A NEW PROOF OF THE ATOMIC DECOMPOSITION OF HARDY SPACES

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Abstract. A new proof is given of the atomic decomposition of Hardy spaces H^p, $0 < p \le 1$, in the classical setting on $\mathbb{R}^n$. The new method can be used to establish atomic decomposition of maximal Hardy spaces in general and nonclassical settings.

1. Introduction

The study of the real-variable Hardy spaces H^p, $0 < p \le 1$, on $\mathbb{R}^n$ was pioneered by Stein and Weiss [6] and a major step forward in developing this theory was made by Fefferman and Stein in [3], see also [5]. Since then there has been a great deal of work done on Hardy spaces. The atomic decomposition of H^p was first established by Coifman [1] in dimension $n = 1$ and by Latter [4] in dimensions $n > 1$.

The purpose of this article is to give a new proof of the atomic decomposition of the H^p spaces in the classical setting on $\mathbb{R}^n$. Our method does not use the Calderón-Zygmund decomposition of functions and an approximation of the identity as the classical argument does, see [5]. The main advantage of the new proof over the classical one is that it is amenable to utilization in more general and nonclassical settings. For instance, it is used in [2] for establishing the equivalence of maximal and atomic Hardy spaces in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property.

Notation. For a set $E \subset \mathbb{R}^n$ we will denote $E + B(0,\delta) := \cup_{x\in E} B(x,\delta)$, where $B(x,\delta)$ stands for the open ball centered at x of radius δ. We will also use the notation $cB(x,\delta) := B(x, c\delta)$. Positive constants will be denoted by $c, c_1, \dots$ and they may vary at every occurrence; $a \sim b$ will stand for $c_1 \le a/b \le c_2$.

1.1. Maximal operators and H^p spaces.

We begin by recalling some basic facts about Hardy spaces on $\mathbb{R}^n$. For a complete account of Hardy spaces we refer the reader to [5].

Given $\varphi \in \mathcal{S}$ with $\mathcal{S}$ being the Schwartz class on $\mathbb{R}^n$ and $f \in \mathcal{S}'$ one defines

$$M_\varphi f(x) := \sup_{t>0} |\varphi_t * f(x)| \text{ with } \varphi_t(x) := t^{-n}\varphi(t^{-1}x), \quad \text{and} \tag{1.1}$$

$$M^*_{\varphi,a} f(x) := \sup_{t>0} \sup_{y\in\mathbb{R}^n, |x-y|\le at} |\varphi_t * f(y)|, \quad a \ge 1. \tag{1.2}$$

We now recall the grand maximal operator. Write

$$\mathcal{P}_N(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N \max_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|$$

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and denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S} : \mathcal{P}_N(\varphi) \le 1\}.$$

The grand maximal operator is defined by

$$\mathcal{M}_N f(x) := \sup_{\varphi \in \mathcal{F}_N} M^*_{\varphi,1} f(x), \quad f \in \mathcal{S}'. \tag{1.3}$$

It is easy to see that for any $\varphi \in \mathcal{S}$ and $a \ge 1$ one has

$$M^*_{\varphi,a} f(x) \le a^N \mathcal{P}_N(\varphi) \mathcal{M}_N f(x), \quad f \in \mathcal{S}'. \tag{1.4}$$

Definition 1.1. *The space H^p, $0 < p \le 1$, is defined as the set of all bounded distributions $f \in \mathcal{S}'$ such that the Poisson maximal function $\sup_{t>0} |P_t * f(x)|$ belongs to L^p; the quasi-norm on H^p is defined by*

$$\|f\|_{H^p} := \big\| \sup_{t>0} |P_t * f(\cdot)| \big\|_{L^p}. \tag{1.5}$$

As is well known the following assertion holds, see [3, 5]:

Proposition 1.2. *Let $0 < p \le 1$, $a \ge 1$, and assume $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \varphi \ne 0$. Then for any $N \ge \lfloor \frac{n}{p} \rfloor + 1$*

$$\|f\|_{H^p} \sim \|M^*_{\varphi,a} f\|_{L^p} \sim \|\mathcal{M}_N f\|_{L^p}, \quad \forall f \in H^p. \tag{1.6}$$

1.2. **Atomic H^p spaces.** A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_{L^\infty} \le |B|^{-1/p}$, and

(iii) $\int_{\mathbb{R}^n} x^\alpha a(x) dx = 0$ for all α with $|\alpha| \le n(p^{-1} - 1)$.

The atomic Hardy space H^p_A, $0 < p \le 1$, is defined as the set of all distributions $f \in \mathcal{S}'$ that can be represented in the form

$$f = \sum_{j=1}^\infty \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^\infty |\lambda_j|^p < \infty, \tag{1.7}$$

$\{a_j\}$ are atoms, and the convergence is in $\mathcal{S}'$. Set

$$\|f\|_{H^p_A} := \inf_{f = \sum_j \lambda_j a_j} \Big(\sum_{j=1}^\infty |\lambda_j|^p \Big)^{1/p}, \quad f \in H^p_A. \tag{1.8}$$

2. Atomic decomposition of H^p spaces

We now come to the main point in this article, that is, to give a new proof of the following classical result [1, 4], see also [5]:

Theorem 2.1. *For any $0 < p \le 1$ the continuous embedding $H^p \subset H^p_A$ is valid, that is, if $f \in H^p$, then $f \in H^p_A$ and*

$$\|f\|_{H^p_A} \le c \|f\|_{H^p}, \tag{2.1}$$

where $c > 0$ is a constant depending only on p, n. This along with the easy to prove embedding $H^p_A \subset H^p$ leads to $H^p = H^p_A$ and $\|f\|_{H^p} \sim \|f\|_{H^p_A}$ for $f \in H^p$.

Proof. We first derive a simple *decomposition* identity which will play a central rôle in this proof. For this construction we need the following

Lemma 2.2. *For any $m \geq 1$ there exists a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(0,1)$, $\hat{\varphi}(0) = 1$, and $\partial^\alpha \hat{\varphi}(0) = 0$ for $0 < |\alpha| \leq m$. Here $\hat{\varphi}(x) := \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi} dx$.*

Proof. We will construct a function φ with the claimed properties in dimension $n = 1$. Then a normalized dilation of $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$ will have the claimed properties on $\mathbb{R}^n$.

For the univariate construction, pick a smooth "bump" ϕ with the following properties: $\phi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi \subset [-1/4, 1/4]$, $\phi(x) > 0$ for $x \in (-1/4, 1/4)$, and ϕ is even. Let $\Theta(x) := \phi(x + 1/2) - \phi(x - 1/2)$ for $x \in \mathbb{R}$. Clearly Θ is odd.

We may assume that $m \geq 1$ is even, otherwise we work with $m + 1$ instead. Denote $\Delta_h^m := (T_h - T_{-h})^m$, where $T_h f(x) := f(x + h)$.

We define $\varphi(x) := \frac{1}{x}\Delta_h^m \Theta(x)$, where $h = \frac{1}{8m}$. Clearly, $\varphi \in C^\infty(\mathbb{R})$, φ is even, and $\operatorname{supp}\varphi \subset [-\frac{7}{8}, -\frac{1}{8}] \cup [\frac{1}{8}, \frac{7}{8}]$. It is readily seen that for $\nu = 1, 2, \dots, m$

$$\hat{\varphi}^{(\nu)}(\xi) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1} \Delta_h^m \Theta(x) e^{-i\xi x} dx$$

and hence

$$\hat{\varphi}^{(\nu)}(0) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1} \Delta_h^m \Theta(x) dx = (-i)^{\nu+m} \int_{\mathbb{R}} \Theta(x) \Delta_h^m x^{\nu-1} dx = 0.$$

On the other hand,

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) dx = 2\int_0^\infty x^{-1} \Delta_h^m \Theta(x) dx = 2(-1)^m \int_{1/4}^{3/4} \Theta(x) \Delta_h^m x^{-1} dx.$$

However, for any sufficiently smooth function f we have $\Delta_h^m f(x) = (2h)^m f^{(m)}(\xi)$, where $\xi \in (x - mh, x + mh)$. Hence,

$$\Delta_h^m x^{-1} = (2h)^m m! (-1)^m \xi^{-m-1} \quad \text{with} \quad \xi \in (x - mh, x + mh) \subset [1/8, 7/8].$$

Consequently, $\hat{\varphi}(0) \neq 0$ and then $\hat{\varphi}(0)^{-1}\varphi(x)$ has the claimed properties. $\square$

With the aid of the above lemma, we pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ with the following properties: $\operatorname{supp}\varphi \subset B(0,1)$, $\hat{\varphi}(0) = 1$, and $\partial^\alpha\hat{\varphi}(0) = 0$ for $0 < |\alpha| \leq K$, where K is sufficiently large. More precisely, we choose $K \geq n/p$.

Set $\psi(x) := 2^n\varphi(2x) - \varphi(x)$. Then $\hat{\psi}(\xi) = \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi)$. Therefore, $\partial^\alpha\hat{\psi}(0) = 0$ for $|\alpha| \leq K$ which implies $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for $|\alpha| \leq K$. We also introduce the function $\tilde{\psi}(x) := 2^n\varphi(2x) + \varphi(x)$. We will use the notation $h_k(x) := 2^{kn}h(2^k x)$.

Clearly, for any $f \in \mathcal{S}'$ we have $f = \lim_{j\to\infty} \varphi_j * \varphi_j * f$ (convergence in $\mathcal{S}'$), which leads to the following representation: For any $j \in \mathbb{Z}$

$$\begin{aligned} f &= \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \big[\varphi_{k+1} * \varphi_{k+1} * f - \varphi_k * \varphi_k * f\big] \\ &= \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \big[\varphi_{k+1} - \varphi_k\big] * \big[\varphi_{k+1} + \varphi_k\big] * f. \end{aligned}$$

Thus we arrive at

$$f = \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \psi_k * \tilde{\psi}_k * f, \quad \forall f \in \mathcal{S}' \ \ \forall j \in \mathbb{Z} \quad \text{(convergence in } \mathcal{S}'). \tag{2.2}$$

Observe that $\operatorname{supp}\psi_k \subset B(0, 2^{-k})$ and $\operatorname{supp}\tilde{\psi}_k \subset B(0, 2^{-k})$.

In what follows we will utilize the grand maximal operator $\mathcal{M}_N$, defined in (1.3) with $N := \lfloor \frac{n}{p} \rfloor + 1$. The following claim follows readily from (1.4): If $\phi \in \mathcal{S}$, then for any $f \in \mathcal{S}'$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$

$$|\phi_k * f(y)| \le c\mathcal{M}_N f(x) \quad \text{for all} \quad y \in \mathbb{R}^n \;\; with \;\; |y - x| \le 2^{-k+1}, \tag{2.3}$$

where the constant $c > 0$ depends only on $\mathcal{P}_N(\phi)$ and N.

Let $f \in H^p$, $0 < p \le 1$, $f \ne 0$. We define

$$\Omega_r := \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^r\}, \quad r \in \mathbb{Z}. \tag{2.4}$$

Clearly, Ω_r is open, $\Omega_{r+1} \subset \Omega_r$, and $\mathbb{R}^n = \cup_{r\in\mathbb{Z}}\Omega_r$. It is easy to see that

$$\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p d\mu(x) \le c\|f\|_{H^p}^p. \tag{2.5}$$

From (2.5) we get $|\Omega_r| \le c2^{-pr}\|f\|_{H^p}^p$ for $r \in \mathbb{Z}$. Therefore, for any $r \in \mathbb{Z}$ there exists $J > 0$ such that $\|\varphi_j * \varphi_j * f\|_\infty \le c2^r$ for $j < -J$. Consequently, $\|\varphi_j * \varphi_j * f\|_\infty \to 0$ as $j \to -\infty$, which implies

$$f = \lim_{K\to\infty} \sum_{k=-\infty}^{K} \psi_k * \tilde{\psi}_k * f \quad (\text{convergence in } \mathcal{S}'). \tag{2.6}$$

Assuming that $\Omega_r \ne \emptyset$ we write

$$E_{rk} := \{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2^{-k+1}\} \setminus \{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\}.$$

By (2.5) it follows that $|\Omega_r| < \infty$ and hence there exists $s_r \in \mathbb{Z}$ such that $E_{rs_r} \ne \emptyset$ and $E_{rk} = \emptyset$ for $k < s_r$. Evidently $s_r \le s_{r+1}$. We define

$$F_r(x) := \sum_{k\ge s_r} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy, \quad x \in \mathbb{R}^n,\ r \in \mathbb{Z}, \tag{2.7}$$

and more generally

$$F_{r,\kappa_0,\kappa_1}(x) := \sum_{k=\kappa_0}^{\kappa_1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy, \quad s_r \le \kappa_0 \le \kappa_1 \le \infty. \tag{2.8}$$

It will be shown in Lemma 2.3 below that the functions F_r and F_{r,κ_0,κ_1} are well defined and $F_r, F_{r,\kappa_0,\kappa_1} \in L^\infty$.

Note that $\operatorname{supp}\psi_k \subset B(0, 2^{-k})$ and hence

$$\operatorname{supp}\Big(\int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy\Big) \subset E_{rk} + B(0, 2^{-k}). \tag{2.9}$$

On the other hand, clearly $2B(y, 2^{-k}) \cap (\Omega_r \setminus \Omega_{r+1}) \ne \emptyset$ for each $y \in E_{rk}$, and $\mathcal{P}_N(\tilde{\psi}) \le c$. Therefore, see (2.3), $|\tilde{\psi}_k * f(y)| \le c2^r$ for $y \in E_{rk}$, which implies

$$\Big\|\int_E \psi_k(\cdot - y)\tilde{\psi}_k * f(y)dy\Big\|_\infty \le c2^r, \quad \forall E \subset E_{rk}. \tag{2.10}$$

Similarly,

$$\Big\|\int_E \varphi_k(\cdot - y)\tilde{\varphi}_k * f(y)dy\Big\|_\infty \le c2^r, \quad \forall E \subset E_{rk}. \tag{2.11}$$

We collect all we need about the functions F_r and F_{r,κ_0,κ_1} in the following

Lemma 2.3. *(a) We have*

$$E_{rk} \cap E_{r'k} = \emptyset \quad \text{if } r \neq r' \quad \text{and} \quad \mathbb{R}^n = \cup_{r\in\mathbb{Z}} E_{rk}, \quad \forall k \in \mathbb{Z}. \tag{2.12}$$

(b) There exists a constant $c > 0$ such that for any $r \in \mathbb{Z}$ and $s_r \le \kappa_0 \le \kappa_1 \le \infty$

$$\|F_r\|_\infty \le c2^r, \quad \|F_{r,\kappa_0,\kappa_1}\|_\infty \le c2^r. \tag{2.13}$$

(c) The series in (2.7) and (2.8) (if $\kappa_1 = \infty$) converge point-wise and in distributional sense.

(d) Moreover,

$$F_r(x) = 0, \quad \forall x \in \mathbb{R}^n \setminus \Omega_r, \ \forall r \in \mathbb{Z}. \tag{2.14}$$

Proof. Identities (2.12) are obvious and (2.14) follows readily from (2.9).

We next prove the left-hand side inequality in (2.13); the proof of the right-hand side inequality is similar and will be omitted. Consider the case when $\Omega_{r+1} \neq \emptyset$ (the case when $\Omega_{r+1} = \emptyset$ is easier). Write

$$U_k = \big\{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2^{-k+1}\big\}, \quad V_k = \big\{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\big\}.$$

Observe that $E_{rk} = U_k \setminus V_k$.

From (2.9) it follows that $|F_r(x)| = 0$ for $x \in \mathbb{R}^n \setminus \cup_{k\ge s_r}(E_{rk} + B(0, 2^{-k}))$. We next estimate $|F_r(x)|$ for $x \in \cup_{k\ge s_r}(E_{rk} + B(0, 2^{-k}))$. Two cases present themselves here.

Case 1: $x \in \big[\cup_{k\ge s_r}(E_{rk} + B(0, 2^{-k}))\big] \cap \Omega_{r+1}$. Then there exist $\nu, \ell \in \mathbb{Z}$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu). \tag{2.15}$$

Due to $\Omega_{r+1} \subset \Omega_r$ we have $V_k \subset U_k$, implying $(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) = \emptyset$ if $\nu < \ell$. We consider two subcases depending on whether $\nu \ge \ell + 3$ or $\ell \le \nu \le \ell + 2$.

(a) Let $\nu \ge \ell + 3$. We claim that (2.15) yields

$$B(x, 2^{-k}) \cap E_{rk} = \emptyset \quad \text{for} \quad k \ge \nu + 2 \ \text{ or } \ k \le \ell - 1. \tag{2.16}$$

Indeed, if $k \ge \nu + 2$, then $E_{rk} \subset \Omega_r \setminus V_{\nu+2}$, which implies (2.16), while if $k \le \ell - 1$, then $E_{rk} \subset U_{\ell-1}$, again implying (2.16).

We also claim that

$$B(x, 2^{-k}) \subset E_{rk} \quad \text{for} \quad \ell + 2 \le k \le \nu - 1. \tag{2.17}$$

Indeed, clearly

$$(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) \subset (U_{k-1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_{\nu+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies (2.17).

From (2.9) and (2.16)- (2.17) it follows that

$$\begin{aligned} F_r(x) &= \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy \\ &\quad + \sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy + \sum_{k=\nu-1}^{\nu+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \end{aligned}$$

However,

$$\sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = \sum_{k=\ell+2}^{\nu-2} \big[\varphi_{k+1} * \varphi_{k+1} * f(x) - \varphi_k * \varphi_k * f(x)\big]$$
$$= \varphi_{\nu-1} * \varphi_{\nu-1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x)$$
$$= \int_{E_{r,\nu-1}} \varphi_{\nu-1}(x-y)\varphi_{\nu-1} * f(y)dy - \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y)\varphi_{\ell+2} * f(y)dy.$$

Combining the above with (2.10) and (2.11) we obtain $|F_r(x)| \le c2^r$.

(b) Let $\ell \le \nu \le \ell+2$. Just as above we have

$$F_r(x) = \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = \sum_{k=\ell}^{\ell+3} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy$$

We use (2.10) to estimate each of these four integrals and again obtain $|F_r(x)| \le c2^r$.

Case 2: $x \in \Omega_r \setminus \Omega_{r+1}$. Then there exists $\ell \ge s_r$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}).$$

Just as in the proof of (2.16) we have $B(x, 2^{-k}) \cap E_{rk} = \emptyset$ for $k \le \ell - 1$, and as in the proof of (2.17) we have

$$(U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies $B(x, 2^{-k}) \subset E_{rk}$ for $k \ge \ell+2$. We use these and (2.9) to obtain

$$F_r(x) = \sum_{k=\ell}^{\infty} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy$$
$$= \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy + \sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy.$$

For the last sum we have

$$\sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = \lim_{\nu\to\infty} \sum_{k=\ell+2}^{\nu} \psi_k * \tilde{\psi}_k * f(x)$$
$$= \lim_{\nu\to\infty} \big(\varphi_{\nu+1} * \varphi_{\nu+1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x)\big)$$
$$= \lim_{\nu\to\infty} \Big(\int_{E_{r,\nu+1}} \varphi_{\nu+1}(x-y)\varphi_{\nu+1} * f(y)dy - \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y)\varphi_{\ell+2} * f(y)dy\Big).$$

From the above and (2.10)-(2.11) we obtain $|F_r(x)| \le c2^r$.

The point-wise convergence of the series in (2.7) follows from above and we similarly establish the point-wise convergence in (2.8).

The convergence in distributional sense in (2.7) relies on the following assertion: For every $\phi \in \mathcal{S}$

$$\sum_{k \ge s_r} |\langle g_{rk}, \phi\rangle| < \infty, \quad \text{where} \quad g_{rk}(x) := \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \tag{2.18}$$

Here $\langle g_{rk}, \phi\rangle := \int_{\mathbb{R}^n} g_{rk}\overline{\phi}dx$. To prove the above we will employ this estimate:

$$\|\tilde{\psi}_k f\|_\infty \le c2^{kn/p}\|f\|_{H^p}, \quad k \in \mathbb{Z}. \tag{2.19}$$

Indeed, using (1.4) we get

$$|\tilde{\psi}_k f(x)|^p \le \inf_{y:|x-y|\le 2^{-k}} \sup_{z:|y-z|\le 2^{-k}} |\tilde{\psi}_k f(z)|^p \le \inf_{y:|x-y|\le 2^{-k}} c\mathcal{M}_N(f)(y)^p$$
$$\le c|B(x,2^{-k})|^{-1} \int_{B(x,2^{-k})} \mathcal{M}_N(f)(y)^p d\mu(y) \le c2^{kn}\|f\|_{H^p}^p,$$

and (2.19) follows.

We will also need the following estimate: For any $\sigma > n$ there exists a constant $c_\sigma > 0$ such that

$$\Big|\int_{\mathbb{R}^n} \psi_k(x-y)\phi(x)dx\Big| \le c_\sigma 2^{-k(K+1)}(1+|y|)^{-\sigma}, \quad y \in \mathbb{R}^n,\ k \ge 0. \tag{2.20}$$

This is a standard estimate for inner products taking into account that $\phi \in \mathcal{S}$ and $\psi \in C^\infty$, $\operatorname{supp}\psi \subset B(0,1)$, and $\int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0$ for $|\alpha| \le K$.

We now estimate $|\langle g_{rk}, \phi\rangle|$. From (2.19) and the fact that $\psi \in C_0^\infty(\mathbb{R})$ and $\phi \in \mathcal{S}$ it readily follows that

$$\int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)||\phi(x)||\tilde{\psi}_k f(y)|dydx < \infty, \quad k \ge s_r.$$

Therefore, we can use Fubini's theorem, (2.19), and (2.20) to obtain for $k \ge 0$

$$|\langle g_{rk}, \phi\rangle| \le \int_{E_{rk}} \Big|\int_{\mathbb{R}^n} \psi_k(x-y)\phi(x)dx\Big| |\tilde{\psi}_k f(y)|dy$$
$$\le c2^{-k(K+1-n/p)}\|f\|_{H^p} \int_{E_{rk}} (1+|y|)^{-\sigma}dy \le c2^{-k(K+1-n/p)}\|f\|_{H^p}, \tag{2.21}$$

which implies (2.18) because $K \ge n/p$.

Denote $G_\ell := \sum_{k=s_r}^{\ell} g_{rk}$. From the above proof of (b) and (2.13) we infer that $G_\ell(x) \to F_r(x)$ as $\ell \to \infty$ for $x \in \mathbb{R}^n$ and $\|G_\ell\|_\infty \le c2^r < \infty$ for $\ell \ge s_r$. On the other hand, from (2.18) it follows that the series $\sum_{k\ge s_r} g_{rk}$ converges in distributional sense. By applying the dominated convergence theorem one easily concludes that $F_r = \sum_{k\ge s_r} g_{rk}$ with the convergence in distributional sense. □

We set $F_r := 0$ in the case when $\Omega_r = \emptyset$, $r \in \mathbb{Z}$.

Note that by (2.12) it follows that

$$\psi_k * \psi_k * f(x) = \int_{\mathbb{R}^n} \psi_k(x-y)\psi_k * f(y)dy = \sum_{r\in\mathbb{Z}} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy \tag{2.22}$$

and using (2.6) and the definition of F_r in (2.7) we arrive at

$$f = \sum_{r\in\mathbb{Z}} F_r \ \text{ in } \ \mathcal{S}', \ \text{ i.e. } \ \langle f, \phi\rangle = \sum_{r\in\mathbb{Z}} \langle F_r, \phi\rangle, \quad \forall \phi \in \mathcal{S}, \tag{2.23}$$

where the last series converges absolutely. Above $\langle f, \phi\rangle$ denotes the action of f on $\overline{\phi}$. We next provide the needed justification of identity (2.23).

From (2.6), (2.7), (2.22), and the notation from (2.18) we obtain for $\phi \in \mathcal{S}$

$$\langle f, \phi\rangle = \sum_k \langle \psi_k \tilde{\psi}_k f, \phi\rangle = \sum_k \sum_r \langle g_{rk}, \phi\rangle = \sum_r \sum_k \langle g_{rk}, \phi\rangle = \sum_r \langle F_r, \phi\rangle.$$

Clearly, to justify the above identities it suffices to show that $\sum_k \sum_r |\langle g_{rk}, \phi\rangle| < \infty$. We split this sum into two: $\sum_k \sum_r \cdots = \sum_{k\ge 0} \sum_r \cdots + \sum_{k<0} \sum_r \cdots =: \Sigma_1 + \Sigma_2$.

To estimate Σ_1 we use (2.21) and obtain

$$\begin{aligned}\Sigma_1 &\le c\|f\|_{H^p} \sum_{k\ge 0} 2^{-k(K+1-n/p)} \sum_r \int_{E_{rk}} (1+|y|)^{-\sigma} dy \\ &\le c\|f\|_{H^p} \sum_{k\ge 0} 2^{-k(K+1-n/p)} \int_{\mathbb{R}^n} (1+|y|)^{-\sigma} dy \le c\|f\|_{H^p}.\end{aligned}$$

Here we also used that $K \ge n/p$ and $\sigma > n$.

We estimate Σ_2 in a similar manner, using the fact that $\int_{\mathbb{R}^n} |\psi_k(y)|dy \le c < \infty$ and (2.19). We get

$$\begin{aligned}\Sigma_2 &\le c\|f\|_{H^p} \sum_{k<0} 2^{kn/p} \sum_r \int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)|dy|\phi(x)|dx \\ &\le c\|f\|_{H^p} \sum_{k<0} 2^{kn/p} \int_{\mathbb{R}^n} (1+|x|)^{-n-1} dx \le c\|f\|_{H^p}.\end{aligned}$$

The above estimates of Σ_1 and Σ_2 imply $\sum_k \sum_r |\langle g_{rk}, \phi\rangle| < \infty$, which completes the justification of (2.23).

Observe that due to $\int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0$ for $|\alpha| \le K$ we have

$$\int_{\mathbb{R}^n} x^\alpha F_r(x)dx = 0 \quad \text{for } |\alpha| \le K,\ r \in \mathbb{Z}. \tag{2.24}$$

We next decompose each function F_r into atoms. To this end we need a Whitney type cover for Ω_r, given in the following

Lemma 2.4. *Suppose Ω is an open proper subset of $\mathbb{R}^n$ and let $\rho(x) := \mathrm{dist}(x, \Omega^c)$. Then there exists a constant $K > 0$, depending only on n, and a sequence of points $\{\xi_j\}_{j\in\mathbb{N}}$ in Ω with the following properties, where $\rho_j := \mathrm{dist}(\xi_j, \Omega^c)$:*

(a) $\Omega = \cup_{j\in\mathbb{N}} B(\xi_j, \rho_j/2)$.

(b) $\{B(\xi_j, \rho_j/5)\}$ *are disjoint.*

(c) If $B\big(\xi_j, \frac{3\rho_j}{4}\big) \cap B\big(\xi_\nu, \frac{3\rho_\nu}{4}\big) \ne \emptyset$, *then* $7^{-1}\rho_\nu \le \rho_j \le 7\rho_\nu$.

(d) For every $j \in \mathbb{N}$ *there are at most* K *balls* $B\big(\xi_\nu, \frac{3\rho_\nu}{4}\big)$ *intersecting* $B\big(\xi_j, \frac{3\rho_j}{4}\big)$.

Variants of this simple lemma are well known and frequently used. To prove it one simply selects $\{B(\xi_j, \rho(\xi_j)/5)\}_{j\in\mathbb{N}}$ to be a maximal disjoint subcollection of $\{B(x, \rho(x)/5)\}_{x\in\Omega}$ and then properties (a)-(d) follow readily, see [5], pp. 15-16.

We apply Lemma 2.4 to each set $\Omega_r \ne \emptyset$, $r \in \mathbb{Z}$. Fix $r \in \mathbb{Z}$ and assume $\Omega_r \ne \emptyset$. Denote by $B_j := B(\xi_j, \rho_j/2)$, $j = 1, 2, \dots,$ the balls given by Lemma 2.4, applied to Ω_r, with the additional assumption that these balls are ordered so that $\rho_1 \ge \rho_2 \ge \cdots$. We will adhere to the notation from Lemma 2.4. We will also use the more compact notation $\mathcal{B}_r := \{B_j\}_{j\in\mathbb{N}}$ for the set of balls covering Ω_r.

For each ball $B \in \mathcal{B}_r$ and $k \ge s_r$ we define

$$E^B_{rk} := E_{rk} \cap \big(B + 2B(0, 2^{-k})\big) \quad \text{if} \quad B \cap E_{rk} \ne \emptyset \tag{2.25}$$

and set $E^B_{rk} := \emptyset$ if $B \cap E_{rk} = \emptyset$.

We also define, for $\ell = 1, 2, \dots,$

$$R^{B_\ell}_{rk} := E^{B_\ell}_{rk} \setminus \cup_{\nu>\ell} E^{B_\nu}_{rk} \quad \text{and} \tag{2.26}$$

$$F_{B_\ell}(x) := \sum_{k \geq s_r} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \tag{2.27}$$

Lemma 2.5. *For every $\ell \geq 1$ the function F_{B_ℓ} is well defined, more precisely, the series in* (2.27) *converges point-wise and in distributional sense. Furthermore,*

$$\operatorname{supp} F_{B_\ell} \subset 7B_\ell, \tag{2.28}$$

$$\int_{\mathbb{R}^n} x^\alpha F_{B_\ell}(x)dx = 0 \quad \textit{for all } \alpha \textit{ with } |\alpha| \leq n(p^{-1}-1), \tag{2.29}$$

and

$$\|F_{B_\ell}\|_\infty \leq c_\sharp 2^r, \tag{2.30}$$

where the constant $c_\sharp$ is independent of r, ℓ.

In addition, for any $k \geq s_r$

$$E_{rk} = \cup_{\ell \geq 1} R_{rk}^{B_\ell} \quad \textit{and} \quad R_{rk}^{B_\ell} \cap R_{rk}^{B_m} = \emptyset, \quad \ell \neq m. \tag{2.31}$$

Hence

$$F_r = \sum_{B \in \mathcal{B}_r} F_B \quad (\textit{convergence in } \mathcal{S}'). \tag{2.32}$$

Proof. Fix $\ell \geq 1$. Observe that using Lemma 2.4 we have $B_\ell \subset \Omega_r^c + B(0, 2\rho_\ell)$ and hence $E_{rk}^{B_\ell} := \emptyset$ if $2^{-k+1} \geq 2\rho_\ell$. Define $k_0 := \min\{k : 2^{-k} < \rho_\ell\}$. Hence $\rho_\ell/2 \leq 2^{-k_0} < \rho_\ell$. Consequently,

$$F_{B_\ell}(x) := \sum_{k \geq k_0} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \tag{2.33}$$

It follows that $\operatorname{supp} F_{B_\ell} \subset B\big(\xi_\ell, (7/2)\rho_\ell\big) = 7B_\ell$, which confirms (2.28).

To prove (2.30) we will use the following

Lemma 2.6. *For an arbitrary set $S \subset \mathbb{R}^n$ let $S_k := \{x \in \mathbb{R}^n : \operatorname{dist}(x, S) < 2^{-k+1}\}$ and set*

$$F_S(x) := \sum_{k \geq \kappa_0} \int_{E_{rk} \cap S_k} \psi_k(x-y)\tilde{\psi}_k * f(y)dy \tag{2.34}$$

for some $\kappa_0 \geq s_r$. Then $\|F_S\|_\infty \leq c2^r$, where $c > 0$ is a constant independent of S and κ_0. Moreover, the above series converges in $\mathcal{S}'$.

Proof. From (2.9) it follows that $F_S(x) = 0$ if $\operatorname{dist}(x, S) \geq 3 \times 2^{-\kappa_0}$

Let $x \in S$. Evidently, $B(x, 2^{-k}) \subset S_k$ for every k and hence

$$\begin{aligned} F_S(x) &= \sum_{k \geq \kappa_0} \int_{E_{rk} \cap B(x, 2^{-k})} \psi_k(x-y)\tilde{\psi}_k * f(y)dy \\ &= \sum_{k \geq \kappa_0} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy = F_{r,\kappa_0}(x). \end{aligned}$$

On account of Lemma 2.3 (b) we obtain $|F_S(x)| = |F_{r,\kappa_0}(x)| \leq c2^r$.

Consider the case when $x \in S_\ell \setminus S_{\ell+1}$ for some $\ell \ge \kappa_0$. Then $B(x, 2^{-k}) \subset S_k$ if $\kappa_0 \le k \le \ell - 1$ and $B(x, 2^{-k}) \cap S_k = \emptyset$ if $k \ge \ell + 2$. Therefore,

$$\begin{aligned} F_S(x) &= \sum_{k=\kappa_0}^{\ell-1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy + \sum_{k=\ell}^{\ell+1} \int_{E_{rk}\cap S_k} \psi_k(x-y)\tilde{\psi}_k * f(y)dy \\ &= F_{r,\kappa_0,\ell-1}(x) + \sum_{k=\ell}^{\ell+1} \int_{E_{rk}\cap S_k} \psi_k(x-y)\tilde{\psi}_k * f(y)dy, \end{aligned}$$

where we used the notation from (2.8). By Lemma 2.3 (b) and (2.10) it follows that $|F_S(x)| \le c2^r$.

We finally consider the case when $2^{-\kappa_0+1} \le \operatorname{dist}(x, S) < 3 \times 2^{-\kappa_0}$. Then we have $F_S(x) = \int_{E_{r\kappa_0}\cap S_{\kappa_0}} \psi_{\kappa_0}(x-y)\tilde{\psi}_{\kappa_0} * f(y)dy$ and the estimate $|F_S(x)| \le c2^r$ is immediate from (2.10).

The convergence in $\mathcal{S}'$ in (2.34) is established as in the proof of Lemma 2.3. $\square$

Fix $\ell \ge 1$ and let $\{B_j : j \in \mathcal{J}\}$ be the set of all balls $B_j = B(\xi_j, \rho_j/2)$ such that $j > \ell$ and

$$B\Big(\xi_j, \frac{3\rho_j}{4}\Big) \cap B\Big(\xi_\ell, \frac{3\rho_\ell}{4}\Big) \ne \emptyset.$$

By Lemma 2.4 it follows that $\#\mathcal{J} \le K$ and $7^{-1}\rho_\ell \le \rho_j \le 7\rho_\ell$ for $j \in \mathcal{J}$. Define

$$k_1 := \min\Big\{k : 2^{-k+1} < 4^{-1} \min\big\{\rho_j : j \in \mathcal{J} \cup \{\ell\}\big\}\Big\}. \tag{2.35}$$

From this definition and $2^{-k_0} < \rho_\ell$ we infer

$$2^{-k_1+1} \ge 8^{-1} \min\big\{\rho_j : j \in \mathcal{J} \cup \{\ell\}\big\} > 8^{-2}\rho_\ell > 8^{-2}2^{-k_0} \implies k_1 \le k_0 + 7. \tag{2.36}$$

Clearly, from (2.35)

$$B_j + 2B(0, 2^{-k}) \subset B\big(\xi_j, 3\rho_j/4\big), \quad \forall k \ge k_1, \ \forall j \in \mathcal{J} \cup \{\ell\}. \tag{2.37}$$

Denote $S := \cup_{j\in\mathcal{J}} B_j$ and $\tilde{S} := \cup_{j\in\mathcal{J}} B_j \cup B_\ell = S \cup B_\ell$. As in Lemma 2.6 we set

$$S_k := S + 2B(0, 2^{-k}) \quad \text{and} \quad \tilde{S}_k := \tilde{S} + 2B(0, 2^{-k}).$$

It readily follows from the definition of k_1 in (2.35) that

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \cup_{\nu>\ell} E_{rk}^{B_\nu} = \big(E_{rk} \cap \tilde{S}_k\big) \setminus \big(E_{rk} \cap S_k\big) \quad \text{for} \quad k \ge k_1. \tag{2.38}$$

Denote

$$\begin{aligned} F_S(x) &:= \sum_{k\ge k_1} \int_{E_{rk}\cap S_k} \psi_k(x-y)\tilde{\psi}_k * f(y)dy, \quad \text{and} \\ F_{\tilde{S}}(x) &:= \sum_{k\ge k_1} \int_{E_{rk}\cap \tilde{S}_k} \psi_k(x-y)\tilde{\psi}_k * f(y)dy. \end{aligned}$$

From (2.38) and the fact that $S \subset \tilde{S}$ it follows that

$$F_{B_\ell}(x) = F_{\tilde{S}}(x) - F_S(x) + \sum_{k_0 \le k < k_1} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde{\psi}_k * f(y)dy.$$

By Lemma 2.6 we get $\|F_S\|_\infty \le c2^r$ and $\|F_{\tilde{S}}\|_\infty \le c2^r$. On the other hand from (2.36) we have $k_1 - k_0 \le 7$. We estimate each of the (at most 7) integrals above using (2.10) to conclude that $\|F_{B_\ell}\|_\infty \le c2^r$.

We deal with the convergence in (2.27) and (2.32) as in the proof of Lemma 2.3.

Clearly, (2.29) follows from the fact that $\int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0$ for all α with $|\alpha| \le K$.

Finally, from Lemma 2.4 we have $\Omega_r \subset \cup_{j\in\mathbb{N}} B_\ell$ and then (2.31) is immediate from (2.25) and (2.26). $\square$

We are now prepared to complete the proof of Theorem 2.1. For every ball $B \in \mathcal{B}_r$, $r \in \mathbb{Z}$, provided $\Omega_r \ne \emptyset$, we define $B^\star := 7B$,

$$a_B(x) := {c_\sharp}^{-1}|B^\star|^{-1/p}2^{-r}F_B(x) \quad \text{and} \quad \lambda_B := c_\sharp|B^\star|^{1/p}2^r,$$

where $c_\sharp > 0$ is the constant from (2.30). By (2.28) $\operatorname{supp} a_B \subset B^\star$ and by (2.30)

$$\|a_B\|_\infty \le {c_\sharp}^{-1}|B^\star|^{-1/p}2^{-r}\|F_B\|_\infty \le |B^\star|^{-1/p}.$$

Furthermore, from (2.29) it follows that $\int_{\mathbb{R}^n} x^\alpha a_B(x)dx = 0$ if $|\alpha| \le n(p^{-1}-1)$. Therefore, each a_B is an atom for H^p.

We set $\mathcal{B}_r := \emptyset$ if $\Omega_r = \emptyset$. Now, using the above, (2.23), and Lemma 2.5 we get

$$f = \sum_{r\in\mathbb{Z}} F_r = \sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} F_B = \sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} \lambda_B a_B,$$

where the convergence is in $\mathcal{S}'$, and

$$\sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} |\lambda_B|^p \le c\sum_{r\in\mathbb{Z}} 2^{pr}\sum_{B\in\mathcal{B}_r}|B| = c\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c\|f\|^p_{H^p},$$

which is the claimed atomic decomposition of $f \in H^p$. Above we used that $|B^\star| = |7B| = 7^n|B|$. $\square$

Remark 2.7. *The proof of Theorem 2.1 can be considerably simplified and shortened if one seeks to establish atomic decomposition of the H^p spaces in terms of q-atoms with $p < q < \infty$ rather than ∞-atoms as in Theorem 2.1, i.e. atoms satisfying $\|a\|_{L^q} \le |B|^{1/q-1/p}$ with $q < \infty$ rather than $\|a\|_{L^\infty} \le |B|^{-1/p}$. We will not elaborate on this here.*

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