SPECIAL ULTRAFILTERS AND COFINAL SUBSETS OF $({}^{\omega}\omega, <^*)$

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Abstract. The interplay between ultrafilters and unbounded subsets of ${}^{\omega}\omega$ with the order $<^*$ of strict eventual domination is studied. Among the tools are special kinds of non-principal ("free") ultrafilters on ω. These include simple P-points; that is, ultrafilters with a base that is well-ordered with respect to the reverse of the order $\subset^*$ of almost inclusion. It is shown that the cofinality of such a base must be either $\mathfrak{b}$, the least cardinality of $<^*$-unbounded ("undominated") set, or $\mathfrak{d}$, the least cardinality of a $<^*$-cofinal ("dominating") set.

The small uncountable cardinal $\pi\mathfrak{p}$ is introduced. Consequences of $\mathfrak{b} < \pi\mathfrak{p}$ and of $\mathfrak{r} < \mathfrak{d}$ are explored; in particular, both imply $\mathfrak{b} < \mathfrak{d}$. Here $\mathfrak{r}$ is the reaping number, and is also the least cardinality of a π-base for a free ultrafilter. Both of these inequalities are shown to occur if there exist simple P-points of different cofinalities (in other words, if $\mathfrak{b} < \mathfrak{d}$ and there exist simple $P_{\mathfrak{b}}$-points and $P_{\mathfrak{d}}$-points), but this is a long-standing open problem.

Six axioms on nonprincipal ultrafilters on ω and the relationships between them are discussed along with various models of set theory in which one or more are known to hold (or are known to fail). The strongest of these, Axiom 1, is that for every free ultrafilter $\mathcal{U}$ and for every $<^*$-unbounded $<^*$-chain C of increasing functions in ${}^{\omega}\omega$, C is also unbounded in the ultraproduct ${}^{\omega}\omega, <_{\mathcal{U}}$. The other axioms replace one or both quantifiers with "there exists." The negation of Axiom 3 in a model provides a family of normal sequentially compact spaces whose product is not countably compact. The question of whether such a family exists in ZFC, even with "normal" weakened to "regular", is a famous unsolved problem of set-theoretic topology, known as the Scarborough-Stone problem.

There is an extensive literature on the subject of special ultrafilters on a denumerable set, usually the set ω unless circumstances suggest a different underlying set. Some kinds of ultrafilters, such as the ZFC-independent Ramsey ultrafilters, have a strong interplay with the structure of the set ${}^{\omega}\omega$ of sequences $\sigma : \omega \to \omega$ under the order $<^*$ of eventual domination.

This paper is an expansion of a set of privately circulated notes from the mid-80's, cited in [28] and [13] *inter alia*. It focuses on a number of other classes of ultrafilters that have strong interplay with ${}^{\omega}\omega$. One is the class of *simple P-points*. These are the free ("non-principal") ultrafilters with bases that are totally ordered by the order $\subset^*$ of almost inclusion.

It is ZFC-independent whether simple P-points exist, but if they do, then the cofinality of a base, with respect to the reverse of $\subset^*$, must be either $\mathfrak{b}$ or $\mathfrak{d}$. Here $\mathfrak{b}$ is the least cardinality of a $<^*$-unbounded ("undominated") family in ${}^{\omega}\omega$ and $\mathfrak{d}$ is the least cardinality of a cofinal ("dominating") family in ${}^{\omega}\omega$. In 1983, the author asked Saharon Shelah whether there could be a model of $\mathfrak{b} < \mathfrak{d}$ with both simple $P_{\mathfrak{b}}$ and simple $P_{\mathfrak{d}}$ points. A model believed to be of this sort was published in 1987 [9], but a flaw in the proof was found about a decade and

2010 *Mathematics Subject Classification.* Primary 03C20, 03E04, 03E20, 03E35. Secondary 03E10, 54D20.
Key words and phrases. ultrafilter, simple P-point, pseudo-P_{κ}-point, small uncountable cardinals, ${}^{\omega}\omega$, eventual domination, unbounded, dominating.

a half ago by Alan Dow. There is a ongoing project by Heike Mildenberger and Saharon Shelah to construct a valid model, but so far there is no assurance of success.

Fortunately, there do exist models of $\mathfrak{b} < \mathfrak{d}$ where there are simple $P_{\mathfrak{b}}$-points and others where there are simple $P_{\mathfrak{d}}$-points. Some are described in Sections 2 and 3, respectively. These sections also describe generalizations of these two kinds of simple P-points. They are, respectively, ultrafilters with a π-base of cardinality $< \mathfrak{d}$, and pseudo-P_κ-points where $\kappa > \mathfrak{b}$. The presence of either kind entails $\mathfrak{b} < \mathfrak{d}$, along with most of the conclusions that follow from the existence of the kind of model described in the preceding paragraph.

In Sections 4 through 6, we draw on the information in the earlier sections in analyzing six main axioms, and several auxiliary ones, and various combinations of them and their negations. The main axioms pertain to whether $<^*$-unbounded chains are also $<_{\mathcal{U}}$-unbounded for some or all free ultrafilters $\mathcal{U}$. Although some combinations are known to be consistent and others impossible, many open problems remain. The final section deals with generalizations, especially the substitution of directed subsets of ${}^\omega\omega$ for chains.

1. Preliminaries

Among the mathematical objects influenced by the ultrafilters studied here are some standard "small uncountable cardinals" [17], [29]. All cardinals in the following list are easily shown to be uncountable, and are obviously $\leq \mathfrak{c} = 2^\omega$.

$\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F}$ is a "dominating" (cofinal) subset of $({}^\omega\omega, <^*)\}$

$\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F}$ is an "undominated" (unbounded) subset of $({}^\omega\omega, <^*)\}$

$[\pi]\mathfrak{u} = min\{|\mathcal{B}| : \mathcal{B}$ is a $[\pi$-$]$base for a free ("non-principal") ultrafilter on $\omega\}$

$\mathfrak{s} = min\{|\mathcal{S}| : \mathcal{S}$ is a splitting family of subsets of $\omega\}$

$\mathfrak{r} = min\{|\mathcal{R}| : \mathcal{R}$ is a refining family of subsets of $\omega\}$

Here, $<^*$ refers to eventual strict domination: $f <^* g$ means that there exists $k \in \omega$ such that $f(n) < g(n)$ for all $n \geq k$. A set S is said to *split* a set A if $A \setminus S$ and $A \cap S$ are both infinite, and a family $\mathcal{S}$ is *splitting* if for each $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that S splits A. As usual, $[\omega]^\omega$ refers to the family of infinite subsets of ω.

A set S is said to *reap* a collection $\mathcal{A} \subset [\omega]^\omega$ if S splits every member of $\mathcal{A}$. A collection $\mathcal{R} \subset [\omega]^\omega$ is said to be *refining* if for every $A \in [\omega]^\omega$ there exists $R \in \mathcal{R}$ such that either $R \subset^* A$ or $R \subset^* (\omega \setminus A)$. It is easy to see that a refining family cannot be reaped, or as some say, it is "unsplittable," and, conversely, if $\mathcal{R}$ cannot be reaped, it is a refining family. The symbol $\mathfrak{r}$, originally suggested by the author to Erik van Douwen for different reasons, now has these natural mnemonics provided for it.

A collection $\mathcal{B} \subset [\omega]^\omega$ is a π-*base* for an ultrafilter $\mathcal{U}$ if for each $A \in \mathcal{U}$ there is $B \in \mathcal{B}$ such that $B \subset A$. Unlike for bases, there is no requirement that $B \in \mathcal{U}$, and the same collection could be a π-base for many different ultrafilters; in fact, $[\omega]^\omega$ is a π-base for every ultrafilter on ω. It is easy to see that $\mathfrak{r} \leq \pi\mathfrak{u}$, and Bohuslav Balcar [3, Proposition 1.6] showed that, in fact, $\mathfrak{r} = \pi\mathfrak{u}$.

Already when [17] was written, it was known that $\mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{s} \leq \mathfrak{d}$, and $\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{u}$. It has long been known that no other inequalities hold between these cardinals in ZFC. However, the existence of simple P-points gives rise to others.

Definition 1.1. Let κ be a regular cardinal. A free ultrafilter on ω is a *simple P_κ-point* if it has a totally ordered base under the order $\subset^*$ (where $A \subset^* B$ means that $A \setminus B$ is finite and $B \setminus A$ is infinite), of cofinality κ in the reverse order of $\subset^*$. A *simple P-point* is a simple P_κ-point for some (necessarily unique, and uncountable) cardinal κ.

Obviously, if there is a simple P_κ-point, then $\mathfrak{u} \leq \kappa$; also, $\kappa \leq \mathfrak{s}$. This latter inequality extends to pseudo-P_κ-points.

Definition 1.2. Let κ be a cardinal number. An ultrafilter $\mathcal{U}$ on ω is a *pseudo-P_κ-point* if, given any $\mathcal{A} \subset \mathcal{U}$, such that $|\mathcal{A}| < \kappa$, there exists an infinite *pseudo-intersection* of $\mathcal{A}$; that is, there exists $B \in [\omega]^\omega$ such that $B \subset^* A$ for all $A \in \mathcal{A}$.

Lemma 1.3. *If there is a pseudo-P_κ-point, then $\kappa \leq \mathfrak{s}$.*

Proof. Suppose $\kappa > \mathfrak{s}$, and let $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{s}\}$ be splitting. Let $\mathcal{U}$ be a free ultrafilter on ω. For each $\alpha < \mathfrak{s}$, let $U_\alpha \in \mathcal{U}$ be whichever of $\{S_\alpha, \omega \setminus S_\alpha\}$ is in $\mathcal{U}$. If $\mathcal{U}$ is a pseudo-P_κ-point, let B be a pseudo-intersection of $\{S_\alpha : \alpha < \mathfrak{s}\}$; then B is not split by any $S \in \mathcal{S}$, a contradiction. □

Later [Theorem 2.3] we will see that there are only two possibilities for a simple P_κ-point: $\kappa = \mathfrak{b}$ and $\kappa = \mathfrak{d}$. In the former case, $\mathfrak{r} = \mathfrak{u} = \mathfrak{b}$ while in the latter case, $\mathfrak{s} = \mathfrak{d}$. So in any model of $\mathfrak{b} < \mathfrak{d}$ where there are simple $P_\mathfrak{b}$-points and simple $P_\mathfrak{d}$-points, $\mathfrak{s}$ and $\mathfrak{d}$ are strictly greater than all of $\mathfrak{b}, \mathfrak{r}$, and $\mathfrak{u}$. These inequalities have been long known to be consistent, but here we have an easy unified way of seeing this, *if* the existence of such a pair of simple P-points is consistent.

Problem 1. Is there a model of ZFC with simple P-points of different cofinalities with respect to the reverse $\subset^*$ order?

Fortunately, several inequalities follow just from the existence of an ultrafilter with a π-base of cardinality $< \mathfrak{d}$, and also from that of a pseudo-$P_{\mathfrak{b}^+}$-point.

Pseudo-P_κ-points have also been called "almost P_κ-points" by Alan Dow. They were introduced in [21], and [22] gave some information about pseudo-$P_\mathfrak{b}$-points in various models.

Notation 1.4. *Let $\mathcal{U}$ be a free ultrafilter on ω.*

$$\pi\mathfrak{p}(\mathcal{U}) = min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{U} \text{ and } \neg\exists B \in [\omega]^\omega(\forall A \in \mathcal{A}(B \subset^* A))\}$$

and

$$\pi\mathfrak{p} = sup\{\pi\mathfrak{p}(\mathcal{U}) : \mathcal{U} \text{ is a free ultrafilter on } \omega\}.$$

Obviously, $\pi\mathfrak{p}(\mathcal{U}) = sup\{\kappa : \mathcal{U}$ is a pseudo-P_κ-point$\}$. Problem (4) of [13] asked whether $\pi\mathfrak{p}(\mathcal{U})$ could consistently be singular for some free ultrafilter $\mathcal{U}$. As far as the author knows, this is still unsolved.

If we can always take B to be in $\mathcal{U}$ in Definition 1.2, then we have the definition of a P_κ-*point*. The two concepts are not equivalent: although the existence of a P_{ω_1}-point (in other words, of a P-point) is ZFC-independent, *every* free ultrafilter on ω is a pseudo-P_{ω_1}-point [30, 3.27]. Moreover:

Lemma 1.5. *The cardinal $\mathfrak{p}$ is the greatest cardinal κ for which every free ultrafilter is a pseudo-P_κ-point.*

Proof. The usual definition of $\mathfrak{p}$ is that it is the least cardinal κ for which there is $\mathcal{B} \subset [\omega]^\omega$ such that $\mathcal{B}$ has no infinite pseudo-intersection, yet $|\bigcap \mathcal{A}| = \omega$ for every $\mathcal{A} \in [\mathcal{B}]^{<\omega}\}$.

If $\mathcal{U}$ is a free ultrafilter on ω, then $|\bigcap \mathcal{A}| = \omega$ for every $\mathcal{A} \in [\mathcal{U}]^{<\omega}$, so $\pi\mathfrak{p}(\mathcal{U}) \geq \mathfrak{p}$; [1] in other words, $\mathcal{U}$ is a pseudo-$P_\mathfrak{p}$-point. On the other hand, if $\mathcal{B}$ is as in the definition of $\mathfrak{p}$, and $\mathcal{U}$ is a free ultrafilter extending $\mathcal{B}$, then $\pi\mathfrak{p}(\mathcal{U}) \leq |\mathcal{B}|$. The case where $|\mathcal{B}| = \mathfrak{p}$ then gives a free ultrafilter that is not a pseudo-$P_{\mathfrak{p}^+}$ point. □

The following is immediate from Lemma 1.5 and Lemma 1.3.

Theorem 1.6. $\mathfrak{p} \leq \pi\mathfrak{p} \leq \mathfrak{s}$.

Both inequalities can be (consistently) strict. As implied earlier, there are even models where $\mathfrak{b} < \pi\mathfrak{p}$. [As is well known, $\mathfrak{p} \leq \mathfrak{b}$.] In [13], Joerg Brendle and Saharon Shelah showed that, for all regular cardinals κ, it is consistent that $\pi\mathfrak{p} = \kappa$ while $\mathfrak{s} = \mathfrak{c} = \kappa^+$. Brendle has shown [private communication] that the models involved also satisfy $\mathfrak{h} = \kappa^+$. But the following problem is open:

Problem 2. **(a)** Is $\mathfrak{h} \leq (\pi\mathfrak{p})^+$ true in ZFC? **(b)** Is $\mathfrak{s} \leq (\pi\mathfrak{p})^+$ true in ZFC?

The cardinal $\mathfrak{h}$ is especially attractive in this context because it is intimately related to ultrafilters:

Definition 1.7. $\mathfrak{h}$ *is the minimum height of a tree π-base for the free ultrafilters on ω with respect to the reverse $\subset^*$ order. [In other words, the tree includes a π-base for each free ultrafilter.]*

The cardinal $\mathfrak{h}$ was introduced in [2], where it is shown that $\mathfrak{p} \leq \mathfrak{h} \leq \mathfrak{s}$ and $\mathfrak{h} \leq \mathfrak{b}$, and that $\mathfrak{h}$ is the distributivity of the poset $([\omega]^\omega, \subset^*)$. Other early information on $\mathfrak{h}$ may be found in [29]. A recent characterization is that it is the least cardinality of a topological space which is countably compact but not sequentially compact [8].

Now we introduce a class of inequalities that will play a central role in what is to come.

Notation 1.8. *If $\mathcal{U}$ is a free ultrafilter on ω, and $f, g \in {}^\omega\omega$, then $f <_\mathcal{U} g$ means that $f < g$ in the ultrapower ${}^\omega\omega/\mathcal{U}$. That is, $\{n : f(n) < g(n)\} \in \mathcal{U}$. We write $f =_\mathcal{U} g$ if $\{n : f(n) = g(n)\} \in \mathcal{U}$.*

It is obvious that $=_\mathcal{U}$ is an equivalence relation, that $f <_\mathcal{U} g <^* h$ and $f <^* g \leq_\mathcal{U} h$ both imply $f <_\mathcal{U} h$, and that $<_\mathcal{U}$ induces a strict total order on the equivalence classes modulo $\mathcal{U}$. It is also easy to show that $f <^* g$ iff $f <_\mathcal{U} g$ for *every* free ultrafilter $\mathcal{U}$ on ω. Therefore, a $<^*$-cofinal ("dominating") family in ${}^\omega\omega$ will also be $<_\mathcal{U}$-cofinal [henceforth written simply as "$\mathcal{U}$-cofinal"] for every $\mathcal{U}$, while every $\mathcal{U}$-cofinal (equivalently, $\mathcal{U}$-unbounded) family will in turn be $<^*$-unbounded. In particular, $\mathfrak{b} \leq cf({}^\omega\omega/\mathcal{U}) \leq \mathfrak{d}$ for all free ultrafilters $\mathcal{U}$ on ω. On the other hand, it is consistent that there is a $<^*$-unbounded family of cardinality $\aleph_1$, while every $\mathcal{U}$-unbounded family is of cardinality $\geq \aleph_2$ for all free ultrafilters $\mathcal{U}$ on ω. This happens, for example, in any model of NCF (Near Coherence of Filters); see Section 2.

[1] It is an accident of notation that $\pi\mathfrak{p} \geq \mathfrak{p}$ while $\pi\mathfrak{u} \leq \mathfrak{u}$, and both inequalities can be strict: see the note after Theorem 1.6.

Notation 1.9. *Given an infinite subset A of ω, its enumeration function (*i.e.*, the unique order-preserving bijection from ω to A) will be designated ψ_A.*

The following lemma is reminiscent of the way, given a subgroup H of G, all the elements of G can be obtained by multiplying elements of H by elements in a set that meets each left coset modulo H.

Lemma 1.10. *Let $\mathcal{U}$ be a free ultrafilter on ω with π-base $\mathcal{B}$, and let $F \subset {}^{\omega}\omega$ be a $\mathcal{U}$-unbounded (equivalently, $\mathcal{U}$-cofinal) family of nowhere decreasing functions. Then $({}^{\omega}\omega, <^*)$ has a dominating family which is an order-preserving image of $(F, <^*) \times (\mathcal{B}, \supset^*)$ [where $\supset^*$ denotes the reverse of $\subset^*$].*

Proof. The family $\{f \circ \psi_B : B \in \mathcal{B},\ f \in F\}$ behaves as desired. Indeed, given any nowhere decreasing $g \in {}^{\omega}\omega$, pick $f \in F$ such that $g <_{\mathcal{U}} f$ and pick $B \in \mathcal{B}$ such that $B \subset \{n \in \omega : g(n) < f(n)$. Since f and g are nowhere decreasing, "the leftward shift of f by B," $f \circ \psi_B$, satisfies $g(n) < f \circ \psi_B(n)$ for all n. □

The following is implicit in the proof of [27, Theorem 2], so we only outline the proof.

Lemma 1.11. *If $\mathcal{B}$ is a π-base for a free ultrafilter on ω, then $\{\psi_B : B \in \mathcal{B}\}$ is unbounded in $({}^{\omega}\omega, <^*)$.*

Outline of proof. Since $\mathcal{B}$ clearly cannot be reaped, we can argue as follows. Suppose there exists g such that $\psi_B <^* g$ for all $B \in \mathcal{B}$. We may assume $g(n) > n$ for all $n \in \omega$. Let $g^{(n)}$ denote the n-fold composition of g with itself. Then $g^{(n)}(0) < \psi_B(g^{(n)}(0)) < g^{(n+1)}(0)$ for all but finitely many n, so $\bigcup_{n=1}^{\infty}[g^{(2n-1)}(0), g^{(2n)}(0))$ splits every member of $\mathcal{B}$, a contradiction. □

A slight variation on this argument gives the ZFC inequality $\mathfrak{s} \leq \mathfrak{d}$ [17].

2. Consequences of $\mathfrak{r} < \mathfrak{d}$

This section studies ultrafilters with "small" π-bases, where "small" means "having $< \mathfrak{d}$ elements." [The section title uses the fact [3, Proposition 1.6.] that $\mathfrak{r} = \pi\mathfrak{u}$.] These occur in any model of NCF. In fact, the following is equivalent to NCF [5] [6]:

For every free ultrafilter $\mathcal{U}$ on ω, there is a finite-to-one mapping $f : \omega \to \omega$ such that $f(\mathcal{U})$ is generated by fewer than $\mathfrak{d}$ sets.

Here, $f(\mathcal{U})$ is the (free) ultrafilter generated by the set of all images $f[A]$ such that $A \in \mathcal{U}$. $f(\mathcal{U})$ is a P-point, as is any ultrafilter with a base of cardinality $< \mathfrak{d}$. The proof of this latter fact is a trivial modification of the one for the case when $\mathfrak{d} = \mathfrak{c}$ [19], [26, pp. 40–41].

In all known models of NCF, $2^{\aleph_0} = \mathfrak{d} = \aleph_2$, and so $\mathfrak{u} = \aleph_1$ there, and a simple induction shows that, if $\mathcal{U}$ is as above, then $f(\mathcal{U})$ is a simple P_{ω_1}-point.

There are also models where $\mathfrak{r}$ and $\mathfrak{d}$ can be "arbitrarily far apart" and "there may be many different π-characters below $\mathfrak{d}$" [13, Corollary 5.6]. In fact, given any regular κ and regular $\lambda > \kappa$, there is a forcing extension preserving cardinals where $\kappa = \mathfrak{u}$ and $\mathfrak{d} = \lambda$ [10].

Lemmas 1.10 and 1.11 do much of the work of establishing:

Theorem 2.1. *If $\mathcal{V}$ is a free ultrafilter on ω with a π-base $\mathcal{B}$ of cardinality $\kappa < \mathfrak{d}$, then:*

(i) $({}^{\omega}\omega, <_{\mathcal{V}})$ has cofinality $\mathfrak{d}$. In particular, $\mathfrak{d}$ is regular.

(ii) $\mathfrak{b} < \mathfrak{d}$.

(iii) Every $<^*$-chain of cofinality $< \mathfrak{d}$ is p-bounded.

(iv) If $\mathcal{V}$ is a simple P-point, then it is a simple $P_{\mathfrak{b}}$-point, so $\mathfrak{r} = \mathfrak{u} = \mathfrak{b} \leq \mathfrak{s}$.

(v) Every $<^*$-unbounded $<^*$-chain which consists of non-decreasing functions and is of cofinality $> \kappa$ is $\mathcal{V}$-unbounded, hence of cofinality $\mathfrak{d}$.

(vi) If there is a $<^*$-unbounded $<^*$-chain of cofinality $> \kappa$, and $\mathcal{V}$ is a simple P-point, then $({}^{\omega}\omega, <^*)$ has a cofinal family of order type $\mathfrak{b} \times \mathfrak{d}$.

(vii) $\mathcal{V}$ is not a semi-Q-point.[2] *In other words, $\{\psi_A : A \in p\}$ is not cofinal in $({}^{\omega}\omega, <_p)$.*

Proof. *(i)* and *(ii)* follow easily from Lemmas 1.10 and 1.11, and the fact, noted earlier, that $cf(\leq_{\mathcal{U}}) \leq \mathfrak{d}$ for all free ultrafilters $\mathcal{U}$. Now *(iii)* is immediate from *(i)*. The proof of *(iv)* is deferred to the next section.

If a chain C as in *(v)* were $\mathcal{V}$-bounded, say $f <_{\mathcal{V}} g$ for all $f \in C$, then for each $f \in C$ there would exist $B_f \in \mathcal{B}$ such that $f(n) < g(n)$ for all $n \in B_f$. From $cf(C) > \kappa$, it follows that $\{f : B_f = B\}$ is cofinal in C for some $B \in \mathcal{B}$. But then $f(n) < g \circ \psi_B(n)$ for all n and all f such that $B_f = B$, because each f in C is nowhere decreasing. Hence $f <^* g \circ \psi_B$ for all $f \in C$, and C is $<^*$-bounded. The last phrase of *(v)* follows from *(i)* and from the rest of *(v)*.

To show *(vi)*, we first replace a given $<^*$-unbounded $<^*$-chain C with one that consists of increasing functions, by replacing each f in C with $\overline{f}$, defined by $\overline{f}(n) = n + max\{f(i) : i \leq n\}$. It is easy to check that $\overline{f}$ is increasing and that this replacement is a $<^*$-unbounded $<^*$-chain. By *(v)* we can now assume that this replacement is $\{f_\alpha : \alpha < \mathfrak{d}\}$ where $f_\alpha <^* f_\beta$ whenever $\alpha < \beta$.

Next, let $\mathcal{B} = \{B_\xi : \xi < \kappa\}$ be a $\subset^*$-decreasing base for $\mathcal{V}$, where κ is regular. If ψ_ξ is the enumeration function of B_ξ, then $\{\psi_\xi : \xi < \kappa\}$ is a $<^*$-increasing, $<^*$-unbounded chain in ${}^{\omega}\omega$.

We thus obtain the $\mathfrak{d} \times \kappa$ array $\{f_\alpha \circ \psi_\xi : \alpha < \mathfrak{d}, \xi < \kappa\}$, in which $f_\alpha \circ \psi_\xi <^* f_\beta \circ \psi_\eta$ if either $\alpha \leq \beta$ and $\xi < \eta$ or $\alpha < \beta$ and $\xi \leq \eta$. It is a routine matter, using $\kappa < \mathfrak{d}$, to cut this array down so that if neither condition holds, $f_\alpha \circ \psi_\xi <^* f_\beta \circ \psi_\eta$ fails. First we throw out a column if cofinally many members are dominated by members of earlier columns, beginning with the preceding column that was not thrown out. Since the columns have greater cofinality than the rows, and each row is $<^*$-unbounded, κ columns will remain. For each $\beta < \kappa$, there is thus a member $f_\alpha \circ \psi_\beta$ of the (reindexed) βth column which is not dominated by any member of any earlier column. If we throw out all the rows below the supremum of these α's, this will leave us with an array of $<^*$-order type $\mathfrak{d} \times \kappa$ which is cofinal in $({}^{\omega}\omega, <^*)$ by Lemma 1.10. Thus every $<^*$-chain of cofinality $< \kappa$ is bounded, and $\kappa = \mathfrak{b}$.

Finally, *(vii)* is easy: if $A \in \mathcal{V}$, then ψ_A is dominated by some ψ_β, and $\beta < \kappa < \mathfrak{d}$. □

[2] Semi-Q-points are also referred to as *rapid ultrafilters*.

Remarks 2.2. (1) The following is immediate from *(i)*: *If $\mathfrak{d}$ is singular, then $\mathfrak{r} \geq \mathfrak{d}$.*

(2) Blass [5] showed *(i)* in the special case where $\mathcal{V}$ has a *base* of cardinality $< \mathfrak{d}$. This was enough to conclude that $\mathfrak{u} \geq \mathfrak{d}$ when $\mathfrak{d}$ is singular.

(3) In *(iv)*, $\mathfrak{b} = \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{s}$ are both possible. In the NCF model of [9], $\mathfrak{b} = \aleph_1$ and $\mathfrak{s} = \aleph_2$, [13] but in the $\mathfrak{u} < \mathfrak{g}$ model of [6] and [11], $\mathfrak{b} = \mathfrak{s} = \aleph_1$.

(4) According to Heike Mildenberger [private communication], it is possible to have models of NCF where all $<^*$-unbounded $<^*$-chains are of cofinality $\mathfrak{b}$ (and hence the hypothesis of *(vi)* is false and so is the conclusion) and others where there exist $<^*$-chains of cofinality $\mathfrak{d}$.

(5) The following is related to *(vii)* and note (2): Mike Canjar [15] showed that if an ultrafilter $\mathcal{V}$ has a *base* of cardinality $< \mathfrak{d}$, then $\mathcal{V}$ is not even above a semi-Q-point in the Rudin-Keisler order.

Problem 3. Can a free ultrafilter have a π-base of cardinality $< \mathfrak{d}$, yet be above a semi-Q-point in the Rudin-Keisler order?

Theorem 2.3. *If there exist simple P_κ-points and simple P_λ-points and $\kappa < \lambda$, then $\kappa = \mathfrak{b}$ and $\lambda = \mathfrak{d}$.*

Proof. A totally ordered base for a P_λ-point gives a $<^*$-unbounded chain of cofinality λ, so clearly $\lambda \leq \mathfrak{d}$, and so $\kappa < \mathfrak{d}$. But then by *(v)* of Theorem 2.1, $\lambda = \mathfrak{d}$. And now *(vi)* gives $\kappa = \mathfrak{b}$. □

Corollary 2.4. *There cannot exist simple P_κ-points for more than two distinct κ.*

3. Pseudo-$P_{\mathfrak{b}^+}$ points

Ultrafilters in this section have no overlap with the ones studied in the previous section. (See the last part of Theorem 3.1 below.) They may even seem to be unrelated to those of the preceding section, yet they exert a peculiarly "dual" influence on ${}^\omega\omega$, and most of the arguments will be similar.

Alan Dow has noted [private communication] that in the $\mathfrak{r} = \kappa$, $\mathfrak{d} = \lambda > \kappa$ model of [10], one could easily turn the $\kappa \times \lambda$ array of intermediate models used in the iteration on its side to give a forcing extension where $\mathfrak{b} = \kappa$ and $\pi\mathfrak{p} = \lambda$ — in particular, there is a pseudo-$P_{\mathfrak{b}^+}$ point. Dow notes that the same forcing results if one substitutes the Blass-Shelah posets at each step of the iteration in [12] for the actual posets used there.

Theorem 3.1. *If $\mathcal{W}$ is a pseudo-P_λ-point and $\lambda > \mathfrak{b}$, then:*

(i) $({}^\omega\omega, <_\mathcal{W})$ has cofinality $\mathfrak{b}$.

(ii) $\mathfrak{b} < \lambda \leq \mathfrak{s} (\leq \mathfrak{d})$.

(iii) Every $<^*$-chain of cofinality $> \mathfrak{b}$ is $\mathcal{W}$-bounded.

(iv) If $\mathcal{W}$ is a simple P-point, then it is a simple $P_\mathfrak{d}$-point, so $\mathfrak{r} \leq \mathfrak{s} = \mathfrak{d}$, and $\mathfrak{d}$ is regular.

(v) Every $<^*$-unbounded $<^*$-chain which consists of nowhere decreasing functions and is of cofinality $< \lambda$ is $\mathcal{W}$-unbounded, hence of cofinality $\mathfrak{b}$.

(vi) If $\mathcal{W}$ is a simple P-point, then $({}^\omega\omega, <^*)$ has a cofinal family of order type $\mathfrak{b} \times \mathfrak{d}$.

(vii) $\mathcal{W}$ is not a semi-Q-point.

(viii) Every π-base for $\mathcal{W}$ is of cardinality $\geq \mathfrak{d}$.

Proof. Part *(ii)* is a restatement of Lemma 1.3.

Next we show *(v)*. If such a chain C were $\mathcal{W}$-bounded, there would exist g such that for each $f \in {}^\omega\omega$, the set $S(f,g) = \{n : f(n) < g(n)\}$ is in $\mathcal{W}$. But then there would be $B \subset^* S(f,g)$ for all $f \in C$, whence $f <^* g \circ \psi_B$ for all $f \in C$, contradicting $<^*$-unboundedness.

To show *(i)*, recall that ZFC is enough to give a $<^*$-unbounded $<^*$-well ordered family of order type $\mathfrak{b}$ in ${}^\omega\omega$, consisting of increasing functions. Now use *(v)*.

Part *(iii)* follows immediately from *(i)*. To show *(viii)*, use *(iii)* and the definition of $\mathfrak{b}$ to show that any $\mathcal{W}$-unbounded $<^*$-chain of nowhere decreasing functions is of cofinality $\mathfrak{b}$. Now use Lemma 1.10 and the definition of $\mathfrak{d}$.

Next we show *(vii)*. Let $\{f_\alpha : \alpha \in \mathfrak{b}\}$ be $<^*$-unbounded. If $\{\psi_A : A \in \mathcal{W}\}$ were dominating, we could build a family of $\mathfrak{b}$ functions ψ_{A_α} by induction, with $f_\alpha <^* \psi_{A_\alpha}$ and $\psi_{A_\beta} <^* \psi_{A_\alpha}$ for all $\beta < \alpha$. But then an infinite pseudo-intersection B of $\{A_\alpha : \alpha < \mathfrak{b}\}$ would give a ψ_B dominating all the f_α, a contradiction.

The proof of *(vi)* is the same as that for *(vi)* of Theorem 2.1, except that we use an unbounded chain of cofinality $\mathfrak{b}$ for the rows, and the ψ_B from a totally ordered base $\mathcal{B}$ of q for the columns.

Finally, we show *(iv)* and use it to show *(iv)* of Theorem 2.1.

If $\mathcal{W}$ is a simple P-point, *(viii)* implies that it is a $P_\mathfrak{d}$-point, and now we can use $\mathfrak{s} \le \mathfrak{d}$ and $\mathfrak{r} \le \lambda \le \mathfrak{s}$ [see Lemma 1.3] to conclude that $\mathcal{W}$ is a simple $P_\mathfrak{d}$-point. [A peculiarity of our terminology is that a simple P_λ-point is a P_κ-point for all $\kappa < \lambda$, but is *not* a simple P_κ-point if $\kappa < \lambda$, because every base must be of cardinality $\ge \lambda$. So we do need the extra information about $\mathfrak{s}$ here.]

Theorem 2.1 *(iv)* now follows: if $\mathcal{V}$ is a simple P_κ-point and $\kappa < \mathfrak{d}$, the preceding paragraph shows that $\mathcal{V}$ cannot be a pseudo-P_λ-point for any $\lambda > \mathfrak{b}$; on the other hand, $\mathfrak{b} \le \mathfrak{u}$, so $\mathcal{V}$ cannot be generated by fewer than $\mathfrak{b}$ members; thus $\kappa = \mathfrak{b}$. □

We now obtain a sharpening of Theorem 2.3.

Corollary 3.2. *If there is a simple P_κ-point, then either $\kappa = \mathfrak{b}$ or $\kappa = \mathfrak{d}$. If $\kappa = \mathfrak{b}$, then $\mathfrak{u} = \mathfrak{r} = \mathfrak{b} \le \mathfrak{s}$, while if $\kappa = \mathfrak{d}$, then $\mathfrak{s} = \mathfrak{d}$.*

Proof. Since $\mathfrak{u} \ge \mathfrak{b}$, $\kappa \ge \mathfrak{b}$. If $\kappa > \mathfrak{b}$, use Theorem 3.1 *(iv)* to conclude that $\kappa = \mathfrak{d}$. The rest follows from the ZFC inequalities $\mathfrak{b} \le \mathfrak{r} \le \mathfrak{u}$ and $\mathfrak{s} \le \mathfrak{d}$ and from Lemma 1.3 and Theorem 2.1 *(iv)*. □

Remark 3.3. In [13, Corollary 1.2], *(viii)* was paraphrased: *For any ultrafilter $\mathcal{U}$, we have either $\pi\mathfrak{p}(\mathcal{U}) \le \mathfrak{b}$ or $\pi\chi(\mathcal{U}) \ge \mathfrak{d}$.* This suggests the following weakening of Problem 1.

Problem 4. Is it consistent to have both $\mathfrak{r} < \mathfrak{d}$ and $\pi\mathfrak{p} > \mathfrak{b}$? In other words, is it consistent to have an ultrafilter $\mathcal{V}$ with a π-base of cardinality $< \mathfrak{d}$ and also a pseudo-$P_{\mathfrak{b}^+}$ point $\mathcal{W}$?

Problem 5. If there is a pseudo-$P_{\mathfrak{b}^+}$ point, is there a $<^*$-unbounded chain of cofinality $> \mathfrak{b}$? what if there is a $P_{\mathfrak{b}^+}$ point?

Of course, if $\mathcal{W}$ is a *simple* $P_{\mathfrak{b}^+}$ point, with $\subset^*$-descending-well-ordered base $\mathcal{B}$, then $\{\psi_B : B \in \mathcal{B}\}$ is a $<^*$-unbounded chain of cofinality $\mathfrak{d} > \mathfrak{b}$.

Another corollary of Theorem 3.1 has to do with more general ultrafilters.

Corollary 3.4. *If $\mathcal{U}$ is an ultrafilter that satisfies $cf({}^\omega\omega, <_\mathcal{U}) < \lambda$, and there is a pseudo-$P_\lambda$ point, then $cf({}^\omega\omega, <_\mathcal{U}) = \mathfrak{b}$.*

Proof. As shown in [7], there can be only one $\kappa < \mathfrak{s}$ such that $cf({}^\omega\omega, <_\mathcal{U}) = \kappa$ for some free ultrafilter $\mathcal{U}$, and so by Theorem 3.1 *(i)*, there is one where $\kappa = \mathfrak{b}$, and $\lambda \leq \mathfrak{s}$ by Lemma 1.3. $\square$

4. Six axioms on unbounded chains

The behavior of $<^*$-unbounded chains modulo ultrafilters in general is the topic of this section, where we survey six axioms and various combinations of them and their negations.

In the following axiom schema, C always stands for a $<^*$-unbounded, $<^*$-well-ordered family of increasing functions, $\mathcal{U}$ stands for a free ultrafilter on ω, and "$\mathcal{U}$-unbounded" means that the chain C is unbounded (equivalently, cofinal) in the order $<_\mathcal{U}$.

1. $\forall C(\forall \mathcal{U}(C$ is $\mathcal{U}$-unbounded$))$
2. $\exists C(\forall \mathcal{U}(C$ is $\mathcal{U}$-unbounded$))$
3. $\exists \mathcal{U}(\forall C(C$ is $\mathcal{U}$-unbounded$))$
4. $\forall \mathcal{U}(\exists C(C$ is $\mathcal{U}$-unbounded$))$
5. $\forall C(\exists \mathcal{U}(C$ is $\mathcal{U}$-unbounded$))$
6. $\exists C(\exists \mathcal{U}(C$ is $\mathcal{U}$-unbounded$))$

Abstract logic gives $1 \implies 2 \implies 4 \implies 6$ and $1 \implies 3 \implies 5 \implies 6$. In addition, Juris Steprāns showed [28] $2 \implies 5$. This has recently been improved by Alan Dow [unpublished] to $2 \implies 3$, so that $1 \implies 2 \implies 3 \implies 5 \implies 6$. As will be seen, no other implications between the axioms hold in ZFC. But there are several problems left, one of the most interesting being:

Problem 6. Do Axiom 3 and Axiom 4 together imply Axiom 2?

Neither of these axioms is sufficient by itself; see Theorem 4.5 below and the comments at the end of Section 6. We will return to Problem 6 in Section 5. In particular, a negative answer to Problem 6 can only be in a model where $\mathfrak{d}$ is singular: see the comment after Theorem 5.4.

Axioms 1 and 2 have easy equivalents: Axiom 2 is equivalent the existence of a *scale*: a well-ordered cofinal family in $({}^\omega\omega, <^*)$. Indeed, any C that witnesses Axiom 2 will be a scale, because if g is not dominated by some member of C, then the sets of the form $D(f, g) = \{n : f(n) \leq g(n)\}$ as f ranges over C from a filter, and if $\mathcal{U}$ is an ultrafilter extending this, then C is $\mathcal{U}$-bounded. Conversely, as remarked in Section 1, any dominating $<^*$-chain is $\mathcal{U}$-cofinal. By this same argument, Axiom 1 is equivalent to *every* $<^*$-unbounded chain of increasing functions being a scale. Models of Axiom 1 include the "dominating reals" model [4], the Laver model for the Borel Conjecture [22], and Peter Dordal's factored Mathias forcing models [16].

As is well known, the existence of a scale is equivalent to $\mathfrak{b} = \mathfrak{d}$, and Martin's Axiom (MA) implies $\mathfrak{p} = \mathfrak{c} = \mathfrak{b}$ and thus Axiom 2 — but not Axiom 1, which implies $\mathfrak{p} < \mathfrak{d}$ [4], [16].

Axiom 3 shares the following property with Axioms 1 and 2.

Lemma 4.1. *Axiom 3 implies that all $<^*$-unbounded $<^*$-chains are of cofinality $\mathfrak{b}$.*

Proof. An elementary cofinality argument shows that Axiom 3 implies that all $<^*$-unbounded $<^*$-chains are of the same cofinality. Now use the well known fact that ZFC implies the existence of a $<^*$-unbounded $<^*$-chain of increasing functions, of cofinality $\mathfrak{b}$. $\square$

In [1], an argument like that for Axiom 2 implying the existence of a scale is used to cover $\omega^* = \beta\omega - \omega$ by nowhere dense simple P-sets in models where there are $<^*$-unbounded $<^*$-chains of different cofinalities — in other words, where Axiom 3 *fails.* The argument goes: for each free ultrafilter $\mathcal{U}$ have C range over a $<^*$-unbounded chain of increasing functions which is $\mathcal{U}$-bounded by g_C. The filter one obtains as above associates naturally with a nowhere dense simple P-set containing $\mathcal{U}$.

In [24], ultrafilters like the above are called *T-points* because each nowhere dense simple P-set in ω^* comes from a complete tower in $[\omega]^\omega$.

Definition 4.2. A *tower* is a $\subset^*$-descending well-ordered sequence in $[\omega]^\omega$. A tower is *complete* if it has no infinite pseudo-intersection. An ultrafilter $\mathcal{U}$ on ω is a *T-point* if there is a complete tower $\mathcal{T} \subset \mathcal{U}$.

For convenience, we introduce:

Axiom 4.3. Axiom T: *Every ultrafilter on ω is a T-point.*

As explained above, $\neg 3$ implies Axiom T. The converse fails as Alan Dow and his then-student Geta Techanie showed a number of years back (unpublished): they showed that the following model even satisfies Axiom 1 + Axiom T.

Example 4.4. Let P be the forcing poset for the countable support iteration of length ω_2 in which the poset for a Miller real is added at limit ordinals of cofinality ω_1 while the poset for a Laver real is added at successor stages and stages of countable cofinality. Adding a generic subset G of P to the ground model produces a forcing extension in which Axiom 1 and Axiom T both hold.

Dow has recently shown [unpublished] that Axiom T holds also in the usual Laver model, which has long been known to satisfy Axiom 1. In other words, the interweaving of Miller forcing can be dispensed with. In particular, this shows that ω^* can be covered by nowhere dense P-sets in Laver's model. This answers a question posed in [22], where it was shown that if ω^* can be covered by nowhere dense P-sets in this model, then it can be covered by nowhere dense simple P-sets.

Models of Axiom T have an additional topological significance. In them, there is a family of sequentially compact regular (in fact, normal) spaces whose product is not countably compact [24]. It is not known whether the existence of such a family can be deduced from ZFC. If "regular" is weakened to "Hausdorff," the answer is Yes [25].

Now, Axiom T fails under CH [1, 1.9.3]. However, CH and (more generally) $\mathfrak{b} = \mathfrak{c}$ provide alternative methods of constructing such families of regular spaces [17, 13.1]. These examples cannot be normal under the PFA, though [23].

A consequence of Theorem 2.1 *(i), (ii), (iii)* and *(v)* is that a model with a free ultrafilter with a π-base of cardinality $< \mathfrak{d}$ either satisfies $\neg\, 4$ (if there is no $<^*$-chain of cofinality $\mathfrak{d}$) or $\neg\, 3$ (if there is one, but then (v) also shows that Axiom 6 holds). The following result is related.

Theorem 4.5. *If $({}^\omega\omega, <^*)$ has a cofinal family of order type $\kappa \times \lambda$, $cf(\kappa) \leq cf(\lambda)$, then $cf(\kappa) = \mathfrak{b}$, $cf(\lambda) = |\lambda| = \mathfrak{d}$, and Axiom 4 holds. (And if $\mathfrak{b} \neq \mathfrak{d}$ then Axioms 2 and 3 fail.)*

Proof. If $\mathfrak{b} = \mathfrak{d}$, then Axiom 2 and hence Axiom 4 hold. Otherwise, Axiom 2 obviously fails, and the existence of a $<^*$-unbounded $<^*$-chain of order type $\mathfrak{b}$ ensures $cf(\kappa) = \mathfrak{b}$, and then $cf(\lambda) = \mathfrak{d} = |\lambda|$ by cofinality in ${}^\omega\omega$. So there is a cofinal family $\{f_{\alpha,\nu} : \alpha < \mathfrak{b}, \nu < \mathfrak{d}\}$, with each $f_{\alpha,\nu}$ increasing. With $\mathcal{U}$ fixed, suppose each row $\{f_{\alpha,\nu} : \alpha < \mathfrak{b}\}$ is $\mathcal{U}$-bounded, say by f_{β_ν,η_ν}. There must exist β such that $\beta = \beta_\nu$ for $\mathfrak{d}$-many distinct $\nu < \mathfrak{d}$. Thus the βth column is $\mathcal{U}$-unbounded, since any $\mathcal{U}$-bound would $\mathcal{U}$-dominate the whole array, negating cofinality in $({}^\omega\omega, <^*)$. So Axiom 4 holds, but Axiom 3 fails due to $cf(\lambda) = \mathfrak{d} > \mathfrak{b}$ in this second case. □

This argument routinely extends to models (which may be constructed by the technique of [18]) where $({}^\omega\omega, <^*)$ has a cofinal family of order type $\lambda_1 \times \cdots \times \lambda_n$. Of course, $\mathfrak{b}$ is the smallest and $\mathfrak{d}$ is the greatest of the $cf(\lambda_i)$.

It is now easy to show that a model with simple P-points of two different cofinalities satisfies $4+5+\neg 2+\neg 3$. Later we will show [Corollary 5.7] that a simple $P_{\mathfrak{b}^+}$ point is enough for this. Even a pseudo-$P_{\mathfrak{b}^+}$ is enough to imply Axiom 6, by Theorem 3.1 *(v)*, while 3.1 *(iii)* shows that Axiom 2 fails.

5. Three more axioms, and another look at models of NCF

To further analyze the axioms of the preceding section, we introduce three more axioms, and recall the following theorem of Mike Canjar:

Theorem 5.1. [14] *There is an ultrafilter $\mathcal{U}$ such that $cf({}^\omega\omega, <_{\mathcal{U}}) = cf(\mathfrak{d})$.*

We continue to follow the format of Axioms 1 through 6:

Axiom 5.5. $\exists\mathcal{U}$ such that $cf({}^\omega\omega, <_{\mathcal{U}}) = \mathfrak{b}$.

Axiom 6.5 $\exists\mathcal{U}\exists C$ such that $cf({}^\omega\omega, <_{\mathcal{U}}) = cf(C)$.

Axiom D. $\exists C$ such that $cf(C) = \mathfrak{d}$.

Clearly, Axiom 6 implies Axiom 6.5. Also, as the numbering suggests,

Lemma 5.2. *Axiom 5 $\implies$ Axiom 5.5 $\implies$ Axiom 6.*

Proof. The first implication follows from the existence of a $<^*$-unbounded $<^*$-chain of increasing functions of cofinality $\mathfrak{b}$, and a simple cofinality argument. For the second implication, let $\mathcal{U}$ be such that $cf({}^\omega\omega, <_{\mathcal{U}}) = \mathfrak{b}$, and replace a $\mathcal{U}$-cofinal set of order type $\mathfrak{b}$ with an unbounded $<^*$-chain of increasing functions by a simple transfinite induction so that each function in the $<^*$-chain dominates all "earlier" functions there and all "earlier" functions in the $\mathcal{U}$-cofinal set. □

Corollary 5.3. *Axiom 4 implies Axiom D if and only if $\mathfrak{d}$ is regular.*

Proof. If $\mathfrak{d}$ is regular, use Theorem 5.1 to get $\mathcal{U}$ such that $cf({}^{\omega}\omega, <_{\mathcal{U}}) = \mathfrak{d}$, and apply Axiom 4. For the converse, note that Axiom D implies $\mathfrak{d}$ is regular. $\square$

Problem 7. Does Axiom 4 imply $\mathfrak{d}$ is regular? Equivalently, does Axiom 4 imply Axiom D?

The converse problem is also unsolved.

Problem 8. Does Axiom D imply Axiom 4?

An affirmative answer to Problem 7 would imply one to Problem 6 ("Do Axiom 3 and Axiom 4 together imply Axiom 2?"), because of:

Theorem 5.4. *Axiom 2 is equivalent to Axiom 3 + Axiom D.*

Proof. Axiom 2 clearly implies Axiom D; in fact, it implies that $cf({}^{\omega}\omega, <_{\mathcal{U}}) = \mathfrak{b} = \mathfrak{d}$ for all free ultrafilters $\mathcal{U}$. And, as stated earlier, it implies Axiom 3. The converse follows from Lemma 4.1. $\square$

Because of this theorem and Corollary 5.3, a negative answer to Problem 6 would have to come in a model with singular $\mathfrak{d}$. and Theorem 5.4.

Problem 9. Is Axiom 2 equivalent to the statement that $cf({}^{\omega}\omega, <_{\mathcal{U}}) = \mathfrak{b}$ for all free ultrafilters $\mathcal{U}$?

By Theorem 5.1, the statement does imply that $\mathfrak{b} = cf(\mathfrak{d})$. So here, too, the answer could be negative only in a model where $\mathfrak{d}$ is singular.

The following lemma was used in [28] to show that Axiom 2 implies Axiom 5.

Lemma 5.5. *If there is a $<^*$-unbounded $<^*$-chain C of cofinality $\mathfrak{d}$, such that $\{c \restriction A : c \in C\}$ is $<^*$-unbounded for all $A \in [\omega]^{\omega}$, then $\exists \mathcal{U}(C$ is $\mathcal{U}$-unbounded).*

Corollary 5.6. *Axiom D implies Axiom 6.*

Proof. If C is a witness for Axiom D, then C is $\mathcal{U}$-unbounded for some free ultrafilter $\mathcal{U}$. This is because of the Lemma 5.5 and the elementary fact that a $<^*$-unbounded $<^*$-chain of increasing functions traces a $<^*$-unbounded family on every infinite subset of ω. $\square$

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Here is another interesting application of Lemma 5.5.

Corollary 5.7. *If there is a simple $P_{\mathfrak{b}^+}$ point, then $4 + 5 + D + \neg 3 + \neg 2$.*

Proof. It follows from Theorem 3.1 *(vi)* and Theorem 4.5 that $4 + \neg 3 + \neg 2$ is satisfied. To show Axiom 5, let $\mathcal{W}$ be a simple $P_{\mathfrak{b}^+}$ point. Theorem 3.1 *(vi)* implies every C as in the axiom schema is of cofinality either $\mathfrak{b}$ or $\mathfrak{d}$. In the first case, use Theorem 3.1 *(v)*. In the second case, use Lemma 5.5.

To show Axiom D, use Theorem 3.1 *(iv)* and the fact that the functions ψ_B from a totally ordered base of $\mathcal{W}$ form a $<^*$-unbounded $<^*$-chain of increasing functions, of cofinality $\mathfrak{d}$. $\square$

Shelah and Steprāns showed [28] that $\neg 6$ and $4 + \neg 5.5$ (hence also $6 + \neg 5.5$ and $6 + \neg 5$) are consistent, leaving us with the following questions where direct implications between numbered axioms are concerned:

Problem 10. Is $6.5 + \neg 6$ consistent? What about $5.5 + \neg 5$?

A model of $6.5 + \neg 6$ would have to satisfy $\mathfrak{d} > \aleph_2$. Indeed, if Axiom 6 (or even Axiom 5.5) fails, then $cf({}^\omega\omega, <_\mathcal{U}) \geq \aleph_2$ for all $\mathcal{U}$. Then if $\mathfrak{d} = \aleph_2$, Axiom 6.5 implies Axiom D and hence Axiom 6.

The various models of NCF cannot serve for Problems 7, 8, or either part of Problem 10, but they do show the consistency of $4 + D + \neg 5.5$ and hence of $4 + D + \neg 5$. The demonstration begins with a simple lemma and corollary.

Lemma 5.8. *Let $f : \omega \to \omega$ be finite-to-one. Let $\{g_\alpha : \alpha < \kappa\}$ be cofinal and well-ordered with respect to $<_{f(\mathcal{U})}$, where $\mathcal{U}$ is a free ultrafilter on ω. Then $\{g_\alpha \circ f : \alpha < \kappa\}$ is cofinal and well-ordered with respect to $<_\mathcal{U}$.*

Proof. If the functions $g_\alpha \circ f$ are not $\mathcal{U}$-cofinal, then there is a function g that is above $g_\alpha \circ f$ on $A_\alpha \in \mathcal{U}$ for all α. Let $h(n) = max\{g(k) : k \in f^{-1}\{0, \dots, n\}\}$. Then on $f[A_\alpha] \in f(\mathcal{U})$, h is above g_α. This contradicts $f(\mathcal{U})$-cofinality of $\{g_\alpha : \alpha < \kappa\}$. The second conclusion is even easier to show. $\square$

Corollary 5.9. *If $f : \omega \to \omega$ is finite-to-one, then $cf({}^\omega\omega, <_\mathcal{U}) = cf({}^\omega\omega, <_{f(\mathcal{U})})$ for every free ultrafilter $\mathcal{U}$ on ω,*

Indeed, $\leq$ is immediate from Lemma 5.8, where we may assume κ is regular, so that $\{g_\alpha \circ f : \alpha < \kappa\}$ cannot have a cofinal subset of strictly smaller cardinality, and neither can any other $\mathcal{U}$-unbounded chain.

Theorem 5.10. *NCF implies that* **(a)** $cf({}^\omega\omega, <_\mathcal{U}) = \mathfrak{d} > \mathfrak{b}$ *for all free ultrafilters $\mathcal{U}$ on ω. Hence it implies $\neg 5.5$.*

Proof. Use the characterization of NCF at the beginning of Section 2; Theorem 2.1 *(i)* and *(ii)*; and Corollary 5.9. $\square$

In contrast, Theorem 3.1 *(i)* says that any pseudo-$P_{\mathfrak{b}^+}$ point is a witness for Axiom 5.5.

Theorem 5.11. *If NCF holds, then $\mathfrak{d}$ is regular, and Axiom 4, Axiom D, and Axiom 6.5 are all equivalent.*

Proof. Regularity of $\mathfrak{d}$ is immediate from Theorems 5.1 and 5.10. By Corollary 5.3, this makes Axiom 4 and Axiom D equivalent. Clearly Axiom 4 implies Axiom 6.5. Conversely, Theorem 5.10 and Axiom 6.5 give a chain C of cofinality $\mathfrak{d}$, hence Axiom D holds, and with it Axiom 4. $\square$

It now follows from Theorems 5.10 and 5.11 and Remarks 2.2(4) that $4 + D + \neg 5.5$ is consistent.

All ZFC implications between Axiom D and numbered axioms have been settled except for Problems 7 and 8. [A model of $3 + \neg D$ will be given in the next section.] However, there are a number of combinations that are still not known to be consistent, including some involving Axiom D or its negation. These are:

$5+\neg 4+\neg 3$, $4+5.5+\neg 5$, $4+D+5.5+\neg 5$, and anything from $4+\neg D$ (Problem 7) to $4+5+\neg D+\neg 3+\neg 2$, and anything from $6.5+\neg 5+\neg 4$ to either $5.5+\neg 5+\neg 4$ or $6+\neg 5.5+\neg 4$.

6. Some applications of Cohen forcing

In this section, we analyze the effect of adding mutually Cohen reals, obtaining various models of $5.5+\neg 4$. As is well known, this kind of forcing cannot add $<^*$-well-ordered chains of cofinality $>\omega_1$, so Axiom D will fail in all but exceptional cases.

In what follows, CTM will be shorthand for "countable transitive model of a sufficiently large fragment of ZFC".

Lemma 6.1. *Let M be CTM, and let $A \subset \omega$ be Cohen over M. If $\mathcal{B} \in M$ is a filterbase on ω, then in $M[A]$, $\mathcal{B}$ extends to a filterbase $\mathcal{B}'$ such that $D(f, \psi_A) \in \mathcal{B}'$ for all $f \in {}^\omega\omega \cap M$. Moreover, if $\mathcal{U} \in M[A]$ is an ultrafilter on ω extending $\mathcal{B}'$, then ${}^\omega\omega \cap M$ is $\mathcal{U}$-dominated by ψ_A.*

Proof. Recall that $D(f,g) = \{n \in \omega : f(n) \le g(n)\}$. The partial order P for adding A to M is $\bigcup\{{}^n 2 : n \in \omega\}$. For each finite subset F of ${}^\omega\omega \cap M$ and each $B \in \mathcal{B}$, and each $m \in \omega$, the following is dense in P:

$$D(F,B,m) = \{p \in P : \exists k \in B \setminus m (\text{the } k\text{th element in } spt(p) \text{ exceeds } f(k) \text{ for all } f \in F\}.$$

[In this definition, it is implicit that there is a kth element in $spt(p)$; of course, the support could be of cardinality $< k$, in which case $p \notin D(F,B,m)$.]

Let G be a P-generic such that $A = spt(\bigcup G)$. Let $\mathcal{D}$ be the collection of all finite intersections of the sets $D(f,\psi_A)$ ($= \{n : f(n) \le$ the $(n+1)$st element of A), as f ranges over ${}^\omega\omega \cap M$. Then each $B \in B$ meets every member of $\mathcal{D}$, and the common refinement of $\mathcal{B}$ and $\mathcal{D}$ satisfies the description of $\mathcal{B}'$. □

Theorem 6.2. *If uncountably many mutually Cohen reals are added to a CTM M, the resulting model satisfies Axiom 5.5. If M has a $<^*$-well-ordered unbounded family of cofinality $>\omega_1$, then Axiom 3 fails. Also, if κ is a regular cardinal $>\omega_1$ in M for which there is no $<^*$-unbounded chain of cofinality κ in M, and if $\ge \kappa$ mutually Cohen reals are added, then Axiom 4 fails.*

Proof. The theorem refers to the usual poset for adding λ Cohen reals side by side, $Fn(\lambda, 2)$. This is equivalent to adding them iteratively with finite supports. Notation and justification of comments in this paragraph and the next can be found in [20, Ch VII, Section 2].

If λ is uncountable, we can separate out exactly $\aleph_1$ of the mutually Cohen reals, listing their enumeration functions as $\{\psi_\alpha : \alpha < \omega_1\}$, and adding the others first to produce a model N to which we add the ψ_α iteratively, with finite supports, to arrive at our final model, characterized as $N[\{\psi_\alpha : \alpha < \omega_1\}] =: N'$.

At the αth stage of the iteration, we use Lemma 6.1 to obtain an ultrafilter $\mathcal{U}_\alpha$ such that ψ_α bounds all functions in $N[\{\psi_\beta : \beta < \alpha\}] \cap {}^\omega\omega$ modulo $\mathcal{U}_\alpha$. This is done so that $\mathcal{U}_\alpha$ extends the earlier $\mathcal{U}_\beta$. Then in the final model N', the union $\mathcal{U}$ of all the $\mathcal{U}_\alpha$ is an ultrafilter since all subsets of ω appear at some initial stage.

Also in N', the ψ_α form a $\mathcal{U}$-unbounded family which can be replaced by a $<^*$-unbounded $<^*$-chain by the usual diagonal argument. Hence $N' \models$ Axiom 5.5 $\wedge\ \mathfrak{b} = \omega_1$. [The latter property is well known to be satisfied when forcing with $Fn(\lambda, 2)$ where λ is uncountable, and we have just given a proof for it.] Hence if M has a $<^*$-well-ordered unbounded family of cofinality $> \omega_1$, then Axiom 3 fails by Lemma 4.1.

This construction generalizes to any uncountable regular κ not exceeding the number of Cohen reals added, giving us an ultrafilter $\mathcal{V}$ such that the functions $\psi_\alpha(\alpha < \kappa)$ form a $\mathcal{V}$-cofinal family, with each ψ_α to $\mathcal{V}$-dominate all functions in $N[\{\psi_\beta : \beta < \alpha\}]$. On the other hand, no unbounded chain in $({}^\omega\omega, <^*)$ that occurred in earlier stages is $<^*$-dominated [1, 2.5]. Now $\mathcal{V}$ gives a witness of $\neg\,4$: no $<^*$-unbounded chain in $({}^\omega\omega, <^*)$ that occurred in earlier stages is $<^*$-dominated, and so there are no unbounded $<^*$-chains of cofinality κ in N'. $\square$

Remark 6.3. There is a superscript on $\mathfrak{d}$ in the following corollary, because $\mathfrak{d}$ can be raised by Cohen forcing.

Corollary 6.4. *If M is a CTM, and more than $\mathfrak{d}^M$ Cohen reals are added, then Axiom 4 fails and so does Axiom D.*

Proof. If $\kappa > \mathfrak{d}^M$, then there is no $<^*$-unbounded $<^*$-chain of cofinality κ in N', but $\mathfrak{d} \geq \kappa$ in the final model, so Axiom D fails. Moreover, if $\mathcal{V}$ is as above, it is a witness for $\neg\,4$. $\square$

Corollary 6.5. *If M is a CTM in which all $<^*$-unbounded chains are of cofinality ω_1, and more than ω_1 mutually Cohen reals are added, then Axiom 4 fails and so does Axiom D.*

Back when the notes of the mid-80's were circulated, the author had a copy of a preprint by the late James Baumgartner which had the following theorem: if exactly $\aleph_2$ Cohen reals are added to a model of CH, then Axiom 3 holds in the forcing extension. Unfortunately, the author has been unable to find this preprint or a reconstruction of the proof. However, around 2002, Alan Dow and his then-student Geta Techanie showed this theorem to be true if one also assumes $\Box_{\aleph_1}$ holds in the ground model. By Corollary 6.5, this gives a model of Axiom 3 $+\neg$ Axiom 4 $+\neg$ Axiom D, and shows that Axiom 3 does not imply Axiom 2 in ZFC.

7. Some generalizations

Shelah and Steprāns [28] have varied the axiom schema used in the preceding sections by letting C range over chains of increasing functions and also chains of arbitrary functions, and replacing $<^*$ with the order $\leq^*$ given by

$$f \leq^* g \text{ if there exists } n \text{ such that } f(i) \leq g(i) \text{ for all } i \geq n.$$

What keeps this from leading to an unmanageable proliferation of axioms is that, as explained in [28], in many cases it makes no difference which order or family of functions is involved, and in some other cases the axioms are simply false. For example, it makes no difference whether increasing or nowhere decreasing functions are used: Axiom n holds in a model if, and only if, its variant for strictly increasing functions is true. This is also true if $\leq^*$ is substituted for $<^*$, and each $<^*$-axiom is equivalent to the corresponding $\leq^*$-axiom.

It also makes no difference to Axioms 2, 4, 6, and 6.5 whether C consists of increasing or arbitrary functions, or whether one uses $<^*$ or $\leq^*$. On the other hand, the modifications of Axioms 1 and 3 are simply false if arbirary functions are used, while Axiom 5 becomes much stronger, and no longer follows from Axiom 2 or even from CH; but OCA suffices for both the $<^*$ version and the (possibly still stronger) $\leq^*$ version [28].

Another generalization, with very different changes, comes when one replaces unbounded chains (denoted C in the axiom schema) with unbounded upwards directed subsets, denoted D below. Let Axiom $n_{\mathcal{D}}$ stand for Axiom n with this modification. Then Axiom $2_{\mathcal{D}}$ is simply true (in ZFC), because we can put a dominating family for D, and it remains dominating in $({}^{\omega}\omega, <_{\mathcal{U}})$ for any free ultrafilter $\mathcal{U}$. And so Axioms $4_{\mathcal{D}}$ and $6_{\mathcal{D}}$ are true, and it is easy to see that $D_{\mathcal{D}}$ and $6.5_{\mathcal{D}}$ are also true.

The situation is quite different for the odd-numbered axioms, because $\forall D$ is more demanding than $\forall C$. So Axioms $1_{\mathcal{D}}$, $3_{\mathcal{D}}$, and $5_{\mathcal{D}}$ imply Axioms 1, 3, and 5 respectively, but the converses are open problems. We have a more fundamental problem with the first two:

Problem 11. Is Axiom $1_{\mathcal{D}}$ or Axiom $3_{\mathcal{D}}$ consistent?

We do have:

Theorem 7.1. *Axiom 2* $\implies$ *Axiom* $5_{\mathcal{D}}$ $\implies$ *Axiom 5.*

Proof. The proof in [28] that Axiom 2 implies Axiom 5 is essentially a proof that Axiom 2 implies Axiom $5_{\mathcal{D}}$. This is because the proof of Lemma 5.5 in [28] makes no use of the fact that C is a $<^*$-chain, only that it is a family of functions whose restriction to each $A \in [\omega]^{\omega}$ is $<^*$-unbounded, and that no $<^*$-unbounded family in ${}^{\omega}\omega$ can be of cardinality $< \mathfrak{d}$. This last stipulation holds because Axiom 2 is equivalent to $\mathfrak{b} = \mathfrak{d}$, and Axiom $5_{\mathcal{D}}$ follows. The fact that $5_{\mathcal{D}} \implies 5$ is a matter of elementary logic. □

Problem 13. Is Axiom 5 equivalent to Axiom $5_{\mathcal{D}}$?

As remarked above, one could put 1 (or 3) in place of both 5's, but Problem 11 takes precedence.

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