

# LOCALLY COMPACT, $\omega_1$ -COMPACT SPACES

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ABSTRACT. An  $\omega_1$ -compact space is a space in which every closed discrete subspace is countable. We give various general conditions under which a locally compact,  $\omega_1$ -compact space is  $\sigma$ -countably compact, i.e., the union of countably many countably compact spaces. These conditions involve very elementary properties.

Many results shown here are independent of the usual (ZFC) axioms of set theory, and the consistency of some may involve large cardinals. For example, it is independent of the ZFC axioms whether a locally compact,  $\omega_1$ -compact space of cardinality  $\aleph_1$  is  $\sigma$ -countably compact. Modulo large cardinals, it is also ZFC-independent whether every hereditarily normal, or every monotonically normal, locally compact,  $\omega_1$ -compact space is  $\sigma$ -countably compact.

As a result, it is also ZFC-independent whether there is a locally compact,  $\omega_1$ -compact Dowker space of cardinality  $\aleph_1$ , or one that does not contain both an uncountable closed discrete subspace and a copy of the ordinal space  $\omega_1$ .

On the other hand, it is a theorem of ZFC that every locally compact, locally connected and connected, monotonically normal space is  $\sigma$ -countably compact. More generally, every locally compact space with a monotonically normal compactification is the topological direct sum of  $\sigma$ -countably compact spaces; and if it is totally disconnected, every summand can be made countably compact.

Set theoretic tools used for the consistency results include  $\clubsuit$ , the Proper Forcing Axiom (PFA), and models generically referred to as “MM(S)[S]”. Most of the work is done by the P-Ideal Dichotomy (PID) axiom, which holds in the latter two cases, and which requires no large cardinal axioms when directly applied to topological spaces of cardinality  $\aleph_1$ , as it is in several theorems.

## 1. INTRODUCTION

A *space of countable extent*, also called an  $\omega_1$ -compact space, is one in which every closed discrete subspace is countable. Obviously, every  $\sigma$ -countably compact space [i.e., the union of countably many countably compact spaces] is  $\omega_1$ -compact. On the other hand, an elementary application of the Baire Category Theorem shows that the space of irrational numbers with the usual topology is not  $\sigma$ -countably compact, but like every other separable metrizable space, it is  $\omega_1$ -compact.

The situation is very different when it comes to locally compact spaces. In an earlier version of this paper due to the first author, he asked:

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**Question 1.1.** *Is there a ZFC example of a locally compact,  $\omega_1$ -compact space of cardinality  $\aleph_1$  that is not  $\sigma$ -countably compact? one that is normal?*

Here too, local compactness makes a big difference: without it, the space of irrationals is a counterexample under CH, while ZFC is enough to show any cardinality  $\aleph_1$  subset of a Bernstein set is a counterexample.

As it is, the second author showed that the answer to Question 1.1 is negative; see Section 2. On the other hand, both the Kunen line and a Souslin tree with the usual topology are consistent normal examples for Question 1.1. In Section 4 we will also show:

**Theorem 1.2.** *Assuming  $\clubsuit$ , there is a locally compact, monotonically normal, locally countable (hence first countable)  $\omega_1$ -compact space of cardinality  $\aleph_1$  which is not  $\sigma$ -countably compact.*

Monotonically normal spaces are, informally speaking, “uniformly normal” [see Definition 3.2 below]. They are hereditarily normal, and this theorem gives another independence result when combined with:

**Theorem 1.3.** *In  $MM(S)[S]$  models, every hereditarily normal, locally compact,  $\omega_1$ -compact space is  $\sigma$ -countably compact.*

An even stronger theorem will be shown in Section 3 along with related results under weaker set-theoretic hypotheses. These will put some limitations on what kinds of Dowker spaces (that is, normal spaces  $X$  such that  $X \times [0, 1]$  is not normal) are possible if one only assumes the usual (ZFC) axioms of set theory. The ZFC axioms do, however, suffice to imply  $\sigma$ -countable compactness for some classes of spaces; see Section 5.

In the light of the negative answer to Question 1.1, it is natural to ask for the least cardinality of a locally compact,  $\omega_1$ -compact space which is not  $\sigma$ -countably compact. This is discussed in Section 6, and a new upper bound of  $\mathfrak{b}$  for this cardinality is given. The last section gives more information of the ZFC example behind this upper bound, and about a problem and a result of Eric van Douwen, under the assumption of  $\mathfrak{b} = \mathfrak{c}$ , that are related to this example.

The individual sections are only loosely connected with each other, and each can be read with minimal reliance on any of the others.

All through this paper, “space” means “Hausdorff topological space.” All of the spaces described are locally compact, hence Tychonoff; and all are also normal, except for a consistent example at the end.

## 2. THE CARDINALITY $\aleph_1$ CASE

The P-Ideal Dichotomy (PID) plays a key role in this section and the following one. It has to do with the following concept. A *P-ideal of countable sets* is a family  $\mathcal{P}$  of sets such that, for every countable subfamily  $\mathcal{Q}$  of  $\mathcal{P}$ , there exists  $P \in \mathcal{P}$  such that  $Q \subset^* P$  for every  $Q \in \mathcal{Q}$ . Here  $Q \subset^* P$  means that  $Q \setminus P$  is finite.

The PID states that, for every P-ideal  $\mathcal{I}$  of subsets of a set  $X$ , either

(A) there is an uncountable  $A \subset X$  such that every countable subset of  $A$  is in  $\mathcal{I}$ , or

(B)  $X$  is the union of countably many sets  $\{B_n : n \in \omega\}$  such that  $B_n \cap I$  is finite for all  $n$  and all  $I \in \mathcal{I}$ .

The routine proofs of the next lemma and theorem were given in [11]:

**Lemma 2.1.** *Let  $X$  be a locally compact Hausdorff space. The countable closed discrete subspaces of  $X$  form a  $P$ -ideal if, and only if, the extra point  $\infty$  of the one-point compactification  $X + 1$  of  $X$  is an  $\alpha_1$ -point; that is, whenever  $\{\sigma_n : n \in \omega\}$  is a countable family of sequences converging to  $\infty$ , then there is a sequence  $\sigma$  converging to  $\infty$  such that  $\text{ran}(\sigma_n) \subseteq^* \text{ran}(\sigma)$  for all  $n$ .*

The key is that an ordinary sequence in  $X$  converges to the extra point of  $X + 1$  if, and only if, its range is a closed discrete subspace of  $X$ .

**Theorem 2.2.** *Assume the PID axiom. Let  $X$  be a locally compact space. Then at least one of the following is true:*

(1<sup>-</sup>)  $X$  is the union of countably many subspaces  $Y_n$  such that each sequence in  $Y_n$  has a limit point in  $X$ .

(2)  $X$  has an uncountable closed discrete subspace

(3<sup>+</sup>) The extra point of  $X + 1$  is not an  $\alpha_1$ -point.

The key here is that (A) goes with (2), (B) goes with (1<sup>-</sup>), and (3<sup>+</sup>) is equivalent to the countable closed subspaces failing to form a  $P$ -ideal, by Lemma 2.1.

The following is well known:

**Lemma 2.3.** *If  $X$  is a space of character  $< \mathfrak{b}$  then every point of  $X$  is an  $\alpha_1$ -point.*

We now have a negative answer to the second part of Question 1.1.

**Theorem 2.4.** *Assume the PID and  $\mathfrak{b} > \aleph_1$ . Then every locally compact,  $\omega_1$ -compact, normal space of cardinality  $\aleph_1$  is  $\sigma$ -countably compact.*

*Proof.* In a locally compact space, character  $\leq$  cardinality. Lemmas 2.1 and 2.3 and  $\omega_1$ -compactness give us alternative (1<sup>-</sup>) Theorem 2.2. The closure of each  $Y_n$  is easily seen to be pseudocompact (i.e., every continuous real-valued function is bounded). In a normal space, every closed pseudocompact subspace is countably compact, cf. [28, 17J 3]. So the closures of the  $Y_n$  witness that  $X$  is  $\sigma$ -countably compact.  $\square$

As shown in [11], the hypothesis of normality in Theorem 2.4 can be greatly weakened to “Property wD”. Also, the proof of Theorem 2.4 clearly extends to show that every locally compact,  $\omega_1$ -compact space of character  $< \mathfrak{b}$  is  $\sigma$ -countably compact. However, this may be a very limited improvement: the PID implies  $\mathfrak{b} \leq \aleph_2$ . This is a theorem of Todorćević, whose proof may be found in [14].

The axioms used in Theorem 2.4 follow from the Proper Forcing Axiom ( $PFA$ ) and hold in  $PFA(S)[S]$  models. Each of these models is formed from a  $PFA(S)$  model by forcing with a coherent Souslin tree  $S$  that is part of the definition of what it means to be a  $PFA(S)$  model. The rest of the definition states that every proper poset  $P$  that does not destroy  $S$  when it is forced with, has the following property. For every family of  $\leq \aleph_1$   $\uparrow$ -dense,  $\uparrow$ -open sets, there

is a  $\downarrow$ -closed,  $\uparrow$ -directed subset of  $P$  that meets them all.<sup>1</sup> The PFA is similarly defined by omitting all mention of  $S$ . What remains is very similar to the well-known definition of Martin's Axiom ( $MA$ ); the only difference is that  $MA$  uses “ccc” instead of “proper.”

In this paper, we will use a slight abuse of language with expressions like  $PFA(S)[S]$  and  $MM(S)[S]$  as though they were axioms. The latter is defined like the former, but with “semi-proper” in place of “proper.”

For our negative answer to the first half of Question 1.1, we needed a strengthening of  $\mathfrak{b} > \aleph_1$  to  $\mathfrak{p} > \aleph_1$ . [It is a theorem of ZFC that  $\mathfrak{p} \leq \mathfrak{b}$ .] In  $PFA(S)[S]$  models,  $\mathfrak{p} = \aleph_1$  while  $\mathfrak{b} = \aleph_2$ . On the other hand, the PFA implies both the PID and  $\mathfrak{p} > \aleph_1$ . And this is enough for our negative answer (Theorem 2.6, below).

The standard definition of the cardinal  $\mathfrak{p}$  is used directly in the proof of the key lemma for Theorem 2.6. It involves families of subsets of  $\omega$  (hence of any denumerable set) with what is known as the Strong Finite Intersection Property (SFIP). This states that every finite subfamily has infinite intersection.

$$\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^\omega \text{ and } \mathcal{F} \text{ has the SFIP but } \neg \exists A \in [\omega]^\omega (A \subset^* F \text{ for all } F \in \mathcal{F})\}$$

**Lemma 2.5.** *Let  $X$  be a  $T_3$  space and let  $Y \subset X$ . Suppose that  $|\overline{Y}| < \mathfrak{p}$ , and that no  $Z \in [Y]^\omega$  is closed discrete in  $X$ . Then there exists a countably compact  $Y' \subset X$  containing  $Y$ .*

*Proof.* For every  $Z \in [Y]^\omega$  let  $x_Z$  be an accumulation point of  $Z$ . Let  $Y' = Y \cup \{x_Z : Z \in [Y]^\omega\}$ . We claim that  $Y'$  is countably compact. Indeed, otherwise there exists a countable  $T = \{t_n : n \in \omega\} \subset Y'$  which is closed discrete in  $Y'$ . If  $Z = T \cap Y$  then  $Z$  is finite, otherwise  $x_Z$  would be an accumulation point of  $T$  in  $Y'$ . So we may assume that  $T \cap Y = \emptyset$ . For every  $n$  fix  $Z_n \in [Y]^\omega$  such that  $t_n = x_{Z_n}$ . Let  $W_n \ni t_n$  be a neighborhood of  $t_n$  in  $\overline{Y}$  such that  $W_n \cap W_m = \emptyset$  for all  $n \neq m$ . [In a  $T_3$  space, every countable discrete subspace extends to a disjoint open collection.]

For every  $y \in Y' \setminus T$  find an neighborhood  $U_y$  of  $y$  in  $\overline{Y}$  such that  $\overline{U_y} \cap T = \emptyset$ . For every  $n \in \omega$ , let  $\mathcal{Z}_n = \{(Z_n \cap W_n) \setminus (\overline{U_y}) : y \in Y' \setminus T\}$ . Note that  $\mathcal{Z}_n$  has the SFIP. Since  $|\mathcal{Z}_n| < \mathfrak{p}$ , there exists an infinite  $Z'_n = \{z'_k : k \in \omega\} \subset Z_n \cap W_n$  such that  $Z'_n \subset^* Z$  for any  $Z \in \mathcal{Z}_n$ . Then  $\sigma_n = \langle z'_k : k \in \omega \rangle$  converges to  $t_n$ , because the range of every subsequence of  $\sigma_n$  has some  $x_A \in Y'$  as a limit point, but there is nothing for  $x_A$  to be except  $t_n$ .

Let  $f_y : \omega \rightarrow \omega$  be such that  $\{z'_k : k \geq f_y(n)\} \cap \overline{U_y} = \emptyset$ . Since  $|\overline{Y}| < \mathfrak{b}$ , there exists  $f : \omega \rightarrow \omega$  such that  $f_y <^* f$  for all  $y \in Y' \setminus T$ . Now  $Z_\omega = \{z'_{f(n)} : n \in \omega\} \subset Y$  has no accumulation points in  $T$ , because  $Z_\omega \cap W_n = \{z'_{f(n)}\}$ . Also,  $Z_\omega \cap U_y$  is finite for each  $y \in Y'$ , so  $x_{Z_\omega}$  does not exist. This contradiction implies that  $Y'$  is countably compact.  $\square$

We can weaken the hypothesis in the statement of Lemma 2.5 by having  $\overline{Y}$  be hereditarily of Lindelöf degree  $< \mathfrak{p}$ . That is, if  $S \subset \overline{Y}$ , and  $\mathcal{U}$  is a relatively open cover of  $S$ , then  $\mathcal{U}$  has a subcover of cardinality  $< \mathfrak{p}$ . Applying this to  $S = Y' \setminus T$  and  $\mathcal{U} = \{U_y : y \in S\}$ , a subcover of cardinality  $< \mathfrak{p}$  is enough to make the argument go through.

<sup>1</sup>The notation is as in [12]. It is the “Israeli” notation, whereby stronger conditions are larger. The topology to which it refers is the one where each  $p \in P$  has the one basic neighborhood  $\{t : t \geq p\}$ .

Now we can finish answering Question 1.1.

**Theorem 2.6.** *Assume the PID and  $\mathfrak{p} > \aleph_1$ . Then every locally compact,  $\omega_1$ -compact space of cardinality  $\aleph_1$  is  $\sigma$ -countably compact.*

*Proof.* Again Lemmas 2.1 and 2.3 and  $\omega_1$ -compactness give us alternative (1<sup>-</sup>) of Theorem 2.2. The rest is clear from Lemma 2.5.  $\square$

Returning to Theorem 2.4, its proof also gives:

**Theorem 2.7.** *Assume the PID. Then every locally compact,  $\omega_1$ -compact normal space of cardinality  $< \mathfrak{b}$  is countably paracompact.*

*Proof.* A normal space  $X$  is countably paracompact if, and only if, for each descending sequence of closed sets  $\langle F_n \rangle_{n=1}^\infty$  with empty intersection, there is a sequence of open sets  $\langle U_n \rangle_{n=1}^\infty$  with empty intersection, with  $F_n \subset U_n$  for all  $n$ . If  $X$  is a countable union of countably compact subsets  $C_m$ , as in Theorem 2.4, then in such a sequence of closed sets  $F_n$ , we can only have  $F_n \cap C_m \neq \emptyset$  for finitely many  $n$ . [Otherwise, countable compactness of  $C_m$  implies  $\bigcap_{n=1}^\infty C_m \cap F_n \neq \emptyset$ .] In any Tychonoff space, every pseudocompact subspace, and hence every countably compact subspace, has pseudocompact closure, and every normal, pseudocompact space is countably compact; and so the complements of the sets  $\overline{C_m}$  form the desired sequence of open sets.  $\square$

The equivalence in the preceding proof is shown in [28, 21.3] and is due to Dowker, who also showed its equivalence with  $X \times [0, 1]$  being normal. In honor of his pioneering work, normal spaces that are *not* countably paracompact are called “Dowker spaces.” Theorem 2.7 thus implies the consistency of there being no locally compact,  $\omega_1$ -compact Dowker spaces of cardinality  $\aleph_1$ . Specialized though this fact is, it is one of the few theorems as to what kinds of Dowker spaces are unattainable in ZFC. Another interesting such result was obtained recently by Tall [25]:

**Theorem 2.8.** *If  $PFA(S)[S]$ , then every locally compact non-paracompact space of Lindelöf number  $\leq \aleph_1$  includes a perfect preimage of  $\omega_1$ .*

Alan Dow has improved “perfect preimage” to “copy” [private communication]. Combining this with Theorem 2.7, we have:

**Corollary 2.9.** *If  $PFA(S)[S]$ , then every locally compact Dowker space of cardinality  $\leq \aleph_1$  includes both a copy of  $\omega_1$  and an uncountable closed discrete subspace.*

The consistency of PID was shown using forcing from a ground model with a supercompact cardinal. There are versions for spaces of weight  $\aleph_1$ , hence all locally compact spaces of cardinality  $\aleph_1$ , which require only the consistency of ZFC. One restricted version of the PID axiom is designated (\*) in [1], and is adequate for Theorem 2.7. But it is still an open problem whether the main results of our next section are ZFC-equiconsistent.

### 3. WHEN HEREDITARY NORMALITY IMPLIES $\sigma$ -COUNTABLE COMPACTNESS

For the main theorem of this section, we recall the following concepts:

**Definition 3.1.** Given a subset  $D$  of a set  $X$ , an *expansion* of  $D$  is a family  $\{U_d : d \in D\}$  of subsets of  $X$  such that  $U_d \cap D = d$  for all  $d \in D$ . A space  $X$  is [*strongly*] *collectionwise Hausdorff* (abbreviated [*s*]cwH) if every closed discrete subspace has an expansion to a disjoint [*resp.* discrete] collection of open sets.

The properties of  $\omega_1$ -[*s*]cwH only require taking care of those  $D$  that are of cardinality  $\leq \omega_1$ .

A well-known, almost trivial fact is that every normal, cwH space is scwH: if  $D$  and  $\{U_d : d \in D\}$  are as in 3.1, let  $V$  be an open set containing  $D$  whose closure is in  $\bigcup\{U_d : d \in D\}$ ; then  $\{U_d \cap V : d \in D\}$  is a discrete open expansion of  $D$ .

The following class of normal spaces plays a big role in this section and in the following one.

**Definition 3.2.** A space  $X$  is *monotonically normal* if there is an operator  $G(-, -)$  assigning to each ordered pair  $\langle F_0, F_1 \rangle$  of disjoint closed subsets an open set  $G(F_0, F_1)$  such that

- (a)  $F_0 \subset G(F_0, F_1)$
- (b) If  $F_0 \subset F'_0$  and  $F'_1 \subset F_1$  then  $G(F_0, F_1) \subset G(F'_0, F'_1)$
- (c)  $G(F_0, F_1) \cap G(F_1, F_0) = \emptyset$ .

Monotone normality is a hereditary property; that is, every subspace inherits the property. This is not so apparent from this definition, but it follows almost immediately from the following criterion, due to Borges.

**Theorem 3.3.** *A space is monotonically normal if, and only if, there is an assignment of an open neighborhood  $h(x, U) =: U_x$  containing  $x$  to each pair  $(x, U)$  such that  $U$  is an open neighborhood of  $x$ , and such that, if  $U_x \cap V_y \neq \emptyset$ , then either  $x \in V$  or  $y \in U$ .*

Every monotonically normal space is (hereditarily) countably paracompact [23, Theorem 2.3] and (hereditarily) scwH: the Borges criterion easily give cwH, and normality does the rest.

The main theorem of this section, Theorem 3.6, also involves the following concepts.

**Definition 3.4.** An  $\omega$ -*bounded space* is one in which every countable subset has compact closure. A space is  $\sigma$ - $\omega$ -*bounded* if it is the union of countably many  $\omega$ -bounded subspaces.

Clearly, every  $\omega$ -bounded space is countably compact. Theorem 3.6 below makes use of the following axiom, which is shown in [11] to follow from PID and whose numbering is aligned with that of Theorem 2.2:

**Axiom 3.5.** *Every locally compact space has either:*

- (1) *A countable collection of  $\omega$ -bounded subspaces whose union is the whole space or*
- (2) *An uncountable closed discrete subspace or*
- (3) *A countable subset with non-Lindelöf closure.*

Of course, (1) and (2) are mutually exclusive, but each is compatible with (3).

Part (iii) of our main theorem is the promised strengthening of Theorem 1.3.

**Theorem 3.6.** *Let  $X$  be a locally compact,  $\omega_1$ -compact space. If either*

- (i)  *$X$  is monotonically normal and the  $P$ -Ideal Dichotomy (PID) axiom holds, or*
- (ii)  *$X$  is hereditarily  $\omega_1$ -scwH, and either  $PFA$  or  $PFA(S)[S]$  hold, or*
- (iii)  *$X$  is hereditarily normal, and  $MM(S)[S]$  holds,*

*THEN  $X$  is  $\sigma$ - $\omega$ -bounded, and is either Lindelöf or contains a copy of  $\omega_1$ .*

*Proof.* The  $\sigma$ - $\omega$ -bounded property in case (i) follows from the fact that, in a monotonically normal space, every countable subset has Lindelöf closure [22], and from Axiom 3.5.

The PID, and hence Axiom 3.5, holds in any model of  $PFA$  or  $PFA(S)[S]$ , so  $\sigma$ - $\omega$ -boundedness in case (ii) follows from the following two facts. First, every point (and hence every countable subset) of a locally compact space has an open Lindelöf neighborhood (see [18, Lemma 1.7]). Second, in these models, every open Lindelöf subset of a hereditarily  $\omega_1$ -scwH space has Lindelöf closure (see [18] and [20], respectively). So again, (3) in Axiom 3.5 fails outright, and  $\sigma$ - $\omega$ -boundedness for (ii) follows from (1) of Axiom 3.5. As for (iii), Dow and Tall [8] have shown that  $MM(S)[S]$  implies that every normal, locally compact space is  $\omega_1$ -cwH, and we can continue as for (ii).

That  $X$  is either Lindelöf or contains a copy of  $\omega_1$  in case (i), is immediate from the following fact, whose proof is deferred to Section 5 (see Corollary 5.6):

*Every locally compact, monotonically normal space is either paracompact or contains a closed copy of a regular uncountable cardinal.*

The Lindelöf alternative for case (i) uses the fact [10, 5.2.17] that every locally compact, paracompact space is the union of a disjoint family of (closed and) open Lindelöf subspaces. Now  $\omega_1$ -compactness makes the family countable, and so  $X$  is Lindelöf.

The same either/or alternative for (ii) and (iii) uses the ZFC theorem that any  $\omega$ -bounded, locally compact space is either Lindelöf or contains a perfect preimage of  $\omega_1$ , along with the reduction of character in Lemma 1.2 of [18], which uses  $MA(\omega_1)$ , which in turn is implied by the  $PFA$ . Moreover, the proof of this lemma carries over to  $PFA(S)[S]$ . This proof shows that in a locally compact, hereditarily  $\omega_1$ -scwH space, every open Lindelöf subset has Lindelöf closure and hereditarily Lindelöf (hence first countable) boundary.

Now, given a perfect surjective map  $\varphi : W \rightarrow \omega_1$ , let  $Y$  be the union of the boundaries of the subsets  $\varphi^{\leftarrow}[0, \alpha)$  where  $\alpha$  is a limit ordinal, and apply to  $Y$  the theorem [7] that  $PFA(S)[S]$  implies that every first countable perfect preimage of  $\omega_1$  contains a copy of  $\omega_1$ . That the  $PFA$  also implies this is a well-known theorem of Balogh, shown in [6].  $\square$

The following theorem can be derived from Theorem 3.6 (ii) in the same way that Theorem 2.7 is derived from Theorem 2.4.

**Theorem 3.7.** *Let  $X$  be a locally compact,  $\omega_1$ -compact, normal, hereditarily  $\omega_1$ -scwH space. If either  $PFA$  or  $PFA(S)[S]$  holds, then  $X$  is countably paracompact.*

To put it positively, every ZFC example of a locally compact, hereditarily  $\omega_1$ -scwH Dowker space must contain an uncountable closed discrete subspace. However, to be absolutely certain of this, we need a negative answer to the second part of the following question:

**Problem 1.** *Do we need the large cardinal strength of PID to obtain any or all of the conclusions of Theorem 3.6 or Theorem 3.7?*

Where Theorem 3.7 is concerned, the other applications of *PFA*, etc. are taken care of by axioms  $MA(\omega_1)$  and  $MA(S)[S]$  [18], [21], both of which are equiconsistent with ZFC. However, we may need something more for the conclusion about the copy of  $\omega_1$ , or for the theorem [8] that  $MM(S)[S]$  implies that every locally compact normal space is  $\omega_1$ -cwH, or for the following corollary of this fact and of Theorem 3.7:

**Corollary 3.8.** *If  $MM(S)[S]$ , then every locally compact, hereditarily normal Dowker space must contain an uncountable closed discrete subspace.  $\square$*

#### 4. EXAMPLES UNDER $\clubsuit$ FOR THEOREM 1.2

This section features consistent examples of locally compact,  $\omega_1$ -compact, monotonically normal spaces of cardinality  $\aleph_1$  that are not  $\sigma$ -countably compact. The axiom used is  $\clubsuit$ , and the following concept is part of its definition.

**Definition 4.1.** Given a limit ordinal  $\alpha$ , a *ladder at  $\alpha$*  is a strictly ascending sequence of ordinals less than  $\alpha$  whose supremum is  $\alpha$ . Given an ordinal  $\theta$ , a *ladder system on  $\theta$*  is a family

$$\mathcal{L} = \{L_\alpha : \alpha \in \theta, \alpha \text{ is a limit ordinal of countable cofinality}\}$$

where each  $L_\alpha$  is a ladder at  $\alpha$ .

We will use the symbol  $\Lambda$  for the set of countable limit ordinals.

**Axiom 4.2.** *Axiom  $\clubsuit$*  states that there is a ladder system  $\mathcal{L}$  on  $\omega_1$  such that, for any uncountable subset  $S$  of  $\omega_1$ , there is  $L_\alpha \in \mathcal{L}$  such that  $L_\alpha \subset S$ .

We now define a general family  $\mathfrak{S}$  of spaces similar to the two families in [13]. One subfamily of  $\mathfrak{S}$  consists of examples for Theorem 1.2.

**Notation 4.3.** Let  $\mathfrak{S}$  designate the set of topologies  $\tau$  on  $\omega_1$  in which, to each point  $\alpha$  there are associated  $B(\alpha) \subset [0, \alpha]$  and  $B(\alpha, \xi) = B(\alpha) \cap (\xi, \alpha]$  for each  $\xi < \alpha$ , such that:

- (1)  $\{B(\alpha, \xi) : \xi < \alpha\}$  is a base for the neighborhoods of  $\alpha$  [we allow  $\xi = -1$  in case  $\alpha = 0$ ].
- (2) If  $\alpha \in \Lambda$  and  $\beta > \alpha$ , then there exists  $\xi < \alpha$  such that  $B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]$ . [In particular,  $\alpha \in B(\beta)$ .]
- (3) There is a ladder system  $\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$ , such that if  $M_\alpha = \{\xi + 1 : \xi \in L_\alpha\}$ , then  $\alpha = \sup[M_\alpha \cap B(\alpha)] = \sup[M_\alpha \setminus B(\alpha)]$ .

**Examples 4.4.** Let  $\tau$  be a topology defined using  $\mathcal{L}$ , and using the base  $B(\alpha)$  and  $B(\alpha, \xi)$  defined by recursion as follows:

Let  $B(0) = \{0\}$  and, if  $\alpha = \xi + 1$ , let  $B(\alpha) = B(\xi) \cup \{\alpha\}$ . Given  $L_\alpha \in \mathcal{L}$ , let  $M_\alpha = \{\xi + 1 : \xi \in L_\alpha\}$  be bijectively indexed as  $\{\alpha_n : n \in \omega\}$ . If  $\alpha = \nu + \omega$  where  $\nu$  either is 0 or a limit ordinal, let  $B(\alpha) = B(\nu) \cup (\nu, \alpha] \setminus \{\alpha_{2n} : n \in \omega\}$ .



If  $\alpha$  is not of the form  $\nu + \omega$ , let  $S(\alpha, 0) = B(\alpha_0)$  and, for  $n > 0$ , let  $S(\alpha, n) = B(\alpha_n, \alpha_{n-1})$  and let

$$B(\alpha) = \bigcup_{n=0}^{\infty} (S(\alpha, n) \cup \{\alpha_{2n+1} : n \in \omega\}) \setminus \{\alpha_{2n} : n \in \omega\}.$$

*Claim 1.*  $(\omega_1, \tau) \in \mathfrak{S}$ .

⊢ Assuming (2) in Notation 4.3, we first show (1). If  $\alpha \in B(\beta, \eta) \cap B(\gamma, \nu)$  and  $\alpha > \nu$ ,  $\alpha \notin \Lambda$  then obviously  $B(\alpha, \nu) = \{\alpha\}$  works. Otherwise, we can find a basic open neighborhood of  $\alpha$  inside the intersection by using (2). Just choose  $\xi$  greater than  $\eta$  and  $\nu$  and large enough so that  $B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]$  and also  $B(\alpha, \xi) = B(\gamma) \cap (\xi, \alpha]$ .

Next we show (2) by induction. Suppose it is false, and that  $\beta$  is the first ordinal for which there is a limit ordinal  $\alpha < \beta$  where it fails. Clearly  $\beta \in \Lambda$ . Only successor ordinals are in  $[0, \beta] \setminus B(\beta)$ , so  $\alpha \in B(\beta)$ . Let  $\alpha \in S(\beta, n) = B(\gamma, \nu)$ . By minimality of  $\beta$ , there exists  $\mu \geq \nu$  such that  $B(\alpha, \mu) = B(\gamma, \nu) \cap (\mu, \alpha]$ . However,  $S(\beta, n)$  only contains finitely many members of  $L_\beta$ ; once  $\xi$  is above all the members of  $L_\beta$  that are below  $\alpha$ , we must have  $B(\beta) \cap (\xi, \alpha] = B(\gamma, \nu) \cap (\xi, \alpha]$ , and if  $\xi \geq \mu$  then this equals  $B(\alpha, \xi)$ , a contradiction.

Finally, (3) is true by construction since  $\alpha$  is the supremum of the points of  $M_\alpha$  that have odd subscripts and also of the ones that have even subscripts. ⊣

*Claim 2.*  $(\omega_1, \tau)$  is locally compact.

⊢ Obviously,  $B(0) = \{0\}$  and  $B(\xi + 1, \xi)$  are singletons for all successor ordinals  $\xi + 1$ . If  $\beta \in \Lambda$ , then, since  $B(\beta)$  is countable, it is enough to show that  $B(\beta)$  is countably compact.

Let  $A$  be an infinite subset of  $B(\beta)$ . Then  $A$  contains a strictly ascending sequence  $\sigma$  of ordinals. Let  $\sup(\text{ran}(\sigma)) = \alpha$ . If  $\alpha = \beta$  then  $\sigma \rightarrow \alpha$  by (1), while if  $\alpha < \beta$ , then (1) and (2) have the same effect, implying that  $\alpha$  is a limit point of  $A$  in  $B(\beta)$ . ⊣

Theorem 1.2 now follows from:

**Theorem 4.5.** *If  $\mathcal{L}$  witnesses  $\clubsuit$ , then  $(\omega_1, \tau)$  is monotonically normal and  $\omega_1$ -compact but not  $\sigma$ -countably compact.*

*Proof.* Recall the Borges criterion for monotone normality, Theorem 3.3:

*There is an assignment of an open neighborhood  $h(x, U) =: U_x$  containing  $x$  to each pair  $(x, U)$  such that  $U$  is an open neighborhood of  $x$ , and such that, if  $U_x \cap V_y \neq \emptyset$ , then either  $x \in V$  or  $y \in U$ .*

The choice of  $h(z, U) = \{z\}$  for all isolated points  $z$  works for the case where either  $x$  or  $y$  is not a limit ordinal. So it is enough to take care of the case where  $x$  and  $y$  are both limit ordinals and  $x < y$ . Given  $\alpha \in U$ , let  $U_\alpha = B(\alpha, \xi)$  for some  $\xi$  such that  $B(\alpha, \xi) \subset U$ . It follows from (2) that  $x \in U_y$  whenever  $U_x \cap U_y \neq \emptyset$ .

It also follows from (2) that the  $\tau$ -relative topology on  $\Lambda$  is its usual order topology. So, to show  $\omega_1$ -compactness it is enough to show that every uncountable set  $S$  of successor ordinals has an ascending sequence  $\langle s_n : n \in \omega \rangle$  that  $\tau$ -converges to its supremum.

Let  $R = \{\xi : \xi + 1 \in S\}$ . Since  $\mathcal{L}$  witnesses  $\clubsuit$ , there exists  $\alpha \in R$  such that  $L_\alpha \subset R$ . Then  $M_\alpha \subset S$  and the odd-numbered elements of  $M_\alpha$  converge to  $\alpha$ .

Finally, to show that  $(\omega_1, \tau)$  is not  $\sigma$ -countably compact, let  $\omega_1 = \bigcup_{n=1}^{\infty} A_n$ . It is clearly enough to show that any  $A_n$  that contains uncountably many successor ordinals is not countably compact, since there is at least one such  $A_n$ . Let one of these be  $S$ , and let  $R$  and  $M_\alpha$  be as before. Then the even-numbered members of  $M_\alpha$  are an infinite  $\tau$ -closed discrete subset of  $\omega_1$ .  $\square$

## 5. SOME ZFC RESULTS

For the following theorem, shown in [20], we do not even have to assume  $\omega_1$ -compactness; it comes free of charge:

**Theorem 5.1.** *Every locally compact, locally connected and connected, monotonically normal space is  $\sigma$ -countably compact.*

Monotone normality has one strange feature that is not shared by normality and most of its natural strengthenings: it is often not preserved in taking the one-point compactification.<sup>2</sup> Mary Ellen Rudin gave the first example where this preservation fails, and in [13] two general kinds of examples are given. The following lemma makes it easy to show that all these examples, as well as our  $\clubsuit$  examples, fail to have monotonically normal compactifications.

**Lemma 5.2.** *Let  $X$  be a locally compact space whose set  $C$  of nonisolated points is a copy of an ordinal of uncountable cofinality. If  $X$  has a monotonically normal one-point compactification, then  $C$  has a clopen,  $\omega$ -bounded neighborhood.*

*Proof.* We use the original definition of monotone normality. Identify  $C$  with the ordinal  $\theta$  of which it is a copy, and let  $\Gamma = \theta + 1$ , with the last point of  $\theta + 1$  identified with the extra point  $\infty$  in the one-point compactification of  $X$ . For each  $\alpha \in C$ , let  $V_\alpha = G([0, \alpha], \Gamma \setminus [0, \alpha])$ . Let  $V = \bigcup_{\alpha \in \theta} V_\alpha$ . Clearly,  $V$  is a neighborhood of  $C$  in  $X$ , and  $V_\alpha \subset V_\beta$  whenever  $\alpha < \beta$ . Because  $G([\Gamma \setminus [0, \alpha], [0, \alpha])$  is a neighborhood of  $\infty$  disjoint from  $V_\alpha$ , the latter has compact closure in  $X$ . From this it quickly follows that  $V$  is an  $\omega$ -bounded clopen neighborhood of  $C$ . In fact, every union of fewer than  $cf(\theta)$  compact subsets of  $V$  has compact closure.  $\square$

The main theorem of this section (Theorem 5.10) will give an alternative proof that our  $\clubsuit$  examples do not have monotonically normal one-point compactifications. The first step is to recall a remarkable and powerful theorem of Balogh and Rudin and one of its corollaries [2].

**Proposition 5.3.** *Let  $X$  be monotonically normal, and let  $\mathcal{U}$  an open cover of  $X$ . Then  $X = V \cup \bigcup \mathcal{E}$ , where  $\mathcal{E}$  is a discrete family of closed subspaces, each homeomorphic to a stationary subset of a regular uncountable cardinal, and  $V = \bigcup \mathcal{V}$  is the union of countably many collections  $\mathcal{V}_n$  of disjoint open sets, each of which (partially) refines  $\mathcal{U}$ .*

**Corollary 5.4.** *A monotonically normal space is paracompact if, and only if, it does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

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<sup>2</sup>Locally connected, locally compact spaces are an exception: monotone normality is preserved in the one-point compactification [13].

**Lemma 5.5.** *Let  $\delta$  be an ordinal of uncountable cofinality, and let  $E$  be a locally compact stationary subset of  $\delta$ . There is a tail (final segment)  $T$  which is a closed (hence club) subset of  $\delta$ .*

*Proof.* Each  $x \in E$  has an open neighborhood  $H_x$  which meets  $E$  in an interval  $[\beta_x, x]$  whose least element is isolated in  $E$ ; and if  $H_x$  has compact closure, then  $[\beta_x, x]$  is compact. Then the PDL gives us a single  $\beta$  which works for a cofinal subset  $S$  of  $E$ , and so  $\bigcup\{[\beta, x] : x \in S\}$  is a tail of  $E$  and a club in  $\delta$ .  $\square$

Lemma 5.5 gives the strengthening of Corollary 5.4 for locally compact spaces that was used in the proof of Theorem 3.6:

**Corollary 5.6.** *A monotonically normal, locally compact space is paracompact if, and only if, it does not contain a closed copy of a regular uncountable cardinal.*

*Proof.* Necessity is immediate: every ordinal is a stationary subset of itself. Conversely, the  $T$  obtained in Lemma 5.5 has a closed subset that is a copy of the regular uncountable cardinal  $\text{cof}(\delta)$ .  $\square$

**Example 5.7.** *Let  $\delta = \omega_1 \cdot \omega_1$ , let  $T = \{\omega_1 \cdot \xi : \xi \in \omega_1\}$ , and let  $E = \omega \cup T$ . Then  $E$  is locally compact, and  $T$  is a tail of  $E$  that is a copy of  $\omega_1$  and a club in  $\delta$ , but no tail of  $E$  is a tail of  $\delta$ .*

In this example,  $E$  is homeomorphic to  $A = (\omega_1 + \omega) \setminus \{\omega_1\}$  in its relative topology inherited from  $\omega_1 + \omega$ . Note that this relative topology is not equal to the order topology on  $A$ , i.e., the topology on  $A$  generated by the subbase  $\{\alpha \cap A : \alpha \in A\} \cup \{A \setminus (\alpha + 1) : \alpha \in A\}$ , because  $A$  with this topology is homeomorphic to  $\omega_1 + \omega$  and hence is  $\sigma$ -compact, whereas  $A$  with the subspace topology is not Lindelöf. On the other hand, the two topologies coincide on any closed subset of any ordinal in its inherited order.

We are indebted to Heikki Junnila for the concise proof of the following theorem. Recall that a space is a *topological direct sum* of subspaces if these subspaces partition it into open (hence clopen) subspaces. A great many properties of topological spaces need only be verified for the summands to be true for the whole sum.

**Theorem 5.8.** *A subspace of an ordinal is locally compact if, and only if, it is a topological direct sum of countably compact copies of ordinals.*

*Proof.* Sufficiency is obvious. For the converse, let  $\theta$  be an ordinal and let  $A$  be a locally compact subspace of  $\theta$ . The closure  $\overline{A}$  of  $A$  is a copy of an ordinal, and since  $A$  is locally compact, it is open in  $\overline{A}$ .

Working in  $\overline{A}$ , let  $\mathcal{G}$  be the collection of convex components of  $A$ . These are the maximal intervals of  $A$  within  $\overline{A}$ . Each is a copy of an ordinal, and  $\mathcal{G}$  is a partition of  $A$  into relatively clopen subsets. Let  $\mathcal{G}_1 = \{G \in \mathcal{G} : \text{cof}(G) > \omega\}$ . Each member of  $\mathcal{G}_1$  is countably compact. If  $G \in \mathcal{G} \setminus \mathcal{G}_1$ , then  $G$  has a partition into compact open intervals. Replacing each member of  $\mathcal{G} \setminus \mathcal{G}_1$  by these partition members gives us the desired partition of  $A$ .  $\square$

Next, we show a strengthening of Property 5.3 for locally compact spaces that makes use of 5.3 in an essential way.

**Theorem 5.9.** *Let  $\mathcal{U}$  be an open cover of a locally compact monotonically normal space  $X$ . Then  $X$  has a discrete collection  $\mathcal{C}$  of closed copies of regular uncountable cardinals, such that  $X \setminus \bigcup \mathcal{C}$  has a  $\sigma$ -disjoint cover by open sets refining the trace of  $\mathcal{U}$  on  $X \setminus \bigcup \mathcal{C}$ .*

*Proof.* Assume without loss of generality that  $\bar{U}$  is compact for each  $U \in \mathcal{U}$ . Let  $\mathcal{E} = \{E_\alpha : \alpha \in \kappa\}$  be a discrete collection of closed copies of stationary subsets of regular cardinals  $\theta_\alpha$  as in Property 5.3. For each  $\alpha \in \kappa$ , use Theorem 5.8 to break up each  $E_\alpha$  into a discrete family  $\mathcal{K}_\alpha \cup \mathcal{P}_\alpha$ , where each  $P \in \mathcal{P}_\alpha$  is a compact subset of some  $U \in \mathcal{U}$  and each  $K \in \mathcal{K}_\alpha$  is a copy of an ordinal of uncountable cofinality. Let  $\mathcal{K} = \bigcup_{\alpha \in \kappa} \mathcal{K}_\alpha$ .

Given a fixed  $K \in \mathcal{K}$ , let  $B$  be a set of successors (hence isolated points) in  $K$  that is cofinal in  $K$  and of order type  $\text{cof}(K)$ . Let  $C_K$  be the closure of  $B$  in  $K$ , so that  $C_K$  is homeomorphic and order-isomorphic to  $\text{cof}(K)$ . For each  $\zeta \in B$ , let  $\beta(\zeta) = \sup\{\xi \in C_K : \xi < \zeta\}$ .

Then  $K \setminus C_K$  breaks up into the intervals  $(\beta(\zeta), \zeta)$ . Each interval is compact: the maximum element is the immediate predecessor of  $\zeta$ , and the minimum element is the immediate successor of  $\beta(\zeta)$ . So each interval has a finite partition  $\mathcal{P}_K(\zeta)$  into clopen sets, each a subset of some  $U \in \mathcal{U}$ . Let  $\mathcal{P}_K = \bigcup \{\mathcal{P}_K(\zeta) : \zeta \in B\}$ .

Let  $\mathcal{C} = \{C_K : K \in \mathcal{K}\}$  and  $\mathcal{P} = \bigcup \{\mathcal{P}_K : K \in \mathcal{K}\} \cup \bigcup \{\mathcal{P}_\alpha : \alpha \in \kappa\}$ . Then  $\mathcal{P}$  is a relatively discrete partition of  $(\bigcup \mathcal{E}) \setminus (\bigcup \mathcal{C})$  into compact sets. Every monotonically normal space is *collectionwise normal*, which means that every discrete collection of closed subsets can be expanded to a discrete collection of open sets; the proof is similar to that for the cwH property (see the discussion after Theorem 3.3). So  $\mathcal{P}$  can be expanded to a disjoint collection  $\mathcal{V}_\infty$  of open sets, each contained in a member of  $\mathcal{U}$ . Combine this with the  $\mathcal{V}_n$  of Property 5.3 to get the desired  $\sigma$ -disjoint cover.  $\square$

Now we are ready to prove the main theorem of this section.

**Theorem 5.10.** *Every locally compact space with a monotonically normal one-point compactification is a topological direct sum of  $\sigma$ - $\omega$ -bounded spaces, each of which is the union of an open Lindelöf subset and of a discrete family of closed,  $\omega$ -bounded subsets. If the space is totally disconnected, it is a topological direct sum of  $\omega$ -bounded subspaces.*

*Proof.* Let  $X$  be monotonically normal and locally compact, with a monotonically normal one-point compactification. Let  $\mathcal{C}$  be as in Theorem 5.9. For each  $C_\alpha$  there is an open subset  $W_\alpha \supset C_\alpha$  so that  $\{W_\alpha : \alpha < \kappa\}$  is a discrete collection of open sets. Each  $C_\alpha$  converges to the extra point  $\infty$  in the one-point compactification of  $X$ , so there is an open set  $V^\alpha \supset C_\alpha$  with  $\omega$ -bounded closure, built similarly to  $V$  in the proof of Lemma 5.2. Do it so that  $\text{cl}_X V^\alpha \subset W_\alpha$  by using  $V_\xi^\alpha = G([0, \xi], \{\infty\} \cup (X \setminus W_\alpha) \setminus [0, \xi])$ .

The complement of  $\bigcup_{\alpha < \lambda} V^\alpha$  in  $X$  is a closed subset of  $X \setminus \bigcup \mathcal{C}$ , so it is paracompact. By the classical theorem cited earlier, every locally compact, paracompact space can be partitioned into a family of clopen Lindelöf subsets. If, in addition, the space is totally disconnected, it is zero-dimensional [10, 6.2.4]<sup>3</sup>, and each of these Lindelöf subsets can be further partitioned into compact open subsets. In either case, let  $\mathcal{H} = \{H_\alpha : \alpha \in \gamma\}$  be such a partition.

<sup>3</sup>In [10], totally disconnected spaces are called “hereditarily disconnected.”

In the zero-dimensional case, the boundary of each  $V^\alpha$  is partitioned into compact relatively open sets where it meets  $\bigcup \mathcal{H}$ . By countable compactness, there can be only finitely many such sets. Cover the boundary with finitely many compact open subsets of  $W_\alpha$ . The resulting clopen expansions of the  $V^\alpha$  are a discrete collection in  $X$  and the complement of their union can be partitioned into clopen sets as before. The resulting partition of  $X$  is as desired.

In the more general case, we still have the countably compact boundary of each  $V^\alpha$  partitioned into  $X$ -closed subsets of Lindelöf sets. Hence it is compact here too, and only finitely many members of  $\mathcal{H}$  meet it. Now, the following is clearly an equivalence relation for  $\mathcal{K} = \mathcal{H} \cup \{V^\alpha : \alpha < \lambda\}$ . Let  $K_0 \approx K$  iff there is a finite chain  $K_0, \dots, K_n$  such that  $cl_X(K_i) \cap cl_X(K_{i+1}) \neq \emptyset$  and  $K_n = K$ . The closure of each  $V^\alpha$  meets that of only finitely many other members of  $\mathcal{K}$ , while the closure of each  $H \in \mathcal{H}$  meets that of only countably many other members. The union of the members of each equivalence class is open, because the boundary of  $K_0 \in \mathcal{K}$  is covered by its union with all the eligible candidates for  $K_1$ . The unions of members of each equivalence class give the desired partition into  $\sigma$ - $\omega$ -bounded subspaces.  $\square$

This theorem solves Problem 2 of [13], which asked:

**Question 5.11.** *If a locally compact space has a monotonically normal one-point compactification, is it the topological direct sum of  $\omega_1$ -compact subspaces?*

This question was independently answered by Heikki Junnila [private communication] but his proof made use of the monumentally difficult solution by Mary Ellen Rudin of Nikiel's conjecture [24]. The Balogh-Rudin theorem used here also has a very complicated proof, but nowhere near as complicated as the one for solving Nikiel's Conjecture.

Theorem 5.10 actually characterizes the class  $\mathfrak{C}$  (denoted  $\mathcal{LM}$  in [13]) of locally compact spaces with monotonically normal one-point compactifications. In the zero-dimensional case this follows immediately from the theorem [13, 4.2] that every countably compact, locally compact, monotonically normal space is in  $\mathfrak{C}$  and the easy lemma that  $\mathfrak{C}$  is closed under taking topological direct sums. The more general case is shown in the second paragraph in the proof of [13, 4.4], after all use of local connectedness is over.

## 6. THE MINIMUM CARDINALITY THEME

In the wake of the negative answer to Question 1.1, it is natural to ask:

**Problem 2.** *What is the minimum cardinality of a locally compact,  $\omega_1$ -compact space that is not  $\sigma$ -countably compact?*

**Problem 3.** *What is the minimum cardinality of a locally compact,  $\omega_1$ -compact normal space that is not  $\sigma$ -countably compact?*

One might also ask whether it is consistent for the minimum for Problem 3 to be strictly greater than the minimum for Problem 2. These are difficult problems, and each may need to be settled on a model-by-model basis. But we do have some general results. Here is a recent result by the first author [21]:

**Theorem 6.1.** *The least cardinality of a locally compact, normal,  $\omega_1$ -compact space that is not  $\sigma$ -countably compact is no greater than  $\mathfrak{b}$ .*

Previously, the best upper bound for the minimum was  $\mathfrak{c}$ , using one of E.K. van Douwen's "honest submetrizable" examples [5]. Theorem 6.1 still leaves a lot unsaid. On the one hand, the following "echo" of Question 1.1 is made difficult by the fact [14] that PID implies  $\mathfrak{b} \leq \aleph_2$ :

**Problem 4.** *Is there a ZFC example of a locally compact,  $\omega_1$ -compact space of cardinality  $\aleph_2$  that is not  $\sigma$ -countably compact? one that is normal?*

At the opposite extreme, we have:

**Problem 5.** *Can  $\mathfrak{b}$  be "arbitrarily large" and still be the minimum cardinality of a locally compact,  $\omega_1$ -compact space that is not  $\sigma$ -countably compact? of one that is also normal?*

The question of what happens under Martin's Axiom (*MA*) is especially interesting since it implies  $\mathfrak{b} = \mathfrak{c}$ . It also implies that  $\clubsuit$  fails and that there are no Souslin trees. Now a Souslin tree with the interval topology is of cardinality  $\aleph_1$  that is locally compact,  $\omega_1$ -compact and hereditarily collectionwise normal [16, 4.18] (hence hereditarily normal and hereditarily strongly cwH) but is not  $\sigma$ -countably compact.

To show  $\omega_1$ -compactness, use the fact that every closed discrete subspace in a tree with the interval topology is a countable union of antichains, and the fact that a Souslin tree is, by definition, an uncountable tree in which every chain and antichain is countable. To show that a Souslin tree is not the union of countably many countably compact subspaces, use these facts together with the fact that at least one subspace would have to be uncountable, and the observation that the Erdős - Radó theorem implies that every uncountable tree must either contain an infinite antichain or an uncountable chain. [The interval topology on a subtree is not always the relative topology, but the relative topology is finer, so new infinite closed discrete subspaces could arise.]

It is worth noting here that adding a single Cohen real adds a Souslin tree but leaves  $\mathfrak{b}$  as the same aleph that it is in the ground model. So  $\mathfrak{b}$  can be arbitrarily large while the minimum cardinality for Problems 2 and 3 is  $\aleph_1$ .

A Souslin tree is not monotonically normal because every monotonically normal tree is a topological direct sum of copies of ordinals [17], and the following problem is still unsolved:

**Problem 6.** *Is the existence of a Souslin tree enough to produce a locally compact, monotonically normal,  $\omega_1$ -compact space that is not  $\sigma$ -countably compact?*

Most independence results pertaining to monotonically normal spaces revolve around whether a Souslin tree (equivalently, a Souslin line) exists. However, Problem 6 suggests that this may not be the case with the question of which models have a locally compact, monotonically normal,  $\omega_1$ -compact space that is not  $\sigma$ -countably compact. This would already be true if there is a model of  $\clubsuit$  where there are no Souslin trees [the model in [9] turned out to be flawed].

## 7. LOCALLY COMPACT, QUASI-PERFECT PREIMAGES

The examples that make Theorem 6.1 true are the first nontrivial examples for the following problem by van Douwen [4].

**Problem 7.** *Is ZFC enough to imply that each first countable regular space of cardinality at most  $\mathfrak{c}$  is a quasi-perfect image of some locally compact space?*

**Definition 7.1.** A continuous map  $f : X \rightarrow Y$  is *quasi-perfect* if it is surjective, and closed, and each fiber  $f^{-1}\{x\}$  is countably compact.

The following theorem was the key to Theorem 6.1.

**Theorem 7.2.** [21] *Let  $E$  be a stationary, co-stationary subset of  $\omega_1$ . There is a locally compact, normal, quasi-perfect preimage of  $E$ , of cardinality  $\mathfrak{b}$ . If  $\mathfrak{b} = \aleph_1$ , then this preimage can also be locally countable, hence first countable.*

Theorem 6.1 follows from this theorem and from the following simple facts.

**Theorem 7.3.** *Let  $f : Y \rightarrow X$  be a continuous surjective map.*

(i) *If  $X$  is not  $\sigma$ -countably compact, neither is  $Y$ .*

(ii) *If  $X$  is  $\omega_1$ -compact, and  $f$  is closed, and each fiber  $f^{-1}\{x\}$  is  $\omega_1$ -compact, then  $Y$  is  $\omega_1$ -compact.*

*Proof.* Statement (i) easily follows by contrapositive and the elementary fact that the continuous image of a countably compact space is countably compact.

To show (ii), let  $A$  an uncountable subset of  $Y$ . Then either  $A$  meets some fiber in an uncountable subset, in which case it is not closed discrete in  $Y$ , or  $f[A]$  is uncountable and so it is not closed discrete. Let  $B$  be a subset of  $f[A]$  that is not closed, and let  $p \in \overline{B} \setminus B$ .

Let  $A_0 = \{x_b : b \in B\}$  be a subset of  $A$  such that  $f(x_b) = b$  for all  $b \in B$ . Because  $f$  is closed, and  $B$  is not closed, neither is  $A_0$  and, in fact, it has an accumulation point in  $f^{-1}\{p\}$ . □

Theorem 6.1 now follows easily. If  $E$  is co-stationary, then all countably compact subsets of  $E$ , being closed, are countable. So  $E$  cannot be  $\sigma$ -countably compact unless it is countable. And if  $E$  is stationary, it has limit points in the closure of every uncountable subset of  $\omega_1$ , and so it is  $\omega_1$ -compact. It therefore follows from Theorem 7.3 that any quasi-perfect preimage of  $E$  is  $\omega_1$ -compact, but not  $\sigma$ -countably compact.

Problem 7 was motivated by a theorem in 13.4 of [4], which stated that the preimages it asks for do exist if  $\mathfrak{b} = \mathfrak{c}$ . The preimages van Douwen constructed were locally countable. But it is also consistent that some of them are not normal:

**Theorem 7.4.** *Let  $X$  be a locally compact, locally countable, quasi-perfect preimage of the space  $\mathbb{P}$  of irrationals with the relative topology. If the PFA holds, then  $X$  is not normal.*

*Proof.* Let  $p \in \mathbb{P}$  and let  $\pi : X \rightarrow \mathbb{P}$  be a surjective quasi-perfect map. Let  $\{\sigma_\alpha : \alpha < \omega_1\}$  be a family of injective sequences in  $X$  with disjoint images that converge to  $p$ . Let  $A_\alpha$  be the set of all limit points of  $\sigma_\alpha$ . Clearly  $A_\alpha \subset \pi^{-1}\{p\}$ .

*Case 1.*  $A_\alpha$  is uncountable for some  $\alpha$ . Then  $A_\alpha \cup \text{ran}(\sigma_\alpha)$  is a separable, uncountable, countably compact, noncompact subspace of  $X$ , and since it is first countable, the PFA implies it contains a copy of  $\omega_1$  [6]. But also, the PFA implies  $\mathfrak{p} = \aleph_2$ , and the result follows from [15, 3.2], which reads:

*If  $\mathfrak{p} > \aleph_1$ , no separable, normal, first countable, countably compact space can contain a perfect preimage of  $\omega_1$ .*

*Case 2.*  $A_\alpha$  is countable for all  $\alpha$ . This case only requires CH to fail. By induction, build sequences  $\tau_\alpha$  ( $\alpha < \omega_1$ ) in  $X$  whose  $\pi$ -images converge to  $p$ , and such that  $\text{ran}(\sigma_\beta) \subset^* \text{ran}(\tau_\alpha)$  for all  $\beta < \alpha$ . If some  $\tau_\alpha$  has uncountably many limit points, argue as in Case 1. Otherwise, the set  $C_\alpha$  of limit points of each  $\tau_\alpha$  is compact and countable, so it is contained in a countable, compact, open subset  $V_\alpha$  of  $X$ . Let  $V = \bigcup_{\alpha < \omega_1} V_\alpha$ . Since the  $C_\alpha$  form an increasing chain, their union  $C$  is  $\omega$ -bounded, hence countably compact, hence closed in  $X$  by first countability of  $X$ . However, inasmuch as the  $\sigma_\alpha$  have disjoint ranges,  $\pi[N \setminus C]$  is uncountable for any neighborhood  $N$  of  $C$ . And so, the  $\pi$ -image of every closed neighborhood of  $C$  is an uncountable, closed subset of  $\mathbb{P}$ ; hence it must be of cardinality  $\mathfrak{c}$ . But  $|V| = \aleph_1$ , so  $V$  cannot contain a closed neighborhood of  $C$ , contradicting normality.  $\square$

In the absence of  $\mathfrak{b} = \mathfrak{c}$  we still have very little idea of which first countable spaces have locally compact quasi-perfect preimages. It is unknown whether even the space  $\mathbb{P}$  of irrationals has such a preimage if  $\mathfrak{b} < \mathfrak{c}$ . It is also unknown whether there is a locally compact, locally countable, quasi-perfect preimage of  $[0, 1]$  if  $\mathfrak{b} < \mathfrak{c}$ . Such a space would solve the following problem:

**Problem 8.** Is there a ZFC example of a scattered, countably compact, regular space that can be mapped continuously onto  $[0, 1]$ ?

See [19] for discussion of this problem, including an explanation why the answer is affirmative if “regular” is omitted.

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