)
\]
where as usual \( 1/p + 1/p' = 1 \).

Fix \( \omega \in \Theta_m \) and denote
\[
\Theta^*_m := \{ \theta \in \Theta_m : \rho_m(\omega, \theta) = \nu \}, \quad \nu \geq 0.
\]
Note that by Lemma 2, \( \# \Theta^*_m \leq \nu + 1 \). Therefore, for an arbitrary \( \beta > 0 \),
\[
\sum_{\theta \in \Theta_m} q_1^{\beta \rho_m(\omega, \theta)} \leq \sum_{\nu=0}^{\infty} \sum_{\theta \in \Theta^*_m} q_1^{\beta \nu} \leq \sum_{\nu=0}^{\infty} \# \Theta^*_m q_1^{\beta \nu} \leq c \sum_{\nu=0}^{\infty} \nu + 1 \right|^\beta q_1^{\beta \nu} \leq c < \infty. \quad (3.11)
\]
We use (3.11) in (3.10) with \( \beta = p'/2 \) and integrate to obtain
\[
\left\| \sum_{\theta \in \Theta^*_m} c_\theta f_\theta \right\|_{L_p(\triangle^*)} \leq c \left( \sum_{\theta \in \Theta^*_m} \|c_\theta g_\theta\|_{L_p(\triangle^*)}^p \right)^{1/p}. \quad (3.12)
\]
We need estimate \( \|g_\theta\|_{L_p(\triangle^*)} \). To this end we define, for \( \theta \in \Theta_m \) and \( \triangle \in \Theta_j \) \((m \geq j)\),
\[
\rho_m(\theta, \triangle^*) := \inf \{ \rho_m(\theta, \theta^m_\triangle) : x \in (\triangle^*)^\circ \} \quad (3.13)
\]
and
\[
\rho_j(\triangle, \theta) := \inf \{ \rho_j(\theta^j_\triangle) : x \in \triangle^\circ, \; y \in \theta^\circ \}.
\]
We next show that
\[
\|g\|_{L^p(E)}^p \leq c \|f\|_{L^p(E)}^p \rho_2(\triangle), \quad \text{where } 0 < q_2 < 1. \tag{3.14}
\]

Denote briefly \( r := \rho_m(\theta, \triangle^*) \) and let \( E_r := \{ x \in E : \rho_m(\theta, \omega^m) \geq r \} \). Also, let \( \Theta_m^\nu := \{ \eta \in \Theta_m : \rho_m(\theta, \eta) = \nu \} \). Then \( E_r = \bigcup_{\nu=r}^{\infty} \bigcup_{\eta \in \Theta_m^\nu} \eta \). Evidently, \( \triangle^* \subset E_r \) and hence
\[
\|g\|_{L^p(\triangle^*)}^p \leq \sum_{\nu=r}^{\infty} \sum_{\eta \in \Theta_m^\nu} \|g\|_{L^p(\eta)}^p.
\]

Further, we use the definition of \( g \theta \) to obtain
\[
\|g\|_{L^p(\triangle^*)}^p \leq c |\theta|^{-p/2} \sum_{\nu=r}^{\infty} \sum_{\eta \in \Theta_m^\nu} |\eta| \rho_1^{\nu} \leq c |\theta|^{1-p/2} \sum_{\nu=r}^{\infty} \sum_{\eta \in \Theta_m^\nu} (|\eta|/|\theta|) \rho_1^{\nu p/2}.
\]

By (2.3), \(|\eta|/|\theta| \leq c(\nu + 1)^s\) and by Lemma 2, \( \# \Theta_m^\nu \leq c(\nu + 1)^t \). Consequently,
\[
\|g\|_{L^p(\triangle^*)}^p \leq c |\theta|^{1-p/2} \sum_{\nu=r}^{\infty} \# \Theta_m^\nu (\nu + 1)^s \rho_1^{\nu p/2} \leq c |\theta|^{1-p/2} \sum_{\nu=r}^{\infty} (\nu + 1)^{t+s} \rho_1^{\nu p/2} \tag{3.15}
\]
for some \( 0 < q_2 < 1 \). Now taking into account that \( \|f\|_{L^p(E)} \approx |\theta|^{1/p-1/2} \) by (2.20) and \( \rho_m(\theta, \triangle^*) \geq \rho_j(\triangle, \theta) - 1 \), since \( m \geq j \), we conclude that (3.15) yields (3.14).

Combining (3.9) with (3.12) and (3.14), we obtain
\[
|\triangle|^{-\alpha} S_\triangle(f)_p \leq c \sum_{m=j}^{\infty} \left( \sum_{\delta \in \Theta_m^\nu} (|\theta|/|\triangle|)^{\alpha p} |\theta|^{-\alpha p} \|f\|_{L^p(E)}^{\rho_2(\triangle, \theta)} \right)^{1/p}.
\]

Let \( \omega \in \Theta_j \) be such that \( \theta \subset \omega \). Then by (2.1)-(2.2), \(|\theta|/|\omega| \leq c \rho^{m-j}\) and using (2.3), \(|\omega|/|\triangle| \leq c(\rho_j(\triangle, \omega) + 1)^s \leq c(\rho_j(\triangle, \theta) + 1)^s\). Therefore,
\[
(|\theta|/|\triangle|)^{\alpha p} \rho_2^{\rho_2(\triangle, \theta)} \leq c_1 (\rho_j(\triangle, \theta) + 1)^{\alpha p} \rho_2^{\rho_2(\triangle, \theta)} \rho^{\rho(m-j)} \leq c_2 \rho_2^{\rho_2(\triangle, \theta)} \rho^{\rho(m-j)},
\]
for some \( 0 < q_3 < 1 \). Thus
\[
|\triangle|^{-\alpha} S_\triangle(f)_p \leq c \sum_{m=j}^{\infty} \rho^{\rho(m-j)} \left( \sum_{\delta \in \Theta_m^\nu} A_2^{\rho_2(\triangle, \theta)} \right)^{1/p}.
\]
where \( A_\theta := |\theta|^{-\alpha} \| c_\theta f_\theta \|_p \). Finally, applying Hölder’s inequality, we get

\[
(|\Delta|^{-\alpha} S_\Delta (f)_p)^p \leq c \left( \sum_{m=j}^{\infty} \rho^{\alpha p(m-j)/2} \sum_{\theta \in \Theta_m^*} A_\theta^{p} q_j^{(\Delta, \theta)} \right) \left( \sum_{m=j}^{\infty} \rho^{\alpha p'(m-j)/2} \right)^{p'/p'} \\
\leq c \sum_{m=j}^{\infty} \rho^{\alpha p(m-j)/2} \sum_{\theta \in \Theta_m^*} A_\theta^{p} q_j^{(\Delta, \theta)},
\]

(3.16)

since \( 0 < \rho < 1 \).

We are now prepared to prove (3.8). Using (3.16) in the definition of \( |f|_{B^pq} \), we have

\[
|f|_{B^pq}^q = \sum_{j=0}^{\infty} \left( \sum_{\Delta \in T_j} (|\Delta|^{-\alpha} S_\Delta (f)_p)^p \right)^{q/p} \\
\leq c \sum_{j=0}^{\infty} \left( \sum_{m=j}^{\infty} \rho^{\alpha p(m-j)/2} \sum_{\theta \in \Theta_m^*} A_\theta^{p} q_j^{(\Delta, \theta)} \right)^{q/p} \\
\leq c \sum_{j=0}^{\infty} \left( \sum_{m=j}^{\infty} \frac{\rho^{\alpha p(m-j)/2}}{\rho^{\alpha q(m-j)/2}} \sum_{\theta \in \Theta_m^*} A_\theta^{p} \sum_{\Delta \in T_j} q_j^{(\Delta, \theta)} \right)^{q/p},
\]

where we once switched the order of summation. Similarly as in (3.11), we have

\[
\sum_{\Delta \in T_j} q_j^{(\Delta, \theta)} \leq c < \infty.
\]

On the other hand, using Hölder’s inequality,

\[
\left( \sum_{m=j}^{\infty} \frac{\rho^{\alpha p(m-j)/2}}{\rho^{\alpha q(m-j)/2}} \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p} \leq \left( \sum_{m=j}^{\infty} \rho^{\alpha p(m-j)/4} \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p} \left( \sum_{m=j}^{\infty} \rho^{\alpha q(m-j)/4} \right)^{1/q} \\
\leq c \sum_{m=j}^{\infty} \rho^{\alpha q(m-j)/4} \left( \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p},
\]

where \( \gamma > 1 \) is determined from \( p/q + 1/\gamma = 1 \). Consequently,

\[
|f|_{B^pq}^q \leq c \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \rho^{\alpha q(m-j)/4} \left( \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p} \\
\leq c \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p} \sum_{j=0}^{m} \rho^{\alpha q(m-j)/4} \\
\leq c \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m^*} A_\theta^{p} \right)^{q/p},
\]
where we once switched the order of summation and used that $0 < \rho < 1$. Therefore, $\|f\|_{B^p_{\rho q}} \leq c \|f\|_{B^p_{\rho q}}^{\frac{p}{q}}$.

It remains to show that $\|f\|_p \leq c \|E\|_\alpha \|f\|_{B^p_{\rho q}}$. Exactly as in (3.12), we obtain

$$\|f\|_p \leq \sum_{j=0}^{\infty} \left| \sum_{\theta \in \Theta^*_m} c_\theta f\theta \right|_p \leq c \sum_{j=0}^{\infty} \left( \sum_{\theta \in \Theta^*_m} \|c_\theta f\theta\|_p^p \right)^{1/p},$$

and continuing as in the proof of Lemma 3 we arrive at $\|f\|_p \leq c \|E\|_\alpha \|f\|_{B^p_{\rho q}}$. Thus (3.8) is established.

(b) We next show that

$$\|f\|_{B^p_{\rho q}} \leq c \|f\|_{B^p_{\rho q}},$$

provided $\|f\|_{B^p_{\rho q}} < \infty$. We again consider only the most complicated case when $1 < p < q < \infty$. We first estimate $|c_\theta(f)|$, where $c_\theta(f) := \langle f, \theta \rangle$. If $f \in L_p(E)$, then by (2.17)

$$f = Q_0 f + \sum_{j=1}^{\infty} (Q_j(f) - Q_j-1(f)) =: \sum_{j=0}^{\infty} q_j. \tag{3.18}$$

Fix $\theta \in \Theta^*_m (m \geq 1)$. Then since $f_\theta \perp S_{m-1}$ and $q_j \in S_j$,

$$|c_\theta(f)| \leq \int_E |f_\theta(x) \sum_{j=m}^{\infty} q_j(x)| dx \leq \sum_{j=m}^{\infty} \int_E |f_\theta(x)q_j(x)| dx.$$

Denote $g_\theta(x) := q_1^{\rho_m(\theta, \theta')}\frac{1}{2}$ and $h_\theta(x) := |\theta|^{-1/2} q_1^{\rho_m(\theta, \theta')/2}$. Exactly as in the estimate of $\|g_\theta\|_{L_{p}(\Delta^*)}^p$ above (see (3.15) and also the estimate of $\|f_\theta\|_p$ in [9]) we have $\|h_\theta\|_\tau \approx |\theta|^{1/\tau - 1/2}$ for $0 < \tau \leq \infty$. By (2.18), $|f_\theta(x)| \leq c g_\theta(x) h_\theta(x), x \in E$. Using the above and Hölder’s inequality, we obtain

$$|c_\theta(f)| \leq c \sum_{j=m}^{\infty} \|q_\Delta g_\theta\|_p \|h_\theta\|_{p'} \leq c |\theta|^{1/p' - 1/2} \sum_{j=m}^{\infty} \left( \sum_{\Delta \in S_j} \|q_\Delta g_\theta\|_{L_{p}(\Delta)}^p \right)^{1/p} \leq c |\theta|^{1/2 - 1/p} \sum_{j=m}^{\infty} \left( \sum_{\Delta \in S_j} \|q_\Delta\|_{L_{p}(\Delta)}^p q_1^{\rho_m(\theta, \Delta')/2} \right)^{1/p}, \tag{3.19}$$

where $\rho_m(\theta, \Delta)$ is defined as in (3.13).

For $\Delta \in S_j$, we denote by $\Delta'$ the only triangle in $T_{j-1}$ such that $\Delta \subset \Delta'$. Then by (2.14),

$$\|q_\Delta\|_{L_{p}(\Delta')} \leq \|f - Q_1(f)\|_{L_{p}(\Delta')} + \|f - Q_{j-1}(f)\|_{L_{p}(\Delta')} \leq c (S_{\Delta}(f)_p + S_{\Delta'}(f)_p)$$

and evidently $\rho_m(\theta, \Delta') \leq \rho_m(\theta, \Delta)$. Using this in (3.19), we obtain

$$|c_\theta(f)| \leq c |\theta|^{1/2 - 1/p} \sum_{j=m-1}^{\infty} \left( \sum_{\Delta \in S_j} S_{\Delta}(f)_p q_1^{\rho_m(\theta, \Delta')/2} \right)^{1/p}.$$
and since \( \|f\|_p \approx |\theta|^{1/p - 1/2} \), we have

\[
|\theta|^{-\alpha p} \|c_\theta(f) f_\theta\|_p^p \leq c|\theta|^{-\alpha p} \left( \sum_{j=m-1}^{\infty} \left( \sum_{\Delta \in T_j} S_\Delta(f)^p \cdot q_{\rho_0(\theta, \Delta)^2} \right)^{1/p} \right)^p.
\] (3.20)

If \( \theta \in \Theta_0 \), then

\[
\|c_\theta(f) f_\theta\|_p = |\langle f, f_\theta \rangle| \leq \|f\|_p \|f_\theta\|_{p'} \leq c\|f\|_p
\]

and by (2.1)-(2.2), \( |\theta| \geq c|E| \) for \( \theta \in \Theta_0 \), where \( c > 0 \) depends on the parameters of \( T \) (including \( T_0 \)). Therefore,

\[
|\theta|^{-\alpha} \|c_\theta(f) f_\theta\|_p \leq c|E|^{-\alpha} \|f\|_p. \tag{3.21}
\]

From (3.20)-(3.21), we infer

\[
(\|f\|_{B^p_{p'}(T)})^q = \sum_{m=0}^{\infty} \left( \sum_{\theta \in \Theta_m} |\theta|^{-\alpha} \|c_\theta(f) f_\theta\|_p^p \right)^{q/p} \leq c(\|E\|^{-\alpha} \|f\|_p)^q
\]

\[
+ c \sum_{m=1}^{\infty} \left( \sum_{\theta \in \Theta_m} |\theta|^{-\alpha} \left( \sum_{j=m-1}^{\infty} \left( \sum_{\Delta \in T_j} S_\Delta(f)^p \cdot q_{\rho_0(\theta, \Delta)^2} \right)^{1/p} \right)^p \right)^{q/p}
\]

\[
\leq c A_E^q + c \sum_{m=1}^{\infty} \left( \sum_{\theta \in \Theta_m} \left( \sum_{j=m-1}^{\infty} \left( \sum_{\Delta \in T_j} \left( \frac{|\Delta|}{|\theta|} \right)^{\alpha p} A_{\Delta} q_{\rho_0(\theta, \Delta)^2} \right)^{1/p} \right)^{p/q} \right)^{q/p},
\]

where \( A_E := |E|^{-\alpha} \|f\|_p \) and \( A_{\Delta} := |\Delta|^{-\alpha} S_\Delta(f)_p \). As in (a) it is readily seen that, for \( \theta \in \Theta_m \) and \( \Delta \in T_j \) with \( j \geq m - 1 \), we have \( |\Delta|/|\theta| \leq c p^{-m}(\rho_0(\theta, \Delta) + 1)^{s} \). Therefore,

\[
(\|f\|_{B^p_{p'}(T)})^q \leq c A_E^q + c \sum_{m=1}^{\infty} \left( \sum_{\theta \in \Theta_m} \left( \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)} \sigma_j^{1/p} \right)^{p/q} \right)^{q/p}, \tag{3.22}
\]

where \( \sigma_j := \sum_{\Delta \in T_j} A_{\Delta}^p q_{\rho_0(\theta, \Delta)}^{p} \) for some \( q_1 < q_* < 1 \).

Applying Hölder’s inequality, we obtain

\[
\left[ \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)} \sigma_j^{1/p} \right]^p \leq \left( \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)p/2} \sigma_j \right) \left( \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)p'/2} \right)^{p/p'}
\]

\[
\leq c \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)p/2} \sigma_j.
\]

Substituting this in (3.22), we find

\[
(\|f\|_{B^p_{p'}(T)})^q \leq c A_E^q + c \sum_{m=1}^{\infty} \left( \sum_{\theta \in \Theta_m} \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)p/2} \sum_{\Delta \in T_j} A_{\Delta}^p q_{\rho_0(\theta, \Delta)} \right)^{q/p}
\]

\[
\leq c A_E^q + c \sum_{m=1}^{\infty} \left( \sum_{j=m-1}^{\infty} \rho^{\alpha(j-m)p/2} \sum_{\Delta \in T_j} \sum_{\theta \in \Theta_m} q_{\rho_0(\theta, \Delta)} \right)^{q/p},
\]
where we once switched the order of summation. Exactly as in (3.11), we have
\[ \sum_{\theta \in \Theta^*} q_{m(\theta, \Delta)}^\ell \leq c < \infty. \]
We insert this above and apply once again Hölder’s inequality to obtain
\[
\left( \sum_{\theta \in \Theta^*} q_{m(\theta, \Delta)}^\ell \right)^{q/p} \leq c A_{qE}^p + c \sum_{j=0}^{\infty} \left( \sum_{\Delta \in T_j} A_{\Delta}^p \right)^{q/p} \leq c A_{qE}^p + c \sum_{j=0}^{\infty} \left( \sum_{\Delta \in T_j} A_{\Delta}^p \right)^{q/p} \leq c \|f\|_{B^{\alpha q}_p}(T),
\]
where \( p/q + 1/\gamma = 1 \). Since \( 0 < \rho < 1 \), the last sum above is \( \leq c < \infty \). We switch one last time the order of summation and obtain
\[
\left( \sum_{\theta \in \Theta^*} q_{m(\theta, \Delta)}^\ell \right)^{q/p} \leq c A_{qE}^p + c \sum_{j=0}^{\infty} \left( \sum_{\Delta \in T_j} A_{\Delta}^p \right)^{q/p} \leq c \|f\|_{B^{\alpha q}_p}(T),
\]
which completes the proof of (3.17).

4. Nonlinear Approximation from Franklin Bases

In this section, we apply the characterization of B-spaces via Franklin bases from the previous section to nonlinear \( n \)-approximation.

Suppose that \( F_T \) is a Franklin basis generated by an LR-triangulation \( T \) of a compact polygonal domain \( E \) in \( \mathbb{R}^2 \). We let \( F_n \) denote the nonlinear set of all functions \( g \) of the form

\[
g = \sum_{\theta \in \Lambda} a_{\theta} f_{\theta},
\]
where \( \Lambda \subset \Theta^* \), \( \#\Lambda \leq n \), and \( \Lambda \) is allowed to vary with \( g \). We denote by \( \sigma^F_n(f)_p \) the error of best \( L_p \)-approximation to \( f \in L_p(E) \) from \( F_n \):

\[
\sigma^F_n(f)_p := \inf_{g \in F_n} \|f - g\|_p.
\]

We shall use the machinery of Jackson-Bernstein estimates to characterize the approximation spaces generated by \( (\sigma^F_n(f)_p) \), \( 1 < p < \infty \). To this end we need the B-spaces \( B^{\alpha}_p(T) := B^{\alpha q}_p(T) \), where \( \alpha > 0 \) and \( \tau \) is determined by \( 1/\tau = \alpha + 1/p \).

**Theorem 3 (Jackson estimate).** If \( f \in B^{\alpha}_p(T) \), then

\[
\sigma^F_n(f)_p \leq c n^{-\alpha} \|f\|_{B^{\alpha}_p(T)}
\]

where \( c \) depends only on \( \alpha \), \( p \), and the parameters of \( T \).
Here it is crucial that the space $B^\alpha_\tau(T)$ is embedded in $L_p$, namely, if $f \in B^\alpha_\tau(T)$, then $f \in L_p$ and $\|f\|_p \leq c\|f\|_{B^\alpha_\tau(T)}$, see [7, 8]. In fact, $B^\alpha_\tau(T)$ lies on the Sobolev embedding line.

For the proof of Theorem 3 one uses the scheme of the proof of Theorem 3.4 from [7] combined with the vector valued maximal inequality (2.7) from Proposition 1. The following (embedding) estimate plays an important role: If $f = \sum_{\theta \in \Theta^*} c_\theta f_\theta$ and $0 < s < 1$, then

$$\|f\|_p \leq c \left\| \sum_{\theta \in \Theta^*} (M_T^c c_\theta \tilde{1}_\theta)(\cdot) \right\|_p \leq c \left\| \sum_{\theta \in \Theta^*} |c_\theta| \tilde{1}_\theta(\cdot) \right\|_p,$$

where we used (2.19) and the maximal inequality (2.7). We now invoke Theorem 3.3 from [7] and obtain

$$\|f\|_p \leq c \left( \sum_{\theta \in \Theta^*} \|c_\theta \tilde{1}_\theta\|_p^s \right)^{1/\tau} \leq c \left( \sum_{\theta \in \Theta^*} \|c_\theta f_\theta\|_p^s \right)^{1/\tau} \leq c\|f\|_{B^\alpha_\tau(T)},$$

which is the above mentioned embedding. We skip further details.

**Theorem 4 (Bernstein estimate).** If $g \in F_n$, then

$$\|g\|_{B^\alpha_\tau(T)} \leq c n^\alpha \|g\|_p$$

(4.2)

where $c$ depends only on $\alpha$, $p$, and the parameters of the $T$.

The proof of this theorem can be carried out exactly as in the wavelet case by utilizing the fact that $F_T$ is an unconditional bases for $L_p$ ($1 < p < \infty$) and the localization properties of the Franklin functions given in Proposition 2 (see e.g. [4], Theorem 6). The proof relies on the important fact [9] that if $f \in L_p(E)$, $1 < p < \infty$, and $f = \sum_{\theta \in \Theta} c_\theta f_\theta$, then

$$\|f\|_p \approx \left\| \sum_{\theta \in \Theta^*} |c_\theta|^2 \tilde{1}_\theta(\cdot) \right\|_p^{1/2}.$$

We omit the details.

One can now follow the standard lines to obtain direct and inverse estimates for $\sigma^F_n(f)_p$. To this end, denote by $K(f, t)_p := K(f, t; L_p, B^\alpha_\tau(T))$ the $K$-functional defined by $K(f, t)_p := \inf_{g \in B^\alpha_\tau(T)} \|f - g\|_p + t\|g\|_{B^\alpha_\tau(T)}$, $t > 0$.

By standard arguments (see e.g. [10]), the Jackson and Bernstein estimates (4.1)-(4.2) imply the following direct and inverse estimates: For $f \in L_p(E)$ one has

$$\sigma^F_n(f)_p \leq c K(f, n^{-\alpha})_p$$

(4.3)

and

$$K(f, n^{-\alpha})_p \leq c n^{-\alpha} \left( \sum_{\nu=1}^n \frac{1}{\nu^\alpha (\nu^\alpha \sigma^F_\nu(f)_p)^\tau} \right)^{1/\tau'} + \|f\|_p,$$

(4.4)
where $\tau^* := \min\{\tau, 1\}$.

We define the approximation space $A^\gamma_q(\mathcal{F}_T, L_p)$ to be the set of all functions $f \in L_p(E)$ such that

$$
\|f\|_{A^\gamma_q} := \|f\|_p + \left( \sum_{n=1}^{\infty} (n^\gamma \sigma_n^\mathcal{F}(f)_p)^{q} \frac{1}{n} \right)^{1/q} < \infty
$$

(4.5)

with the usual modification when $q = \infty$.

The following characterization of the approximation spaces $A^\gamma_q$ is immediate from estimates (4.3)-(4.4).

**Theorem 5.** If $0 < \gamma < \alpha$ and $0 < q \leq \infty$, then

$$
A^\gamma_q(\mathcal{F}_T, L_p) = (L_p, B^\alpha(\mathcal{T}))_{\frac{\gamma}{\alpha}, q}
$$

with equivalent (quasi-)norms, where $(L_p, B^\alpha(\mathcal{T}))_{\frac{\gamma}{\alpha}, q}$ is the real interpolation space between $L_p$ and $B^\alpha(\mathcal{T})$ (see e.g. [1]).

In one specific case the approximation space $A^\alpha_q(\mathcal{F}_T, L_p)$ can be identified as a B-space:

**Theorem 6.** Assuming that $1 < p < \infty$, $\alpha > 0$, and $1/\tau := \alpha + 1/p$, we have

$$
A^\alpha(\mathcal{F}_T, L_p) = B^\alpha(\mathcal{T})
$$

(4.6)

with equivalent norms.

The proof is a mere repetition of the proof of Theorem 3.4 in [3] and will be omitted.

Finally, we want to compare the nonlinear $n$-term approximation from $\mathcal{F}_T$ with the $n$-term approximation from $\Phi_T$ (Courant elements). Let $\Sigma_n$ denote the set of all functions $g$ of the form $g = \sum_{\varphi \in \mathcal{M}} a_{\varphi}\varphi$, where $\mathcal{M} \subset \Theta$, $\# \mathcal{M} \leq n$.

We denote by $\sigma_n^\Phi(f)_p$ the error of best $L_p$-approximation to $f \in L_p(E)$ from $\Sigma_n$:

$$
\sigma_n^\Phi(f)_p := \inf_{g \in \Sigma_n} \|f - g\|_p.
$$

Let $A^\gamma_q(\Phi_T, L_p)$ be the approximation space generated by $(\sigma_n^\Phi(f)_p)$, defined similarly as in (4.5). As is shown in [7], the approximation space $A^\gamma_q(\Phi_T, L_p)$ has precisely the same characterization as the one from Theorem 5. Consequently, $A^\gamma_q(\mathcal{F}_T, L_p) = A^\gamma_q(\Phi_T, L_p)$ with equivalent (quasi-)norms, if $1 < p < \infty$, for all $\gamma > 0$ and $0 < q \leq \infty$.

We close by noting that results similar to the results from Theorems 3 - 6 hold true for nonlinear $n$-term approximation from $\mathcal{F}_T$ in $H_1(E, \mathcal{T})$ [9], the Hardy space generated by an LR-triangulation $\mathcal{T}$ of $E$. For the proofs one utilizes the fact that the Franklin system $\mathcal{F}_T$ is an unconditional basis for $H_1(E, \mathcal{T})$ (see [9]) and the techniques used above.
References


GEORGE KYRIAZIS
Department of Mathematics and Statistics
University of Cyprus
1678 Nicosia
CYPRUS
E-mail: kyriazis@ucy.ac.cy

KYUNGWON PARK
Department of Mathematics
University of South Carolina
Columbia, SC 29208
USA
E-mail: kpark001@math.sc.edu

PENCHO PETRUSHEV
Department of Mathematics
University of South Carolina
Columbia, SC 29208
USA
E-mail: pencho@math.sc.edu