LOCALLY COMPACT, ω_1-COMPACT SPACES

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Abstract. An ω_1-compact space is a space in which every closed discrete subspace is countable. We give various general conditions under which a locally compact, ω_1-compact space is σ*-countably compact, i.e.,* the union of countably many countably compact spaces. These conditions involve very elementary properties.

Many results shown here are independent of the usual (ZFC) axioms of set theory, and the consistency of some may involve large cardinals. For example, it is independent of the ZFC axioms whether every locally compact, ω_1-compact space of cardinality $\aleph_1$ is σ-countably compact. Whether $\aleph_1$ can be replaced with $\aleph_2$ is a difficult unsolved problem. Modulo large cardinals, it is also ZFC-independent whether every hereditarily normal, or every monotonically normal, locally compact, ω_1-compact space is σ-countably compact.

As a result, it is also ZFC-independent whether there is a locally compact, ω_1-compact Dowker space of cardinality $\aleph_1$, or one that does not contain both an uncountable closed discrete subspace and a copy of the ordinal space ω_1.

On the other hand, it is a theorem of ZFC that every locally compact, locally connected and connected, monotonically normal space is σ-countably compact. More generally, every locally compact space with a monotonically normal compactification is the topological direct sum of σ-countably compact spaces; and if it is totally disconnected, every summand can be made countably compact.

Set theoretic tools used for the consistency results include ♣, the Proper Forcing Axiom (PFA), and models generically referred to as "MM(S)[S]". Most of the work is done by the P-Ideal Dichotomy (PID) axiom, which holds in the latter two cases, and which requires no large cardinal axioms when directly applied to topological spaces of cardinality $\aleph_1$, as it is in several theorems.

1. Introduction

A *space of countable extent*, also called an ω_1*-compact space*, is one in which every closed discrete subspace is countable. Obvious examples of ω_1-compact spaces are countably compact spaces (because in them every closed discrete subspace is finite), and σ*-countably compact spaces*, *i.e.,* the union of countably many countably compact spaces. On the other hand, an elementary application of the Baire Category Theorem shows that the space of irrational numbers with the usual topology is not σ-countably compact, but like every other separable metrizable space, it is ω_1-compact.

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The situation is very different when it comes to locally compact spaces. In an earlier version of this paper due to the first author, he asked:

Question 1.1. *Is there a ZFC example of a locally compact, ω_1-compact space of cardinality $\aleph_1$ that is not σ-countably compact? one that is normal?*

Here too, local compactness makes a big difference: without it, the space of irrationals is a counterexample under CH, while ZFC is enough to show any cardinality $\aleph_1$ subset of a Bernstein set is a counterexample.

As it is, the second author showed that the answer to Question 1.1 is negative; see Section 2. On the other hand, both the Kunen line and a Souslin tree with the usual topology are consistent normal examples for Question 1.1. In Section 4 we will also show:

Theorem 1.2. *Assuming ♣, there is a locally compact, monotonically normal, locally countable (hence first countable) ω_1-compact space of cardinality $\aleph_1$ which is not σ-countably compact.*

Monotonically normal spaces are, informally speaking, "uniformly normal" [see Definition 3.2 below]. They are hereditarily normal, and this theorem gives another independence result when combined with:

Theorem 1.3. *In MM(S)[S] models, every hereditarily normal, locally compact, ω_1-compact space is σ-countably compact.*

An even stronger theorem will be shown in Section 3 along with related results under weaker set-theoretic hypotheses. These will put some limitations on what kinds of Dowker spaces (that is, normal spaces X such that $X \times [0,1]$ is not normal) are possible if one only assumes the usual (ZFC) axioms of set theory. The ZFC axioms do, however, suffice to imply σ-countable compactness for some classes of spaces; see Section 5.

In the light of the negative answer to Question 1.1, it is natural to ask for the least cardinality of a locally compact, ω_1-compact space which is not σ-countably compact. This is discussed in Section 6, and a new upper bound of $\mathfrak{b}$ for this cardinality is given.

The last section gives more information of the ZFC example behind the upper bound and about a problem and a result of Eric van Douwen, under the assumption of $\mathfrak{b} = \mathfrak{c}$, that are related to this example.

In between, Section 7 gives some interesting counterpoints to the problems in Section 6, by discussing questions about the greatest cardinality of a locally countable, normal, countably compact space.

The individual sections are only loosely connected with each other, and each can be read with minimal reliance on any of the others.

All through this paper, "space" means "Hausdorff topological space." All of the spaces described are locally compact, hence Tychonoff; and all are also normal, except for a consistent example at the end.

2. The cardinality $\aleph_1$ case

The P-Ideal Dichotomy (PID) plays a key role in this section and the following one. It has to do with the following concept. *A P-ideal of countable sets* is a family $\mathcal{P}$ of sets such that, for every countable subfamily $\mathcal{Q}$ of $\mathcal{P}$, there exists $P \in \mathcal{P}$ such that $Q \subset^* P$ for every $Q \in \mathcal{Q}$. Here $Q \subset^* P$ means that $Q \setminus P$ is finite.

The PID states that, for every P-ideal $\mathcal{I}$ of subsets of a set X, either

(A) there is an uncountable $A \subset X$ such that every countable subset of A is in $\mathcal{I}$, or

(B) X is the union of countably many sets $\{B_n : n \in \omega\}$ such that $B_n \cap I$ is finite for all n and all $I \in \mathcal{I}$.

The routine proofs of the next lemma and theorem were given in [12]:

Lemma 2.1. *Let X be a locally compact Hausdorff space. The countable closed discrete subspaces of X form a P-ideal if, and only if, the extra point ∞ of the one-point compactification $X+1$ of X is an α_1-point; that is, whenever $\{\sigma_n : n \in \omega\}$ is a countable family of sequences converging to ∞, then there is a sequence σ converging to ∞ such that $ran(\sigma_n) \subseteq^* ran(\sigma)$ for all n.*

The key is that an ordinary sequence in X converges to the extra point of $X+1$ if, and only if, its range is a closed discrete subspace of X.

Theorem 2.2. *Assume the PID axiom. Let X be a locally compact space. Then at least one of the following is true:*

(1^-) *X is the union of countably many subspaces Y_n such that each sequence in Y_n has a limit point in X.*

(2) *X has an uncountable closed discrete subspace*

(3^+) *The extra point of $X+1$ is not an α_1-point.*

The key here is that (A) goes with (2), (B) goes with (1^-), and (3^+) is equivalent to the countable closed discrete subspaces failing to form a P-ideal, by Lemma 2.1.

The following is well known:

Lemma 2.3. *If X is a space of character $< \mathfrak{b}$ then every point of X is an α_1-point.*

We now have a negative answer to the second part of Question 1.1.

Theorem 2.4. *Assume the PID and $\mathfrak{b} > \aleph_1$. Then every locally compact, ω_1-compact, normal space of cardinality $\aleph_1$ is σ-countably compact.*

Proof. In a locally compact space, character $\leq$ cardinality. Lemmas 2.1 and 2.3 and ω_1-compactness give us alternative (1^-) Theorem 2.2. The closure of each Y_n is easily seen to be pseudocompact (i.e., every continuous real-valued function is bounded). In a normal space, every closed pseudocompact subspace is countably compact, cf. [35, 17J 3]. So the closures of the Y_n witness that X is σ-countably compact. □

As shown in [12], the hypothesis of normality in Theorem 2.4 can be greatly weakened to "Property wD". Also, the proof of Theorem 2.4 clearly extends to show that every locally compact, ω_1-compact space of character $< \mathfrak{b}$ is σ-countably compact. However, this may be a very limited improvement: the PID implies $\mathfrak{b} \leq \aleph_2$. This is a theorem of Todorčević, whose proof may be found in [18].

The axioms used in Theorem 2.4 follow from the Proper Forcing Axiom (PFA) and hold in $PFA(S)[S]$ models. Each of these models is formed from a $PFA(S)$ model by forcing with a coherent Souslin tree S that is part of the definition of what it means to be a $PFA(S)$ model. The rest of the definition states that every proper poset P that does not destroy S when it is forced with, has the following property. For every family of $\leq \aleph_1$ $\uparrow$-dense, $\uparrow$-open sets, there is a $\downarrow$-closed, $\uparrow$-directed subset of P that meets them all. [1] The PFA is similarly defined by omitting all mention of S. What remains is very similar to the well-known definition of Martin's Axiom (MA); the only difference is that MA uses "ccc" instead of "proper."

In this paper, we will use a slight abuse of language with expressions like $PFA(S)[S]$ and $MM(S)[S]$ as though they were axioms. The latter is defined like the former, but with "semi-proper" in place of "proper."

For our negative answer to the first half of Question 1.1, we needed a strengthening of $\mathfrak{b} > \aleph_1$ to $\mathfrak{p} > \aleph_1$. [It is a theorem of ZFC that $\mathfrak{p} \leq \mathfrak{b}$.] In $PFA(S)[S]$ models, $\mathfrak{p} = \aleph_1$ while $\mathfrak{b} = \aleph_2$. On the other hand, the PFA implies both the PID and $\mathfrak{p} > \aleph_1$. And this is enough for our negative answer (Theorem 2.6, below).

The standard definition of the cardinal $\mathfrak{p}$ is used directly in the proof of the key lemma for Theorem 2.6. It involves families of subsets of ω (hence of any denumerable set) with what is known as the Strong Finite Intersection Property (SFIP). This states that every finite subfamily has infinite intersection.

$$\mathfrak{p} = min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^\omega \text{ and } \mathcal{F} \text{ has the SFIP but } \neg\exists A \in [\omega]^\omega (A \subset^* F \text{ for all } F \in \mathcal{F})\}$$

Lemma 2.5. *Let X be a T_3 space and let $Y \subset X$. Suppose that $|\overline{Y}| < \mathfrak{p}$, and that no $Z \in [Y]^\omega$ is closed discrete in X. Then there exists a countably compact $Y' \subset X$ containing Y.*

Proof. For every $Z \in [Y]^\omega$ let x_Z be an accumulation point of Z. Let $Y' = Y \cup \{x_Z : Z \in [Y]^\omega\}$. *We claim that Y' is countably compact.* Indeed, otherwise there exists a countable $T = \{t_n : n \in \omega\} \subset Y'$ which is closed discrete in Y'. If $Z = T \cap Y$ then Z is finite, otherwise x_Z would be an accumulation point of T in Y'. So we may assume that $T \cap Y = \emptyset$. For every n fix $Z_n \in [Y]^\omega$ such that $t_n = x_{Z_n}$. Let $W_n \ni t_n$ be a neighborhood of t_n in $\overline{Y}$ such that $W_n \cap W_m = \emptyset$ for all $n \neq m$. [In a T_3 space, every countable discrete subspace extends to a disjoint open collection.]

For every $y \in Y' \setminus T$ find an neighborhood U_y of y in $\overline{Y}$ such that $\overline{U_y} \cap T = \emptyset$. For every $n \in \omega$, let $\mathcal{Z}_n = \{(Z_n \cap W_n) \setminus (\overline{U_y}) : y \in Y' \setminus T\}$. Note that $\mathcal{Z}_n$ has the SFIP. Since $|\mathcal{Z}_n| < \mathfrak{p}$, there exists an infinite $Z'_n = \{z^n_k : k \in \omega\} \subset Z_n \cap W_n$ such that $Z'_n \subset^* Z$ for any $Z \in \mathcal{Z}_n$. Then $\sigma_n = \langle z^n_k : k \in \omega\rangle$ converges to t_n, because the range of every subsequence of σ_n has some $x_A \in Y'$ as a limit point, but there is nothing for x_A to be except t_n.

Let $f_y : \omega \to \omega$ be such that $\{z^n_k : k \geq f_y(n)\} \cap \overline{U_y} = \emptyset$. Since $|\overline{Y}| < \mathfrak{b}$, there exists $f : \omega \to \omega$ such that $f_y <^* f$ for all $y \in Y' \setminus T$. Now $Z_\omega = \{z^n_{f(n)} : n \in \omega\} \subset Y$ has no

[1]The notation is as in [13]. It is the "Israeli" notation, whereby stronger conditions are larger. The topology to which it refers is the one where each $p \in P$ has the one basic neighborhood $\{t : t \geq p\}$.

accumulation points in T, because $Z_\omega \cap W_n = \{z^n_{f(n)}\}$. Also, $Z_\omega \cap U_y$ is finite for each $y \in Y'$, so x_{Z_ω} does not exist. This contradiction implies that Y' is countably compact. □

We can weaken the hypothesis in the statement of Lemma 2.5 by having $\overline{Y}$ be hereditarily of Lindelöf degree $< \mathfrak{p}$. That is, if $S \subset \overline{Y}$, and $\mathcal{U}$ is a relatively open cover of S, then $\mathcal{U}$ has a subcover of cardinality $< \mathfrak{p}$. Applying this to $S = Y' \setminus T$ and $\mathcal{U} = \{U_y : y \in S\}$, a subcover of cardinality $< \mathfrak{p}$ is enough to make the argument go through.

Now we can finish answering Question 1.1.

Theorem 2.6. *Assume the PID and $\mathfrak{p} > \aleph_1$. Then every locally compact, ω_1-compact space of cardinality $\aleph_1$ is σ-countably compact.*

Proof. Again Lemmas 2.1 and 2.3 and ω_1-compactness give us alternative (1^-) of Theorem 2.2. The rest is clear from Lemma 2.5. □

Returning to Theorem 2.4, its proof also gives:

Theorem 2.7. *Assume the PID. Then every locally compact, ω_1-compact normal space of cardinality $< \mathfrak{b}$ is countably paracompact.*

Proof. A normal space X is countably paracompact if, and only if, for each descending sequence of closed sets $\langle F_n \rangle_{n=1}^\infty$ with empty intersection, there is a sequence of open sets $\langle U_n \rangle_{n=1}^\infty$ with empty intersection, with $F_n \subset U_n$ for all n. If X is a countable union of countably compact subsets C_m, as in Theorem 2.4, then in such a sequence of closed sets F_n, we can only have $F_n \cap C_m \neq \emptyset$ for finitely many n. [Otherwise, countable compactness of C_m implies $\bigcap_{n=1}^\infty C_m \cap F_n \neq \emptyset$.] In any Tychonoff space, every pseudocompact subspace, and hence every countably compact subspace, has pseudocompact closure, and every normal, pseudocompact space is countably compact; and so the complements of the sets $\overline{C_m}$ form the desired sequence of open sets. □

The equivalence in the preceding proof is shown in [35, 21.3] and is due to Dowker, who also showed its equivalence with $X \times [0, 1]$ being normal. In honor of his pioneering work, normal spaces that are *not* countably paracompact are called "Dowker spaces." Theorem 2.7 thus implies the consistency of there being no locally compact, ω_1-compact Dowker spaces of cardinality $\aleph_1$. Specialized though this fact is, its is one of the few theorems as to what kinds of Dowker spaces are unattainable in ZFC. Another interesting such result was obtained recently by Tall [31]:

Theorem 2.8. *If $PFA(S)[S]$, then every locally compact non-paracompact space of Lindelöf number $\leq \aleph_1$ includes a perfect preimage of ω_1.*

Alan Dow has improved "perfect preimage" to "copy" [private communication]. Combining this with Theorem 2.7, we have:

Corollary 2.9. *If $PFA(S)[S]$, then every locally compact Dowker space of cardinality $\leq \aleph_1$ includes both a copy of ω_1 and an uncountable closed discrete subspace.*

The consistency of PID was shown using forcing from a ground model with a supercompact cardinal. There are versions for spaces of weight $\aleph_1$, hence all locally compact spaces of cardinality $\aleph_1$, which require only the consistency of ZFC. One restricted version of the PID axiom is designated $(*)$ in [1], and is adequate for Theorem 2.7. But it is still an open problem whether the main results of our next section are ZFC-equiconsistent.

3. When hereditary normality implies σ-countable compactness

For the main theorem of this section, we recall the following concepts:

Definition 3.1. Given a subset D of a set X, an *expansion* of D is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = d$ for all $d \in D$. A space X is *[strongly] collectionwise Hausdorff* (abbreviated *[s]cwH*) if every closed discrete subspace has an expansion to a disjoint [*resp.* discrete] collection of open sets.

The properties of ω_1-*[s]cwH* only require taking care of those D that are of cardinality $\leq \omega_1$.

A well-known, almost trivial fact is that every normal, cwH space is scwH: if D and $\{U_d : d \in D\}$ are as in 3.1, let V be an open set containing D whose closure is in $\bigcup\{U_d : d \in D\}$; then $\{U_d \cap V : d \in D\}$ is a discrete open expansion of D.

The following class of normal spaces plays a big role in this section and in the following one.

Definition 3.2. A space X is *monotonically normal* if there is an operator $G(\_,\_)$ assigning to each ordered pair $\langle F_0, F_1 \rangle$ of disjoint closed subsets an open set $G(F_0, F_1)$ such that
(a) $F_0 \subset G(F_0, F_1)$
(b) If $F_0 \subset F_0'$ and $F_1' \subset F_1$ then $G(F_0, F_1) \subset G(F_0', F_1')$
(c) $G(F_0, F_1) \cap G(F_1, F_0) = \emptyset$.

Monotone normality is a hereditary property; that is, every subspace inherits the property. This is not so apparent from this definition, but it follows almost immediately from the following criterion, due to Borges.

Theorem 3.3. *A space is monotonically normal if, and only if, there is an assignment of an open neighborhood $h(x, U) =: U_x$ containing x to each pair (x, U) such that U is an open neighborhood of x, and such that, if $U_x \cap V_y \neq \emptyset$, then either $x \in V$ or $y \in U$.*

Every monotonically normal space is (hereditarily) countably paracompact [29, Theorem 2.3] and (hereditarily) scwH: the Borges criterion easily give cwH, and normality does the rest.

The main theorem of this section, Theorem 3.6, also involves the following concepts.

Definition 3.4. An *ω-bounded space* is one in which every countable subset has compact closure. A space is *σ-ω-bounded* if it is the union of countably many ω-bounded subspaces.

Clearly, every ω-bounded space is countably compact. Theorem 3.6 below makes use of the following axiom, which is shown in [12] to follow from PID and whose numbering is aligned with that of Theorem 2.2:

Axiom 3.5. *Every locally compact space has either:*
(1) *A countable collection of ω-bounded subspaces whose union is the whole space or*
(2) *An uncountable closed discrete subspace or*
(3) *A countable subset with non-Lindelöf closure.*

Of course, (1) and (2) are mutually exclusive, but each is compatible with (3).

Part (iii) of our main theorem is the promised strengthening of Theorem 1.3.

Theorem 3.6. *Let X be a locally compact, ω_1-compact space. If either*
(i) *X is monotonically normal and the P-Ideal Dichotomy (PID) axiom holds, or*
(ii) *X is hereditarily ω_1-scwH, and either PFA or PFA(S)[S] hold, or*
(iii) *X is hereditarily normal, and MM(S)[S] holds,*
THEN X is σ-ω-bounded, and is either Lindelöf or contains a copy of ω_1.

Proof. The σ-ω-bounded property in case (i) follows from the fact that, in a monotonically normal space, every countable subset has Lindelöf closure [28], and from Axiom 3.5.

The PID, and hence Axiom 3.5, holds in any model of PFA or $PFA(S)[S]$, so σ-ω-boundedness in case (ii) follows from the following two facts. First, every point (and hence every countable subset) of a locally compact space has an open Lindelöf neighborhood (see [23, Lemma 1.7]). Second, in these models, every open Lindelöf subset of a hereditarily ω_1-scwH space has Lindelöf closure (see [23] and [26], respectively). So again, (3) in Axiom 3.5 fails outright, and σ-ω-boundedness for (ii) follows from (1) of Axiom 3.5. As for (iii), Dow and Tall [8] have shown that $MM(S)[S]$ implies that every normal, locally compact space is ω_1-cwH, and we can continue as for (ii).

That X is either Lindelöf or contains a copy of ω_1 in case (i), is immediate from the following fact, whose proof is deferred to Section 5 (see Corollary 5.6):

Every locally compact, monotonically normal space is either paracompact or contains a closed copy of a regular uncountable cardinal.

The Lindelöf alternative for case (i) uses the fact [10, 5.2.17] that every locally compact, paracompact space is the union of a disjoint family of (closed and) open Lindelöf subspaces. Now ω_1-compactness makes the family countable, and so X is Lindelöf.

The same either/or alternative for (ii) and (iii) uses the ZFC theorem that any ω-bounded, locally compact space is either Lindelöf or contains a perfect preimage of ω_1, along with the reduction of character in Lemma 1.2 of [23], which uses $MA(\omega_1)$, which in turn is implied by the PFA. Moreover, the proof of this lemma carries over to $PFA(S)[S]$. This proof shows that in a locally compact, hereditarily ω_1-scwH space, every open Lindelöf subset has Lindelöf closure and hereditarily Lindelöf (hence first countable) boundary.

Now, given a perfect surjective map $\varphi : W \to \omega_1$, let Y be the union of the boundaries of the subsets $\varphi^{\leftarrow}[0, \alpha)$ where α is a limit ordinal, and apply to Y the theorem [7] that $PFA(S)[S]$ implies that every first countable perfect preimage of ω_1 contains a copy of ω_1. That the PFA also implies this is a well-known theorem of Balogh, shown in [6]. $\square$

The following theorem can be derived from Theorem 3.6 (ii) in the same way that Theorem 2.7 is derived from Theorem 2.4.

Theorem 3.7. *Let X be a locally compact, ω_1-compact, normal, hereditarily ω_1-scwH space. If either PFA or PFA(S)[S] holds, then X is countably paracompact.*

To put it positively, every ZFC example of a locally compact, hereditarily ω_1-scwH Dowker space must contain an uncountable closed discrete subspace. However, to be absolutely certain of this, we need a negative answer to the second part of the following question:

Problem 1. *Do we need the large cardinal strength of PID to obtain any or all of the conclusions of Theorem 3.6 or Theorem 3.7?*

Where Theorem 3.7 is concerned, the other applications of PFA, etc. are taken care of by axioms $MA(\omega_1)$ and $MA(S)[S]$ [23], [25], both of which are equiconsistent with ZFC. However, we may need something more for the conclusion about the copy of ω_1, or for the theorem [8] that $MM(S)[S]$ implies that every locally compact normal space is ω_1-cwH, or for the following corollary of this fact and of Theorem 3.7:

Corollary 3.8. *If $MM(S)[S]$, then every locally compact, hereditarily normal Dowker space must contain an uncountable closed discrete subspace.* □

4. Examples under ♣ for Theorem 1.2

This section features consistent examples of locally compact, ω_1-compact, monotonically normal spaces of cardinality $\aleph_1$ that are not σ-countably compact. The axiom used is ♣, and the following concept is part of its definition.

Definition 4.1. Given a limit ordinal α, a *ladder at α* is a strictly ascending sequence of ordinals less than α whose supremum is α. Given an ordinal θ, a *ladder system on θ* is a family

$$\mathcal{L} = \{L_\alpha : \alpha \in \theta,\ \alpha \text{ is a limit ordinal of countable cofinality}\}$$

where each L_α is a ladder at α.

We will use the symbol Λ for the set of countable limit ordinals.

Axiom 4.2. *Axiom* ♣ states that there is a ladder system $\mathcal{L}$ on ω_1 such that, for any uncountable subset S of ω_1, there is $L_\alpha \in \mathcal{L}$ such that $L_\alpha \subset S$.

We now define a general family $\mathfrak{S}$ of spaces similar to the two families in [17]. One subfamily of $\mathfrak{S}$ consists of examples for Theorem 1.2.

Notation 4.3. Let $\mathfrak{S}$ designate the set of topologies τ on ω_1 in which, to each point α there are associated $B(\alpha) \subset [0, \alpha]$ and $B(\alpha, \xi) = B(\alpha) \cap (\xi, \alpha]$ for each $\xi < \alpha$, such that:

(1) $\{B(\alpha, \xi) : \xi < \alpha\}$ *is a base for the neighborhoods of α [we allow $\xi = -1$ in case $\alpha = 0$].*

(2) *If $\alpha \in \Lambda$ and $\beta > \alpha$, then there exists $\xi < \alpha$ such that $B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]$. [In particular, $\alpha \in B(\beta)$.]*

(3) *There is a ladder system $\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$, such that if $M_\alpha = \{\xi + 1 : \xi \in L_\alpha\}$, then $\alpha = sup[M_\alpha \cap B(\alpha)] = sup[M_\alpha \setminus B(\alpha)]$.*

Examples 4.4. Let τ be a topology defined using $\mathcal{L}$, and using the base $B(\alpha)$ and $B(\alpha, \xi)$ defined by recursion as follows:

Let $B(0) = \{0\}$ and, if $\alpha = \xi + 1$, let $B(\alpha) = B(\xi) \cup \{\alpha\}$. Given $L_\alpha \in \mathcal{L}$, let $M_\alpha = \{\xi + 1 : \xi \in L_\alpha\}$ be bijectively indexed as $\{\alpha_n : n \in \omega\}$. If $\alpha = \nu + \omega$ where ν either is 0 or a limit ordinal, let $B(\alpha) = B(\nu) \cup (\nu, \alpha] \setminus \{\alpha_{2n} : n \in \omega\}$.

If α is not of the form $\nu + \omega$, let $S(\alpha, 0) = B(\alpha_0)$ and, for $n > 0$, let $S(\alpha, n) = B(\alpha_n, \alpha_{n-1})$ and let

$$B(\alpha) = \bigcup_{n=0}^{\infty} (S(\alpha, n) \cup \{\alpha_{2n+1} : n \in \omega\}) \setminus \{\alpha_{2n} : n \in \omega\}.$$

Claim 1. $(\omega_1, \tau) \in \mathfrak{S}$.

$\vdash$ Assuming (2) in Notation 4.3, we first show (1). If $\alpha \in B(\beta, \eta) \cap B(\gamma, \nu)$ and $\alpha > \nu$, $\alpha \notin \Lambda$ then obviously $B(\alpha, \nu) = \{\alpha\}$ works. Otherwise, we can find a basic open neighborhood of α inside the intersection by using (2). Just choose ξ greater than η and ν and large enough so that $B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]$ and also $B(\alpha, \xi) = B(\gamma) \cap (\xi, \alpha]$.

Next we show (2) by induction. Suppose it is false, and that β is the first ordinal for which there is a limit ordinal $\alpha < \beta$ where it fails. Clearly $\beta \in \Lambda$. Only successor ordinals are in $[0, \beta] \setminus B(\beta)$, so $\alpha \in B(\beta)$. Let $\alpha \in S(\beta, n) = B(\gamma, \nu)$. By minimality of β, there exists $\mu \geq \nu$ such that $B(\alpha, \mu) = B(\gamma, \nu) \cap (\mu, \alpha]$. However, $S(\beta, n)$ only contains finitely many members of L_β; once ξ is above all the members of L_β that are below α, we must have $B(\beta) \cap (\xi, \alpha] = B(\gamma, \nu) \cap (\xi, \alpha]$, and if $\xi \geq \mu$ then this equals $B(\alpha, \xi)$, a contradiction.

Finally, (3) is true by construction since α is the supremum of the points of M_α that have odd subscripts and also of the ones that have even subscripts. $\dashv$

Claim 2. (ω_1, τ) is locally compact.

$\vdash$ Obviously, $B(0) = \{0\}$ and $B(\xi + 1, \xi)$ are singletons for all successor ordinals $\xi + 1$. If $\beta \in \Lambda$, then, since $B(\beta)$ is countable, it is enough to show that $B(\beta)$ is countably compact.

Let A be an infinite subset of $B(\beta)$. Then A contains a strictly ascending sequence σ of ordinals. Let $sup(ran(\sigma)) = \alpha$. If $\alpha = \beta$ then $\sigma \to \alpha$ by (1), while if $\alpha < \beta$, then (1) and (2) have the same effect, implying that α is a limit point of A in $B(\beta)$. $\dashv$

Theorem 1.2 now follows from:

Theorem 4.5. *If $\mathcal{L}$ witnesses $\clubsuit$, then (ω_1, τ) is monotonically normal and ω_1-compact but not σ-countably compact.*

Proof. Recall the Borges criterion for monotone normality, Theorem 3.3:

There is an assignment of an open neighborhood $h(x, U) =: U_x$ containing x to each pair (x, U) such that U is an open neighborhood of x, and such that, if $U_x \cap V_y \neq \emptyset$, then either $x \in V$ or $y \in U$.

The choice of $h(z, U) = \{z\}$ for all isolated points z works for the case where either x or y is not a limit ordinal. So it is enough to take care of the case where x and y are both limit ordinals and $x < y$. Given $\alpha \in U$, let $U_\alpha = B(\alpha, \xi)$ for some ξ such that $B(\alpha, \xi) \subset U$. It follows from (2) that $x \in U_y$ whenever $U_x \cap U_y \neq \emptyset$.

It also follows from (2) that the τ-relative topology on Λ is its usual order topology. So, to show ω_1-compactness it is enough to show that every uncountable set S of successor ordinals has an ascending sequence $\langle s_n : n \in \omega \rangle$ that τ-converges to its supremum.

Let $R = \{\xi : \xi + 1 \in S\}$. Since $\mathcal{L}$ witnesses $\clubsuit$, there exists $\alpha \in R$ such that $L_\alpha \subset R$. Then $M_\alpha \subset S$ and the odd-numbered elements of M_α converge to α.

Finally, to show that (ω_1, τ) is not σ-countably compact, let $\omega_1 = \bigcup_{n=1}^{\infty} A_n$. It is clearly enough to show that any A_n that contains uncountably many successor ordinals is not countably compact, since there is at least one such A_n. Let one of these be S, and let R and M_α be as before. Then the even-numbered members of M_α are an infinite τ-closed discrete subset of ω_1. $\square$

5. Some ZFC results

For the following theorem, shown in [26], we do not even have to assume ω_1-compactness; it comes free of charge:

Theorem 5.1. *Every locally compact, locally connected and connected, monotonically normal space is σ-countably compact.*

Monotone normality has one strange feature that is not shared by normality and most of its natural strengthenings: it is often not preserved in taking the one-point compactification. [2] Mary Ellen Rudin gave the first example where this preservation fails, and in [17] two general kinds of examples are given. The following lemma makes it easy to show that all these examples, as well as our $\clubsuit$ examples, fail to have monotonically normal compactifications.

Lemma 5.2. *Let X be a locally compact space whose set C of nonisolated points is a copy of an ordinal of uncountable cofinality. If X has a monotonically normal one-point compactification, then C has a clopen, ω-bounded neighborhood.*

Proof. We use the original definition of monotone normality. Identify C with the ordinal θ of which it is a copy, and let $\Gamma = \theta + 1$, with the last point of $\theta + 1$ identified with the extra point ∞ in the one-point compactification of X. For each $\alpha \in C$, let $V_\alpha = G([0, \alpha], \Gamma \setminus [0, \alpha])$. Let $V = \bigcup_{\alpha \in \theta} V_\alpha$. Clearly, V is a neighborhood of C in X, and $V_\alpha \subset V_\beta$ whenever $\alpha < \beta$. Because $G([\Gamma \setminus [0, \alpha], [0, \alpha])$ is a neighborhood of ∞ disjoint from V_α, the latter has compact closure in X. From this it quickly follows that V is an ω-bounded clopen neighborhood of C. In fact, every union of fewer than $cf(\theta)$ compact subsets of V has compact closure. $\square$

The main theorem of this section (Theorem 5.10) will give an alternative proof that our $\clubsuit$ examples do not have monotonically normal one-point compactifications. The first step is to recall a remarkable and powerful theorem of Balogh and Rudin and one of its corollaries [2].

[2]Locally connected, locally compact spaces are an exception: monotone normality is preserved in the one-point compactification [17].

Proposition 5.3. *Let X be monotonically normal, and let $\mathcal{U}$ an open cover of X. Then $X = V \cup \bigcup \mathcal{E}$, where $\mathcal{E}$ is a discrete family of closed subspaces, each homeomorphic to a stationary subset of a regular uncountable cardinal, and $V = \bigcup \mathcal{V}$ is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.*

Corollary 5.4. *A monotonically normal space is paracompact if, and only if, it does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Lemma 5.5. *Let δ be an ordinal of uncountable cofinality, and let E be a locally compact stationary subset of δ. There is a tail (final segment) T which is a closed (hence club) subset of δ.*

Proof. Each $x \in E$ has an open neighborhood H_x which meets E in an interval $[\beta_x, x]$ whose least element is isolated in E; and if H_x has compact closure, then $[\beta_x, x]$ is compact. Then the PDL gives us a single β which works for a cofinal subset S of E, and so $\bigcup\{[\beta, x] : x \in S\}$ is a tail of E and a club in δ. □

Lemma 5.5 gives the strengthening of Corollary 5.4 for locally compact spaces that was used in the proof of Theorem 3.6:

Corollary 5.6. *A monotonically normal, locally compact space is paracompact if, and only if, it does not contain a closed copy of a regular uncountable cardinal.*

Proof. Necessity is immediate: every ordinal is a stationary subset of itself. Conversely, the T obtained in Lemma 5.5 has a closed subset that is a copy of the regular uncountable cardinal $cof(\delta)$. □

Example 5.7. *Let $\delta = \omega_1 \cdot \omega_1$, let $T = \{\omega_1 \cdot \xi : \xi \in \omega_1\}$, and let $E = \omega \cup T$. Then E is locally compact, and T is a tail of E that is a copy of ω_1 and a club in δ, but no tail of E is a tail of δ.*

In this example, E is homeomorphic to $A = (\omega_1 + \omega) \setminus \{\omega_1\}$ in its relative topology inherited from $\omega_1 + \omega$. Note that this relative topology is not equal to the order topology on A, i.e., the topology on A generated by the subbase $\{\alpha \cap A : \alpha \in A\} \cup \{A \setminus (\alpha + 1) : \alpha \in A\}$, because A with this topology is homeomorphic to $\omega_1 + \omega$ and hence is σ-compact, whereas A with the subspace topology is not Lindelöf. On the other hand, the two topologies coincide on any closed subset of any ordinal in its inherited order.

We are indebted to Heikki Junnila for the concise proof of the following theorem. Recall that a space is a *topological direct sum* of subspaces if these subspaces partition it into open (hence clopen) subspaces. A great many properties of topological spaces need only be verified for the summands to be true for the whole sum.

Theorem 5.8. *A subspace of an ordinal is locally compact if, and only if, it is a topological direct sum of countably compact copies of ordinals.*

Proof. Sufficiency is obvious. For the converse, let θ be an ordinal and let A be a locally compact subspace of θ. The closure $\overline{A}$ of A is a copy of an ordinal, and since A is locally compact, it is open in $\overline{A}$.

Working in $\overline{A}$, let $\mathcal{G}$ be the collection of convex components of A. These are the maximal intervals of A within $\overline{A}$. Each is a copy of an ordinal, and $\mathcal{G}$ is a partition of A into relatively clopen subsets. Let $\mathcal{G}_1 = \{G \in \mathcal{G} : cof(G) > \omega\}$. Each member of $\mathcal{G}_1$ is is countably compact. If $G \in \mathcal{G} \setminus \mathcal{G}_1$, then G has a partition into compact open intervals. Replacing each member of $\mathcal{G} \setminus \mathcal{G}_1$ by these partition members gives us the desired partition of A. $\square$

Next, we show a strengthening of Proposition 5.3 for locally compact spaces that makes use of 5.3 in an essential way.

Theorem 5.9. *Let $\mathcal{U}$ be an open cover of a locally compact monotonically normal space X. Then X has a discrete collection $\mathcal{C}$ of closed copies of regular uncountable cardinals, such that $X \setminus \bigcup \mathcal{C}$ has a σ-disjoint cover by open sets refining the trace of $\mathcal{U}$ on $X \setminus \bigcup \mathcal{C}$.*

Proof. Assume without loss of generality that $\overline{U}$ is compact for each $U \in \mathcal{U}$. Let $\mathcal{E} = \{E_\alpha : \alpha \in \kappa\}$ be a discrete collection of closed copies of stationary subsets of regular cardinals θ_α as in Property 5.3. For each $\alpha \in \kappa$, use Theorem 5.8 to break up each E_α into a discrete family $\mathcal{K}_\alpha \cup \mathcal{P}_\alpha$, where each $P \in \mathcal{P}_\alpha$ is a compact subset of some $U \in \mathcal{U}$ and each $K \in \mathcal{K}_\alpha$ is a copy of an ordinal of uncountable cofinality. Let $\mathcal{K} = \bigcup_{\alpha \in \kappa} \mathcal{K}_\alpha$.

Given a fixed $K \in \mathcal{K}$, let B be a set of successors (hence isolated points) in K that is cofinal in K and of order type $cof(K)$. Let C_K be the closure of B in K, so that C_K is homeomorphic and order-isomorphic to $cof(K)$. For each $\zeta \in B$, let $\beta(\zeta) = sup\{\xi \in C_K : \xi < \zeta\}$.

Then $K \setminus C_K$ breaks up into the intervals $(\beta(\zeta), \zeta)$. Each interval is compact: the maximum element is the immediate predecessor of ζ, and the minimum element is the immediate successor of $\beta(\zeta)$. So each interval has a finite partition $\mathcal{P}_K(\zeta)$ into clopen sets, each a subset of some $U \in \mathcal{U}$. Let $\mathcal{P}_K = \bigcup\{\mathcal{P}_K(\zeta) : \zeta \in B\}$.

Let $\mathcal{C} = \{C_K : K \in \mathcal{K}\}$ and $\mathcal{P} = \bigcup\{\mathcal{P}_K : K \in \mathcal{K}\} \cup \bigcup\{\mathcal{P}_\alpha : \alpha \in \kappa\}$. Then $\mathcal{P}$ is a relatively discrete partition of $(\bigcup \mathcal{E}) \setminus (\bigcup \mathcal{C})$ into compact sets. Every monotonically normal space is *collectionwise normal*, which means that every discrete collection of closed subsets can be expanded to a discrete collection of open sets; the proof is similar to that for the cwH property (see the discussion after Theorem 3.3). So $\mathcal{P}$ can be expanded to a disjoint collection $\mathcal{V}_\infty$ of open sets, each contained in a member of $\mathcal{U}$. Combine this with the $\mathcal{V}_n$ of Property 5.3 to get the desired σ-disjoint cover. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 5.10. *Every locally compact space with a monotonically normal one-point compactification is a topological direct sum of σ-ω-bounded spaces, each of which is the union of an open Lindelöf subset and of a discrete family of closed, ω-bounded subsets. If the space is totally disconnected, it is a topological direct sum of ω-bounded subspaces.*

Proof. Let X be monotonically normal and locally compact, with a monotonically normal one-point compactification. Let $\mathcal{C}$ be as in Theorem 5.9. For each C_α there is an open subset $W_\alpha \supset C_\alpha$ so that $\{W_\alpha : \alpha < \kappa\}$ is a discrete collection of open sets. Each C_α converges to the extra point ∞ in the one-point compactification of X, so there is an open set $V^\alpha \supset C_\alpha$ with ω-bounded closure, built similarly to V in the proof of Lemma 5.2. Do it so that $c\ell_X V^\alpha \subset W_\alpha$ by using $V^\alpha_\xi = G([0, \xi],\ \{\infty\} \cup (X \setminus W_\alpha) \setminus [0, \xi])$.

The complement of $\bigcup_{\alpha<\lambda} V^\alpha$ in X is a closed subset of $X \setminus \bigcup \mathcal{C}$, so it is paracompact. By the classical theorem cited earlier, every locally compact, paracompact space can be partitioned into a family of clopen Lindelöf subsets. If, in addition, the space is totally disconnected, it is zero-dimensional [10, 6.2.4] [3], and each of these Lindelöf subsets can be further partitioned into compact open subsets. In either case, let $\mathcal{H} = \{H_\alpha : \alpha \in \gamma\}$ be such a partition.

In the zero-dimensional case, the boundary of each V^α is partitioned into compact relatively open sets where it meets $\bigcup \mathcal{H}$. By countable compactness, there can be only finitely many such sets. Cover the boundary with finitely many compact open subsets of W_α. The resulting clopen expansions of the V^α are a discrete collection in X and the complement of their union can be partitioned into clopen sets as before. The resulting partition of X is as desired.

In the more general case, we still have the countably compact boundary of each V^α partitioned into X-closed subsets of Lindelöf sets. Hence it is compact here too, and only finitely many members of $\mathcal{H}$ meet it. Now, the following is clearly an equivalence relation for $\mathcal{K} = \mathcal{H} \cup \{V^\alpha : \alpha < \lambda\}$. Let $K_0 \approx K$ iff there is a finite chain $K_0, \dots K_n$ such that $c\ell_X(K_i) \cap c\ell_X(K_{i+1}) \neq \emptyset$ and $K_n = K$. The closure of each V^α meets that of only finitely many other members of $\mathcal{K}$, while the closure of each $H \in \mathcal{H}$ meets that of only countably many other members. The union of the members of each equivalence class is open, because the boundary of $K_0 \in \mathcal{K}$ is covered by its union with all the eligible candidates for K_1. The unions of members of each equivalence class give the desired partition into σ-ω-bounded subspaces. $\square$

This theorem solves Problem 2 of [17], which asked:

Question 5.11. *If a locally compact space has a monotonically normal one-point compactification, is it the topological direct sum of ω_1-compact subspaces?*

This question was independently answered by Heikki Junnila [private communication] but his proof made use of the monumentally difficult solution by Mary Ellen Rudin of Nikiel's conjecture [30]. The Balogh-Rudin theorem used here also has a very complicated proof, but nowhere near as complicated as the one for solving Nikiel's Conjecture.

Theorem 5.10 actually characterizes the class $\mathfrak{C}$ (denoted $\mathcal{LM}$ in [17]) of locally compact spaces with monotonically normal one-point compactifications. In the zero-dimensional case this follows immediately from the theorem [17, 4.2] that every countably compact, locally compact, monotonically normal space is in $\mathfrak{C}$ and the easy lemma that $\mathfrak{C}$ is closed under taking topological direct sums. The more general case is shown in the second paragraph in the proof of [17, 4.4], after all use of local connectedness is over.

6. The minimum cardinality theme

In the wake of the negative answer to Question 1.1, it is natural to ask:

Problem 2. *What is the minimum cardinality of a locally compact, ω_1-compact space that is not σ-countably compact?*

[3] In [10], totally disconnected spaces are called "hereditarily disconnected."

Problem 3. *What is the minimum cardinality of a locally compact, ω_1-compact normal space that is not σ-countably compact?*

One might also ask whether it is consistent for the minimum for Problem 3 to be strictly greater than the minimum for Problem 2. These are difficult problems, and each may need to be settled on a model-by-model basis. But we do have some general results. The following was shown by the first author [25]:

Theorem 6.1. *The least cardinality of a locally compact, normal, ω_1-compact space that is not σ-countably compact is no greater than $\mathfrak{b}$.*

Previously, the best upper bound for the minimum was $\mathfrak{c}$, using one of E.K. van Douwen's "honest submetrizable" examples [5]. Theorem 6.1 still leaves a lot unsaid. On the one hand, the following "echo" of Question 1.1 is made difficult by the fact [18] that PID implies $\mathfrak{b} \le \aleph_2$:

Problem 4. *Is there a ZFC example of a locally compact, ω_1-compact space of cardinality $\aleph_2$ that is not σ-countably compact? one that is normal?*

Of course, if $\mathfrak{b} = \aleph_2$, Theorem 6.1 gives a consistent example. But more simply, so does the topological direct sum of a compact space of cardinality $\aleph_2$ with any of the consistent examples of a locally compact, ω_1-compact space of cardinality $\aleph_1$ that is not σ-countably compact. Some additional such examples are given below.

The technique of proof of Theorem 2.4 only used the weakening of PID that has $|X| = \aleph_1$, designated $PID_{\aleph_1}$. Unfortunately, to make the same technique work for Problem 4, we would need $PID_{\aleph_2} + \mathfrak{b} > \aleph_2$, and that is not possible, as we next show. First, some notation and a definition.

Given a relation R on ω and $x, y \in \omega^\omega$, we denote the set $\{n \in \omega : x(n)\, R\, y(n)\}$ by $[xRy]$.

Definition 6.2. Let κ, λ be regular cardinals. A (κ,λ)-*pregap* $\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda}\rangle$ is a pair of transfinite sequences $\langle f_\alpha : \alpha < \kappa\rangle$ and $\langle g_\beta : \beta < \lambda\rangle$ of nondecreasing sequences f_α, g_β of natural numbers such that $f_{\alpha_1} \le^* f_{\alpha_2} \le^* g_{\beta_2} \le^* g_{\beta_1}$ for all $\alpha_1 \le \alpha_2 < \kappa$ and $\beta_1 \le \beta_2 < \lambda$. As usual, $f \le^* g$ means that the set $[f > g]$ is finite. A (κ,λ)-pregap is called a (κ,λ)-*gap*, if there is no $h \in \omega^\omega$ such that $f_\alpha \le^* h \le^* g_\beta$ for all α, β.

Lemma 6.3 is a slight improvement of [18, Lemma 1.12], which in its turn is modelled on the second paragraph of [34].

Lemma 6.3. *Suppose that there exists a (κ,λ)-gap such that κ, λ are regular uncountable.*

(1) *If $\kappa > \aleph_1$ then* PID_λ *fails.*
(2) *If $\lambda > \aleph_1$ then* PID_κ *fails.*

Proof. The proofs of the two parts of the lemma are essentially the same, but we present both of them for the sake of completeness.

1. Assume that $\kappa > \aleph_1$ but PID_λ holds. Fix a (κ,λ)-gap $\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda}\rangle$ and set

$$\mathcal{I} = \{A \in [\lambda]^{\aleph_0} \ : \ \exists \alpha \in \kappa\, \forall n \in \omega\, (|\{\beta \in A : [f_\alpha > g_\beta] \subset n\}| < \aleph_0)\}.$$

Note that if α witnesses $A \in \mathcal{I}$, then any $\alpha' > \alpha$ has also this property because $f_\alpha \le^* f_{\alpha'}$. We claim that $\mathcal{I}$ is a P-ideal. Indeed, let $\{A_i : i \in \omega\}$ be a sequence of elements of $\mathcal{I}$, α_i be a witness for $A_i \in \mathcal{I}$, and $\alpha = \sup\{\alpha_i : i \in \omega\}$. Then α witnesses $A_i \in \mathcal{I}$ for all $i \in \omega$. Let $B_i = \{\beta \in A_i : [f_\alpha > g_\beta] \subset i\}$. Then B_i is a finite subset of A_i by the definition of $\mathcal{I}$. Set $A = \bigcup_{i\in\omega} A_i \setminus B_i$ and fix $n \in \omega$. If $\beta \in A_i$ is such that $[f_\alpha > g_\beta] \subset n$ and $i \geq n$, then $\beta \in B_i$. Therefore

$$\{\beta \in A : [f_\alpha > g_\beta] \subset n\} \subset \{\beta \in \bigcup_{i<n} A_i : [f_\alpha > g_\beta] \subset n\},$$

and the latter set is finite by the definition of $\mathcal{I}$ and our choice of α.

Applying PID_λ to $\mathcal{I}$ we conclude that one of the following alternatives is true:

1a. There exists $S \in [\lambda]^{\aleph_1}$ such that $[S]^{\aleph_0} \subset \mathcal{I}$. Passing to an uncountable subset of S if necessary, we can assume that $S = \{\beta_\xi : \xi < \omega_1\}$ and $\beta_\xi < \beta_\eta$ for any $\xi < \eta < \omega_1$. For every ξ we denote by S_ξ the set $\{\beta_\zeta : \zeta < \xi\}$.

By the definition of $\mathcal{I}$ for every ξ there exists $\alpha_\xi \in \kappa$ witnessing for $S_\xi \in \mathcal{I}$. Let $\alpha = \sup\{\alpha_\xi : \xi < \omega_1\}$. There exists $n \in \omega$ such that the set $C = \{\xi < \omega_1 : [f_\alpha > g_{\beta_\xi}] \subset n\}$ is uncountable. Let ξ_0 be the ω-th element of C. Then $\alpha \geq \alpha_{\xi_0}$ and for all $\xi \in C \cap \xi_0$ we have $[f_\alpha > g_{\beta_\xi}] \subset n$. On the other hand, $S_{\xi_0} \in \mathcal{I}$, and hence there should be only finitely many such $\xi \in \xi_0$, a contradiction.

1b. $\lambda = \bigcup_{m\in\omega} S_m$ such that S_m is orthogonal to $\mathcal{I}$ for all $m \in \omega$. This means that for every $m \in \omega$ and $\alpha \in \kappa$ there exists $n_{m,\alpha} \in \omega$ such that $[f_\alpha > g_\beta] \subset n_{m,\alpha}$ for all $\beta \in S_m$. (If there is no such $n = n_{m,\alpha}$, construct a sequence $\langle \beta_n : n \in \omega\rangle \in S_m^\omega$ such that $[f_\alpha > g_{\beta_n}] \not\subset n$ and note that α witnesses $\{\beta_n : n \in \omega\} \in \mathcal{I}$, which is impossible because S_m is orthogonal to $\mathcal{I}$). Since λ is regular uncountable, there exists $m \in \omega$ such that S_m is cofinal in λ. Let $n \in \omega$ be such that the set $J = \{\alpha \in \kappa : n_{m,\alpha} = n\}$ is cofinal in κ. For every k let $h(k) = \max\{f_\alpha(k) : \alpha \in J\}$. From the above it follows that $[g_\beta < h] \subset n$ for all $\beta \in S_m$, and hence h contradicts the fact that $\langle \{f_\alpha\}_{\alpha\in J}, \{g_\beta\}_{\beta\in S_m}\rangle$ is a gap.

2. Assume that $\lambda > \aleph_1$ but PID_κ holds. Fix a (κ, λ)-gap $\langle \{f_\alpha\}_{\alpha<\kappa}, \{g_\beta\}_{\beta<\lambda}\rangle$ and set

$$\mathcal{I} = \{A \in [\kappa]^{\aleph_0} \ : \ \exists \beta \in \lambda\, \forall n \in \omega\, (|\{\alpha \in A : [f_\alpha > g_\beta] \subset n\}| < \aleph_0)\}.$$

Note that if β witnesses $A \in \mathcal{I}$, then any $\beta' > \beta$ has also this property because $f_{\beta'} \leq^* f_\beta$. We claim that $\mathcal{I}$ is a P-ideal. Indeed, let $\{A_i : i \in \omega\}$ be a sequence of elements of $\mathcal{I}$, β_i be a witness for $A_i \in \mathcal{I}$, and $\beta = \sup\{\beta_i : i \in \omega\}$. Then β witnesses $A_i \in \mathcal{I}$ for all $i \in \omega$. Let $B_i = \{\alpha \in A_i : [f_\alpha > g_\beta] \subset i\}$. Then B_i is a finite subset of A_i by the definition of $\mathcal{I}$. Set $A = \bigcup_{i\in\omega} A_i \setminus B_i$ and fix $n \in \omega$. If $\alpha \in A_i$ is such that $[f_\alpha > g_\beta] \subset n$ and $i \geq n$, then $\alpha \in B_i$. Therefore

$$\{\alpha \in A : [f_\alpha > g_\beta] \subset n\} \subset \{\alpha \in \bigcup_{i<n} A_i : [f_\alpha > g_\beta] \subset n\},$$

and the latter set is finite by the definition of $\mathcal{I}$ and our choice of β.

Applying PID_κ to $\mathcal{I}$ we conclude that one of the following alternatives is true:

2a. There exists $S \in [\kappa]^{\aleph_1}$ such that $[S]^{\aleph_0} \subset \mathcal{I}$. Passing to an uncountable subset of S if necessary, we can assume that $S = \{\alpha_\xi : \xi < \omega_1\}$ and $\alpha_\xi < \alpha_\eta$ for any $\xi < \eta < \omega_1$. For every ξ we denote by S_ξ the set $\{\alpha_\zeta : \zeta < \xi\}$.

By the definition of $\mathcal{I}$ for every ξ there exists $\beta_\xi \in \lambda$ witnessing for $S_\xi \in \mathcal{I}$. Let $\beta = \sup\{\beta_\xi : \xi < \omega_1\}$ and note that it witnesses $S_\xi \in \mathcal{I}$ for all $\xi < \omega_1$. On the other hand, there exists $n \in \omega$ such that the set $C = \{\xi < \omega_1 : [f_{\alpha_\xi} > g_\beta] \subset n\}$ is uncountable. Let ξ_0 be the ω-th element of C. Then β fails to witness $S_{\xi_0} \in \mathcal{I}$, a contradiction.

2b. $\kappa = \bigcup_{m\in\omega} S_m$ such that S_m is orthogonal to $\mathcal{I}$ for all $m \in \omega$. This means that for every $m \in \omega$ and $\beta \in \lambda$ there exists $n_{m,\beta} \in \omega$ such that $[f_\alpha > g_\beta] \subset n_{m,\beta}$ for all $\alpha \in S_m$. (If there is no such $n = n_{m,\beta}$, construct a sequence $\langle \alpha_n : n \in \omega\rangle \in S_m^\omega$ such that $[f_{\alpha_n} > g_\beta] \not\subset n$ and note that β witnesses $\{\alpha_n : n \in \omega\} \in \mathcal{I}$, which is impossible because S_m is orthogonal to $\mathcal{I}$). Since κ is regular uncountable, there exists $m \in \omega$ such that S_m is cofinal in κ. Let $n \in \omega$ be such that the set $J = \{\beta \in \lambda : n_{m,\beta} = n\}$ is cofinal in λ. For every k let $h(k) = \min\{g_\beta(k) : \beta \in J\}$. From the above it follows that $[f_\alpha > h] \subset n$ for all $\alpha \in S_m$, and hence h contradicts the fact that $\langle \{f_\alpha\}_{\alpha\in S_m}, \{g_\beta\}_{\beta\in J}\rangle$ is a gap.

□

The proof of the next lemma can be found in [14, page 578].

Lemma 6.4. *If $\mathfrak{b} > \aleph_2$ then there is an $(\aleph_2, \lambda)$-gap for some uncountable regular λ.*

As a corollary we get the following result usually attributed to Todorčević.

Theorem 6.5. $\mathrm{PID}_{\aleph_2}$ *implies* $\mathfrak{b} \le \aleph_2$.

Proof. Suppose that $\mathfrak{b} > \aleph_2$ and nevertheless $\mathrm{PID}_{\aleph_2}$ holds. By Lemma 6.4 there is an $(\aleph_2, \lambda)$-gap for some uncountable regular λ. If $\lambda = \aleph_1$, then $\mathrm{PID}_{\aleph_1}$ fails by Lemma 6.3(1). If $\lambda > \aleph_1$, then $\mathrm{PID}_{\aleph_2}$ fails by Lemma 6.3(2). □

Clearly, new ideas for the solution of Problem 4 are needed!

At the opposite extreme from Problem 4, we have:

Problem 5. *Can $\mathfrak{b}$ be "arbitrarily large" and still be the minimum cardinality of a locally compact, ω_1-compact space that is not σ-countably compact? of one that is also normal?*

The question of what happens under Martin's Axiom (MA) is especially interesting since it implies $\mathfrak{b} = \mathfrak{c}$. It also implies that ♣ fails and that there are no Souslin trees. Now a Souslin tree with the interval topology is of cardinality $\aleph_1$, and is locally compact, ω_1-compact and hereditarily collectionwise normal [21, 4.18] (hence hereditarily normal and hereditarily strongly cwH) but is not σ-countably compact.

To show ω_1-compactness, use the fact that every closed discrete subspace in a tree with the interval topology is a countable union of antichains, and the fact that a Souslin tree is, by definition, an uncountable tree in which every chain and antichain is countable. To show that a Souslin tree is not the union of countably many countably compact subspaces, use these facts together with the fact that at least one subspace would have to be uncountable, and the observation that the Erdős - Radó theorem implies that every uncountable tree must either contain an infinite antichain or an uncountable chain. [The interval topology on a subtree is not always the relative topology, but the relative topology is finer, so new infinite closed discrete subspaces could arise.]

It is worth noting here that adding a single Cohen real adds a Souslin tree but leaves $\mathfrak{b}$ as the same aleph that it is in the ground model. So $\mathfrak{b}$ can be arbitrarily large while the minimum cardinality for Problems 2 and 3 is $\aleph_1$.

A Souslin tree is not monotonically normal because every monotonically normal tree is a topological direct sum of copies of ordinals [22], and the following problem is still unsolved:

Problem 6. *Is the existence of a Souslin tree enough to produce a locally compact, monotonically normal, ω_1-compact space that is not σ-countably compact?*

Most independence results pertaining to monotonically normal spaces revolve around whether a Souslin tree (equivalently, a Souslin line) exists. However, Problem 6 suggests that this may not be the case with the question of which models have a locally compact, monotonically normal, ω_1-compact space that is not σ-countably compact. This would already be true if there is a model of $\clubsuit$ where there are no Souslin trees [the model in [9] turned out to be flawed].

7. A maximum cardinality theme

Since most of the examples mentioned are locally countable, it is natural to inquire what happens if local countability is added to local compactness and ω_1-compactness. The following very simple problem is a dramatic counterpoint to Problem 1.

Problem 7. Is there a ZFC example of a normal, locally countable, countably compact space of cardinality greater than $\aleph_1$?

Although this problem does not mention local compactness, that is easily implied by normality (indeed regularity), local countability, and countable compactness.

In a paper nearing completion, [27], the first author shows that there does exist a consistent example of a locally countable, normal, ω-bounded (hence countably compact) space of cardinality $\aleph_2$, under the axiom $\square_{\aleph_1}$. This axiom is consistent if ZFC is consistent, and the equiconsistency strength of its negation is that of a Mahlo cardinal [14, Exercise 27.2]. But the following question, a counterpoint to Question 1.1, is completely open — no consistency results are known either way:

Problem 8. Is there a normal, locally countable, countably compact space of cardinality greater than $\aleph_2$?

Of course, one could also ask whether there is an upper bound on the cardinalities of normal, locally countable, countably compact spaces. If "countably compact" is strengthened to "ω-bounded" then it is consistent, modulo large cardinals, that there is an upper bound. In fact, it has long been known that every regular, locally countable, ω-bounded space is of cardinality $< \aleph_\omega$ if the Chang Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$ holds [16]. It has recently been shown by Monroe Eskew and Yair Hayut [11] that the consistency of a supercompact cardinal implies that this variant is consistent.

On the other hand, it has long been known that there are examples of regular, locally countable, ω-bounded spaces of cardinality $\aleph_n$ for all finite n just from ZFC, as was shown in [15]. However, the construction in [15] is so general that it could even produce non-normal closed subspaces of cardinality $\aleph_1$, and it is difficult to see how these can be avoided in ZFC without many additional details, if at all.

We conjecture that it is consistent, modulo large cardinals, that every normal, locally compact, countably compact space is of cardinality $< \aleph_\omega$, leaving only the consistency of examples of cardinality $\aleph_n$ unknown for $3 \leq n < \omega$. This is because of the [16] result for the ω-bounded case and the following lemma:

Lemma 7.1. *The PFA implies every normal, locally countable, countably compact space is ω-bounded.*

Proof. This follows quickly from three consequences of the PFA:

(1) every first countable, countably compact space is either compact or contains a copy of ω_1 [6] and

(2) the PFA implies $\mathfrak{p} = \aleph_2$, and:

(3) *If $\mathfrak{p} > \aleph_1$, no separable, normal, first countable, countably compact space can contain a perfect preimage of ω_1.* [19, 3.2].

Now, every locally countable, countably compact regular space is first countable, and (3) implies that if such a space is normal, no countable subset can have a copy of ω_1 in its closure. And now (1) implies that the closure of every countable subspace is compact, meaning that the space is ω-bounded. □

Unfortunately, it is not known whether (1) and (3) are compatible with the Chang Conjecture variant.

8. Locally compact, quasi-perfect preimages

The examples that made Theorem 6.1 true are the first nontrivial examples for the following problem by van Douwen [4].

Problem 9. *Is ZFC enough to imply that each first countable regular space of cardinality at most $\mathfrak{c}$ is a quasi-perfect image of some locally compact space?*

Definition 8.1. A continuous map : $X \to Y$ is *quasi-perfect* if it is surjective, and closed, and each fiber $f^{\leftarrow}\{x\}$ is countably compact.

The following theorem was the key to Theorem 6.1.

Theorem 8.2. [25] *Let E be a stationary, co-stationary subset of ω_1. There is a locally compact, normal, quasi-perfect preimage of E, of cardinality $\mathfrak{b}$. If $\mathfrak{b} = \aleph_1$, then this preimage can also be locally countable, hence first countable.*

Theorem 6.1 follows from this theorem and from the following simple facts.

Theorem 8.3. *Let $f : Y \to X$ be a continuous surjective map.*

(i) *If X is not σ-countably compact, neither is Y.*

(ii) *If X is ω_1-compact, and f is closed, and each fiber $f^{\leftarrow}\{x\}$ is ω_1-compact, then Y is ω_1-compact.*

Proof. Statement (i) easily follows by contrapositive and the elementary fact that the continuous image of a countably compact space is countably compact.

To show (ii), let A an uncountable subset of Y. Then either A meets some fiber in an uncountable subset, in which case it is not closed discrete in Y, or $f^{\rightarrow}[A]$ is uncountable and so it is not closed discrete. Let B be a subset of $f^{\rightarrow}[A]$ that is not closed, and let $p \in \overline{B} \setminus B$.

Let $A_0 = \{x_b : b \in B\}$ be a subset of A such that $f(x_b) = b$ for all $b \in B$. Because f is closed, and B is not closed, neither is A_0 and, in fact, it has an accumulation point in $f^{\leftarrow}\{p\}$. □

Theorem 6.1 now follows easily. If E is co-stationary, then all countably compact subsets of E, being closed, are countable. So E cannot be σ-countably compact unless it is countable. And if E is stationary, it has limit points in the closure of every uncountable subset of ω_1, and so it is ω_1-compact. It therefore follows from Theorem 8.3 that any quasi-perfect preimage of E is ω_1-compact, but not σ-countably compact.

Problem 9 was motivated by a theorem in 13.4 of [4], which stated that the preimages it asks for do exist if $\mathfrak{b} = \mathfrak{c}$. The preimages van Douwen constructed were locally countable. But it is also consistent that some of them are not normal:

Theorem 8.4. *Let X be a locally compact, locally countable, quasi-perfect preimage of the space $\mathbb{P}$ of irrationals with the relative topology. If the PFA holds, then X is not normal.*

Proof. Let $p \in \mathbb{P}$ and let $\pi : X \to \mathbb{P}$ be a surjective quasi-perfect map. Let $\{\sigma_\alpha : \alpha < \omega_1\}$ be a family of injective sequences in X with disjoint images that converge to p. Let A_α be the set of all limit points of σ_α. Clearly $A_\alpha \subset \pi^{\leftarrow}\{p\}$, so $A_\alpha \cup ran(\sigma_\alpha)$ is a separable, uncountable, closed, countably compact, noncompact subspace of X.

Case 1. A_α is uncountable for some α. Then $ran(\sigma_\alpha)$ is a countable subspace of $A_\alpha \cup ran(\sigma_\alpha)$ that does not have compact closure, but now Lemma 7.1 shows that $A_\alpha \cup ran(\sigma_\alpha)$, and hence X, is not normal.

Case 2. A_α is countable for all α. This case only requires CH to fail. By induction, build sequences $\tau_\alpha (\alpha < \omega_1)$ in X whose π-images converge to p, and such that $ran(\sigma_\beta) \subset^* ran(\tau_\alpha)$ for all $\beta < \alpha$. If some τ_α has uncountably many limit points, argue as in Case 1. Otherwise, the set C_α of limit points of each τ_α is compact and countable, so it is contained in a countable, compact, open subset V_α of X. Let $V = \bigcup_{\alpha<\omega_1} V_\alpha$. Since the C_α form an increasing chain, their union C is ω-bounded, hence countably compact, hence closed in X by first countability of X. However, inasmuch as the σ_α have disjoint ranges, $\pi[N \setminus C]$ is uncountable for any neighborhood N of C. And so, the π-image of every closed neighborhood of C is an uncountable, closed subset of $\mathbb{P}$; hence it must be of cardinality $\mathfrak{c}$. But $|V| = \aleph_1$, so V cannot contain a closed neighborhood of C, contradicting normality. $\square$

The italicized consequence of $\mathfrak{p} > \aleph_1$ in the proof of Lemma 7.1 is actually equivalent to it. The familiar Franklin-Rajagopalan space $\gamma\mathbb{N}$ has a countable dense set of isolated points, and is normal, locally compact, and locally countable, hence first countable. It can be made countably compact if, and only if, $\mathfrak{p} = \aleph_1$. In [20] there is an extended treatment of when $\gamma\mathbb{N}$ can be hereditarily normal. Theorem 3.6 excludes $MM(S)[S]$ models, but we do not know whether these models can be substituted for the PFA in Theorem 8.4. In these models, $\mathfrak{p} = \aleph_1$ while $\mathfrak{b} = \mathfrak{c} = \aleph_2$.

In the absence of $\mathfrak{b} = \mathfrak{c}$, we still have very little idea of which first countable spaces have locally compact quasi-perfect preimages. It is even unknown whether there is a model of $\mathfrak{b} < \mathfrak{c}$ in which the space $\mathbb{P}$ of irrationals has such a preimage, or whether there is a model in which it does not have one. It is also unknown whether there is a locally compact, locally countable, quasi-perfect preimage of $[0,1]$ if $\mathfrak{b} < \mathfrak{c}$. Such a space would solve the following problem:

Problem 10. Is there a ZFC example of a scattered, countably compact, regular space that can be mapped continuously onto $[0,1]$?

See [24] for discussion of this problem, including an explanation why the answer is affirmative if "regular" is omitted.

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