The structure theory of T_5 and related locally compact, locally connected spaces under the PFA and PFA(S)[S]

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Abstract. A detailed structure theorem is shown for locally compact, locally connected, hereditarily normal spaces and for normal, locally compact, locally connected, hereditarily ω_1-scwH spaces in models of PFA(S)[S], and for the latter kinds of spaces in models of PFA. Corollaries include a powerful refinement theorem like that for monotonically normal spaces, and the corollary that the spaces involved are [hereditarily] collectionwise normal and [hereditarily] countably paracompact. Among the problems left unsolved and discussed at the end is the ambious question of whether it is consistent that hereditarily normal, locally compact, locally connected spaces are actually monotonically normal. An affirmative solution would also solve the problem of consistency of every perfectly normal, locally compact, locally connected space being metrizable and thus also solve a 1935 problem due to Alexandroff.

In [Ny3], the consistency of a strong metrization theorem for manifolds was "proven":

Statement M. *Every T_5 (i.e., hereditarily normal), hereditarily cwH manifold of dimension > 1 is metrizable.*

[Here "cwH" stands for "collectionwise Hausdorff" and "manifold" means "connected space in which each point has a neighborhood homeomporphic to $\mathbb{R}^n$ for some positive integer n"; by Invariance of Domain, n is the same for all points.]

Also in [Ny3], the consistency of the following more general statement was announced:

Statement A. *Every (clopen) component of every locally compact, locally connected, T_5, hereditarily cwH space is either Lindelöf or has uncountably many cut points.*

These results rested on a pair of axioms that turned out to be incompatible. The metrization theorem was salvaged [Ny5] within a week after this discovery, but the consistency of Statement A remained in doubt for a decade. But now we have a theorem of which it is an immediate corollary.

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Main Theorem. *Let X be a locally compact, locally connected space. If either*

(1) PFA(S)[S] *holds and X is T_5, or*

(2) PFA or PFA(S)[S] *holds and X is normal and hereditarily ω_1-scwH,*

then every component of X is the union of an open Lindelöf space L and a countable discrete collection of closed, connected, countably compact noncompact subspaces, each of which includes uncountably many cut points of the component and has exactly one point in the closure of L.

Moreover, each subspace in this collection is a "string of beads" in which there is a set C of cut points of the whole component, such that C is homeomorphic to an ordinal of uncountable cofinality, and each "bead" is the connected 2-point compactification of an open subspace whose boundary consists of the two extra points, successive members of C. The union of these "beads" comprises the entire string.

This theorem recalls the fact that a locally connected space is the topological direct sum of its (connected) components; in other words, its components, which are disjoint, are open (and closed because they are components). Thus, almost any topological property of a locally connected space can be ascertained by examining its components separately.

The more specialized topological properties in the Main Theorem involve the following concepts. An *expansion* of $D \subset X$ is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = \{d\}$ for all $d \in D$. A space X is *[strongly] collectionwise Hausdorff* (abbreviated *[s]cwH*) if every closed discrete subspace of X has an expansion to a disjoint [*resp.* discrete] collection of open sets. A space is ω_1-*[s]cwh* if every closed discrete subspace of of cardinality ω_1 has an expansion to a disjoint [*resp.* discrete] collection of open sets.

A quick corollary of the main theorem is that X is both collectionwise normal and countably paracompact. Under alternative (1) this is (therefore) true of every subspace of X. This and other strong corollaries will be proven in Section 7. Their proofs are self-contained once the Main Theorem is granted, so the interested reader may skip to there now.

All manifolds of dimension > 1 are Lindelöf, hence metrizable, under the conditions in the Main Theorem, because they have no cut points and so the set of countably compact, noncompact subspaces given by the Main Theorem is empty. Thus the following theorem, which strengthens Statement M by significantly relaxing its hypothesis, is a corollary of (1) in the Main Theorem [but was obtained a few months earlier]:

Theorem T. [DT] *Under* PFA(S)[S], *every T_5 manifold of dimension > 1 is metrizable.*

Theorem T was the culmination of years of work by several researchers, especially Frank Tall, showing that the set theoretic and topological axioms which were used to show the consistency of Statement M, and which were derived from the PFA, also hold in PFA(S)[S] models. [Incidentally, PFA(S)[S] is not an axiom in the usual sense, but a recipe for producing certain models of ZFC.] This research also showed that the following axioms hold in PFA(S)[S] models.

Axiom C [*resp.* **Axiom C′**]. *Every locally compact* [resp. *first countable*] *normal space is ω_1-cwH.*

Axiom C′ was adequate for Theorem T, but the Main Theorem requires Axiom C, which was much more difficult to show in PFA(S)[S] models [Ta].

The Main Theorem requires axioms beyond the usual ZFC axioms of set theory. Nonmetrizable T_5, hereditarily scwH manifolds of dimension > 1 have been constructed in many models of the ZFC axioms. The first one, due to Mary Ellen Rudin, is even perfectly normal. It is described in [Ny, Example 3.14].

The three Hausdorff nonmetrizable manifolds-with-boundary of dimension 1 are the long line $\mathcal{L}$ and the open and closed long rays $\mathcal{L}^+$ and $\mathcal{L}^+ \cup \{0\}$ respectively. In the first case, L can be taken to be the real line, and there are two strings S_0 and S_1 such that C_0 can be taken to be the set $\omega_1 \setminus \omega$ of countably infinite ordinals, while C_1 is the copy of C_0 with the ordinals in reverse order. The "beads" are then the copies of $[0,1]$ overlapping at their endpoints.

The open and closed long rays are the subspaces of $\mathcal{L}$ consisting of the points > 0 and ≥ 0, respectively. Each is the union of a Lindelöf subspace and one "string of beads." In the case of the closed long ray, the Lindelöf subspace can be taken to be empty.

It is not known whether the PFA also works for (1) in the Main Theorem. In other words, the following is open:

Problem 1. Does the PFA imply that every T_5 locally compact, locally connected space is (hereditarily) scwH?

Remarkably enough, this remains unanswered if "the PFA" is replaced by "ZFC." Similarly, the following is open:

Problem 2. *Is every locally compact, locally connected normal space scwH?*

It would be enough to prove that any such space is cwH, since every normal cwH space is scwH. By the Main Theorem, and the quick corollary mentioned above, Problem 1 is equivalent to asking that any such space is ω_1-scwH. [Recall that T_5 = hereditarily normal.] Neither local compactness nor local connectedness can be dropped in Problem 1: under MA(ω_1), implied by the PFA, adding $\aleph_1$ points on the x-axis to the open upper half plane in the Niemytzki tangent disk space results in a normal Moore (hence T_5) locally connected space that is not ω_1-cwH, while adding $\aleph_1$ points of the

Cantor set to the full binary tree of height ω has the same effect, except that now the resulting space is locally compact but not locally connected.

In Sections 1, 2, and 3, the only consequence of the PFA used in these sections is MA(ω_1), to give some purely topological axioms, and similarly modest consequences of PFA(S)[S] will used. These consequences will be used on the base case of connected spaces of Lindelöf degree ω_1. The fictitious picture that confronts us there is that of a tree of height ω_1, with at most countably many limbs (but possibly uncountably many twigs) at each fork, and twigs and barbed thorns growing out of its trunk and limbs. In Section 1 we build such a base case space Y inside an arbitrary component of a general space X as in the Main Theorem. In section 2, after some lemmas that will be applied to any such space Y, we obtain a result for the special case of manifolds that was originally obtained via algebraic topology [Ny3].

In Section 3, the bead strings of Y emerge, and are clipped to produce a space T where their remnants look like thorns (perhaps with barbs) growing out of the limbs of the tree. This sets the stage for Section 4, where, for the first time since Section 1, ZFC-independent consequences of the PFA and PFA(S)[S] are brought into play. Due to them, the space T shrivels down to a countable height, looking somewhat like a cycad, with the bead strings of Y emanating from the tips of sharply pointed leaves. This completes the case where the space has Lindelöf degree ω_1 and sets a solid foundation for the general case.

In the remaining sections, there are no new uses for the ZFC-independent axioms. In Section 5, we start to build a tree for the Lindelöf degree $> \omega_1$ case, using some maneuvering to overcome a hurdle at the $\omega_1 + 1$st stage, because the first ω_1 stages may not result in a closed subspace, and we need normality for the Section 4 core collapse.

In Section 6 units like that obtained at the end of Section 5 are used to accelerate the analysis of the space and thus complete the proof of the Main Theorem.

Section 7 features some major consequences of the Main Theorem and of its proof and gives a preview of a forthcoming paper, posing some additional problems. The paper closes with a proof that the Main Theorem holds for a special class of T_5 spaces, the monotonically normal spaces, without using any axioms besides the usual (ZFC) axioms.

All through this paper, "space" means "Hausdorff (T_2) topological space." Concepts not defined here may be found in [E] or [W] or in a reference given when the concept is first discussed.

1. The ω_1-Lindelöf case: canonical sequences and associated trees

Up through Section 5, we will be mostly be considering a non-Lindelöf space Y as in the Main Theorem, with the additional properties of being connected, and of Lindelöf degree ω_1 (that is, every open cover has a subcover of cardinality $\leq \aleph_1$). In a locally compact space, the latter property is easily shown to be equivalent to being the union of $\aleph_1$ open sets with compact closures. Such open sets form the most natural base for a locally compact space, but we will often find it useful to use other kinds of basic open sets. For example:

Lemma 1.1. *Every locally compact space has a base of open Lindelöf subsets. If in addition the space is locally connected, then the open Lindelöf connected subsets form a base.*

Proof. The first sentence has a quick proof using the fact that locally compact spaces are Tychonoff. But here is a unified proof for both sentences.

Let X be locally compact. For each $p \in X$ and open set G such that $p \in G$, let G_0 be an open set such that $\overline{G_0}$ is compact and $p \in G_0$, $\overline{G_0} \subset G$. Suppose G_n has been defined so that $\overline{G_n}$ is compact and $\overline{G_i} \subset G_n$ for all $i < n$. Cover the boundary of G_n with finitely many open subsets whose closures are compact and contained in G. Let G_{n+1} be the union of G_n with these finitely many open sets; then $\overline{G_n} \subset G_{n+1}$ and the induction proceeds through ω. Let $H = \bigcup_{n=0}^{\infty} G_n$; H is an open Lindelöf (because σ-compact) neighborhood of p contained in G.

If X is, in addition, locally connected, let G_0 be connected, and let all the open sets covering the boundary of each G_n be connected and meet G_n. An easy induction shows that each G_n is connected and so H is connected. $\square$

We introduce an extension of the concept of a canonical sequence [Ny1], which is the case $C = \omega_1$ of:

Definition 1.2. *Let θ be an ordinal of uncountable cofinality and let C be a closed unbounded (*"club"*) subset of θ. A* **canonical C-sequence in a space** *X is a well-ordered family $\Sigma = \langle X_\xi : \xi \in C \rangle$ of open subspaces of X such that $\overline{X_\xi}$ is Lindelöf and $\overline{X_\xi} \subset X_\eta$ for all $\xi < \eta$ in C, and $X_\alpha = \bigcup\{X_\xi : \xi < \alpha\}$ for all limit points α of C.*

With a slight abuse of language, we let $\bigcup \Sigma$ mean $\bigcup\{X_\xi : \xi \in C\}$ and will usually suppress the C- prefix.

The proof of the Main Theorem involves the construction of a canonical sequence for the components of a space X as described. In the nontrivial case where X has non-Lindelöf components, the first ω_1 stages will produce a subspace Y as above inside any non-Lindelöf component of X. To get beyond the first ω stages under the PFA, we need the following result. Its proof in [Ny2] has "hereditarily scwH" instead of "hereditarily ω_1-scwH," but the proof goes through unchanged for the present wording.

Theorem 1.3. [MA(ω_1)] *In a locally compact, hereditarily ω_1-scwH space, every open Lindelöf subset has Lindelöf closure and hereditarily Lindelöf, hereditarily separable boundary.* $\square$

In particular, the boundary is first countable. This reduction of character plays an essential role in Section 4.

The proof of Theorem 1.3 in [Ny2, Lemma 1.2] established the fact, true in ZFC, that a hereditarily ω_1-scwH space has the property that every open Lindelöf subset has a boundary of countable spread. The place where Martin's Axiom came in was that MA(ω_1) implies the following ZFC-independent axiom:

Axiom Sz. *Every locally compact space of countable spread is hereditarily separable and hereditarily Lindelöf.*

When using PFA(S)[S], one uses the theorem that its weakening MA_{ω_1}(S)[S] implies Axiom Sz [To]; a different proof is given in [LT].

Theorem 1.4. *Under the hypotheses of the Main Theorem, if X is connected and of Lindelöf degree ($= \ell(X)$) $\geq \omega_1$, then there is a canonical sequence $\Sigma = \langle X_\alpha : \alpha \in \omega_1 \rangle$ in X such that each X_α is connected, open, Lindelöf, and properly contained in X_β whenever $\beta > \alpha$. If X is of Lindelöf degree ω_1, then $\bigcup \Sigma = X$ is obtainable.*

Proof. If we are not aiming for $\{X_\alpha : \alpha \in \omega_1\}$ to cover X (which is impossible anyway if $\ell(X) > \omega_1$) the proof is very much like the proof of Lemma 1.1, but with some numbers increased. We need not be concerned about X_α having compact closure if α is a natural number, and cannot expect it if $\alpha \geq \omega$. We do make X_0 a connected Lindelöf open subspace, and by MA(ω_1) it will have Lindelöf closure.

In general, if X_α has been defined, we cover its boundary with countably many connected Lindelöf open sets, each of which meets the boundary of X_α, and let $X_{\alpha+1}$ be the union of X_α with the added sets. If α is a limit ordinal and X_ξ has been defined for all $\xi < \alpha$, we do what is called for by 1.2, *viz.,* let $X_\alpha = \bigcup\{X_\xi : \xi < \alpha\}$. Since α is countable, X_α is Lindelöf, etc.

To ensure that $\bigcup \Sigma = X$ in case $\ell(X) = \omega_1$, we utilize a theorem and a concept in [W, 26.14, 26.15]. If S is a connected space, and $\mathcal{U}$ is any open cover of S, then any two points p, q of S are *connected by a simple chain in* $\mathcal{U}$.

That is, there is a finite sequence $U_0, \ldots U_n$ of members of $\mathcal{U}$ such that $p \in U_0 \setminus U_1$, $U_i \cap U_j \neq \emptyset \iff |i - j| \leq 1$, and $q \in U_n \setminus U_{n-1}$.

Let $\mathcal{W} = \{W_\xi : \xi < \omega_1\}$. Let $X_0 = W_0$. If X_α has been defined, we proceed as above, but if W_α meets the boundary of X_α we include it in a countable cover of the boundary of X_α. Otherwise, if $W_\alpha \subset X_\alpha$ we define $X_{\alpha+1}$ as before; if, in contrast, $\overline{W_\alpha} \cap X_\alpha = \emptyset$, we also pick a point p_α on the the boundary of X_α and include in $X_{\alpha+1}$ a simple chain in $\mathcal{W}$, $W(0), \ldots W(n)$, such that $p_\alpha \in W(0)$ and $W(n) = W_\alpha$. $\square$

It is useful to picture X as a tree, using the components of each open subspace of the form $X \setminus \overline{X_\xi}$ as the "limbs" that start at the "fork" at the boundary of X_ξ. Because X is locally connected, each component is open. Note that "limbs" never re-connect: if V is a component of $X \setminus \overline{X_\eta}$ and W is a component of $X \setminus \overline{X_\xi}$ and $\eta < \xi$ then either

$W \cap V = \emptyset$ or $W \subset V$; and then, $\overline{W} \subset V$ because X_ξ is an open set containing $\overline{V} \setminus V$. In general, the boundary of each component of $X \setminus \overline{X_\alpha}$ is a nonempty subset of $\overline{X_\alpha} \setminus X_\alpha$.

There are only countably many really large limbs at each fork:

Lemma 1.5. *Under the hypotheses of Theorem 1.4, all but countably many components of $X \setminus \overline{X_\xi}$ are subsets of $X_{\xi+1}$ whenever $\xi < \omega_1$.*

Proof. Let $\mathcal{V}$ be the (disjoint) collection of all components of $X \setminus \overline{X_\xi}$ that meet $X \setminus X_{\xi+1}$. Connectedness of each $V \in \mathcal{V}$ implies that V meets $\overline{X_{\xi+1}} \setminus X_{\xi+1}$. But this set is hereditarily Lindelöf and so cannot contain a family of more than countably many disjoint (relatively) open sets, so $\mathcal{V}$ must be countable. $\square$

Rather than letting $Y = \bigcup\{X_\alpha : \alpha < \omega_1\}$ be obtained by an arbitrary application of the proof of Theorem 1.4, we will utilize Lemma 1.5 and Theorem 1.6 below to produce a fast-growing "tree" inside X.

Theorem 1.6. *Let X be a locally compact, locally connected space in which every open Lindelöf subset has Lindelöf closure, and let V be a connected open subspace of X with Lindelöf boundary. Then there is a connected, Lindelöf, $\overline{V}$-relatively open neighborhood $H(V)$ of $bd(V)$ in $\overline{V}$.*

Proof. Using Lemma 1.1, cover $bd(V)$ with countably many Lindelöf, connected open sets $W_n (n \in \omega)$. Then $bd(V)$ is in the $\overline{V}$-interior of $\bigcup\{W_n \cap \overline{V} : n \in \omega\} = W$. The boundary of $W \cap \overline{V}$ is the union of two disjoint closed sets, $bd(V)$ and the boundary B in V of $W \cap V$ in X. If $W \cap V$ is not connected, each component meets B, so that if H is a connected, Lindelöf, open subset of V containing B, then $H(V) = W \cup H$ is as desired.

To obtain H, let $\mathcal{D}$ be a cover of V by connected Lindelöf open subsets and let $\{D_n : n \in \omega\} \subset \mathcal{D}$ cover B. For each $i \in \omega$ let $\mathcal{C}_i$ be a simple chain in $\mathcal{D}$ from D_i to D_{i+1}. Let $C_i = \bigcup \mathcal{C}_i$ and let $H = \bigcup_{i=0}^\infty D_i \cup C_i$. $\square$

We now build $\{Y_\alpha : \alpha < \omega_1\}$ by induction, using an arbitrary $\{X_\alpha : \alpha < \omega_1\}$ as in Theorem 1.4 as a foundation. Let $Y_0 = X_0$. If $Y_\alpha \supset X_\alpha$ has been defined for $\alpha \in \omega_1$ so that Y_α is connected, open, and Lindelöf, use Lemma 1.1 and Theorem 1.3 and the remarks following 1.3 to get a cover $\{G_n : n \in \omega\}$ of the boundary $\overline{Y_\alpha} \setminus Y_\alpha$ of Y_α by connected, open, Lindelöf subsets of X, each of which meets Y_α, and let $S_{\alpha+1} = Y_\alpha \cup X_{\alpha+1} \cup \bigcup_{n=0}^\infty G_n$.

Then $S_{\alpha+1}$ is Lindelöf, and as in Lemma 1.5, only countably many components of $X \setminus \overline{Y_\alpha}$ meet $X \setminus \overline{S_{\alpha+1}}$. Let V be such a component. If $\overline{V}$ is Lindelöf, let $H(V) = V$, otherwise let $H(V)$ be as in Theorem 1.6. Let

$$Y_{\alpha+1} = \overline{S_{\alpha+1}} \cup \bigcup\{H(V) : \ V \text{ is a component of } X \setminus \overline{Y_\alpha} \text{ that meets } X \setminus \overline{S_{\alpha+1}}\}.$$

Then $Y_{\alpha+1}$ is clearly connected, open and Lindelöf. Letting $Y_\alpha = \bigcup_{\xi<\alpha} Y_\xi$ whenever α is a countable limit ordinal completes the construction of the canonical sequence $\Sigma(Y) = \langle Y_\alpha : \alpha < \omega_1 \rangle$. In particular, $\overline{Y_\alpha}$ is Lindelöf for all α by Theorem 1.3 and the susequent remarks, and it is connected by the elementary fact that the closure of a connected set is connected. Let $Y = \bigcup\{Y_\alpha : \alpha < \omega_1\}$.

Lemma 1.7. *Let $\alpha \in \omega_1$ and let V be a component of $X \setminus \overline{Y_\alpha}$. Then $V \cap Y$ is a component of $Y \setminus \overline{Y_\alpha}$. Moreover, $V \cap Y_\beta$ is connected and $\overline{V \cap Y_\beta} = \overline{V} \cap \overline{Y_\beta}$ for all $\beta > \alpha$. Also, if V has Lindelöf closure in X, then $V \subset Y_{\alpha+1}$.*

Proof. First, suppose V has Lindelöf closure. If $V \cap X \setminus \overline{S_{\alpha+1}} = \emptyset$, then $V \subset S_{\alpha+1} \subset Y_{\alpha+1}$. Otherwise, $H(V) \cap V = V \subset Y_{\alpha+1}$ and so $V \cap Y = V$ is a component of $Y_{\alpha+1} \setminus \overline{Y_\alpha}$ and hence of $Y \setminus \overline{Y_\alpha}$.

If V has non-Lindelöf closure, then $H(V)$ is a connected subset of $Y_{\alpha+1}$ that contains the boundary of $V \setminus S_{\alpha+1}$. Also, $H(V) \cap V$ is an open, connected subset of V and is the only summand in the definition of $Y_{\alpha+1}$ that meets V. Therefore, $H(V) \cap V = H(V) \setminus \overline{Y_\alpha} = V \cap Y_{\alpha+1}$.

The component of $Y \setminus \overline{Y_\alpha}$ that contains $H(V) \cap V$ must be a subset of V; on the other hand, Y is built up by induction in such a way that this component contains all points of $Y \cap V$. Indeed, the boundary of $H(V) \cap V$ in V is the intersection of the boundary of $Y_{\alpha+1}$ with V; and the open summands in the definition of $Y_{\alpha+2}$ are all either subsets of V or disjoint from V, and the union of the former with $Y_{\alpha+1} \cap V$ is connected. The rest of the induction works in the same way, so that $V \cap Y_\beta$ is connected for all $\beta > \alpha$.

Obviously,

$$\overline{V \cap Y_\beta} \subset \overline{V} \cap \overline{Y_\beta} = (\overline{V} \cap Y_\beta) \cup [\overline{V} \cap (\overline{Y_\beta} \setminus Y_\beta)]. \tag{1}$$

Since Y_β is open, it follows that $(\overline{V} \cap Y_\beta) = V \cap Y_\beta \subset \overline{V \cap Y_\beta}$.

As for $\overline{V} \cap (\overline{Y_\beta} \setminus Y_\beta) = (\overline{V} \cap (\overline{Y_\beta}) \setminus Y_\beta$: points of $\overline{Y_\beta} \setminus Y_\beta$ are in the closures of the individual components of $Y_\beta \setminus \overline{Y_\alpha}$ and there is no overlap where the closures meet $\overline{Y_\beta} \setminus Y_\beta$, and so $\overline{V} \cap \overline{Y_\beta}) \setminus Y_\beta \subset \overline{V \cap Y_\beta} \setminus Y_\beta$, and the reverse containment is trivial. ☐

Lemma 1.8. *Let $\alpha < \beta$ and let V_α and V_β be components of $Y \setminus \overline{Y_\alpha}$ and $Y \setminus \overline{Y_\beta}$ respectively, such that $V_\beta \subset V_\alpha$. Then $\overline{V_\alpha} \setminus V_\beta$ and $V_\alpha \setminus V_\beta$ are both connected, and $\overline{V_\alpha \setminus V_\beta} = \overline{V_\alpha} \setminus V_\beta$.*

Proof. Since the closure of a connected set is connected, it is enough to show that $V_\alpha \setminus V_\beta$ is connected and that $\overline{V_\alpha \setminus V_\beta} = \overline{V_\alpha} \setminus V_\beta$.

Clearly, $\overline{V_\alpha} = \overline{V_\alpha \setminus V_\beta} \cup \overline{V_\beta}$, and so

$$\overline{V_\alpha} \setminus V_\beta \text{ is the union of } \overline{V_\alpha \setminus V_\beta} \setminus V_\beta \text{ and } \overline{V_\beta} \setminus V_\beta. \tag{2}$$

but $\overline{V_\alpha \setminus V_\beta} \cap V_\beta = \emptyset$ because V_β is open, so the first summand is simply $\overline{V_\alpha \setminus V_\beta}$, which is thus a subset of $\overline{V_\alpha} \setminus V_\beta$. And since $\overline{V_\beta} \subset V_\alpha$, the second summand is a subset of $V_\alpha \setminus V_\beta \subset \overline{V_\alpha \setminus V_\beta}$, so equality holds.

To show $V_\alpha \setminus V_\beta$ is connected, let $W = (V_\alpha \setminus \overline{Y_\beta}) \setminus V_\beta$ and let $Z = V_\alpha \cap \overline{Y_\beta}$. Clearly, $W \cup Z = V_\alpha \setminus V_\beta$. So we will be done as soon as we show:

Claim 1. *Z is connected.*

and

Claim 2. *$\overline{V} \cap Z \neq \emptyset$ for every component V of W.*

Proof of Claim 1. We will show $Z \subset \overline{V_\alpha \cap Y_\beta}$, from which Claim 1 follows because $V_\alpha \cap Y_\beta$ is connected by Lemma 1.7: in general, if A is a connected subset of a topological space and $A \subset B \subset \overline{A}$, then B is connected.

Now $Y \setminus \overline{Y_\alpha}$ is the topological direct sum of its components, one of which is V_α. So if $A \subset Y \setminus \overline{Y_\alpha}$, the closure of A in $Y \setminus \overline{Y_\alpha}$ is the union of the closures of the sets $A \cap V$ where V ranges over the components of $Y \setminus \overline{Y_\alpha}$. Applying this to $A = Y_\beta$ shows that $Z \subset \overline{V_\alpha \cap Y_\beta}$.

Proof of Claim 2. Let V be any component of W; then V is also a component of $Y \setminus \overline{Y_\beta}$. Hence $\overline{V} \cap \overline{Y_\beta} \neq \emptyset$, but $\overline{V} \subset \overline{V_\alpha}$ and so $\overline{V \cap Z} \neq \emptyset$. □

2. Singleton boundary components and the case of manifolds

As our first step in pinning down L and the bead strings in the Main Theorem, we show what might be called the "Thorn Lemma":

Lemma 2.1. *Let X be a locally compact, locally connected space with a canonical C-sequence $\Sigma = \langle X_\alpha : \alpha \in C \rangle$. Let α be a limit ordinal and let $p \in \overline{X_\alpha} \setminus X_\alpha$. If p is isolated in the relative topology of $\overline{X_\alpha} \setminus X_\alpha$, and G is an open neighborhood of p with compact closure, such that $G \cap \overline{X_\alpha} \setminus X_\alpha = \{p\}$, then there exists $\xi < \alpha$ such that $\overline{G} \cap X_\alpha \setminus X_\nu$ is (nonempty, and) clopen in the relative topology of $X_\alpha \setminus X_\nu$ whenever $\xi \leq \nu < \alpha$.*

Proof. Since G is a neighborhood of p, and Σ is canonical, $G \cap X_\alpha \setminus X_\mu$ is nonempty for all $\mu < \alpha$. If there is no such ξ, then there is an ascending κ-sequence $\langle \xi_\nu : \nu \in \kappa \rangle$ (where κ is the cofinality of α) whose supremum is α and for which there exists $a_\nu \in \overline{G} \setminus G \cap (X_\alpha \setminus X_{\xi_\nu})$. Any complete accumulation point of $A = \{a_\nu : \nu \in \kappa\}$ must be in $\overline{G}$ and can only be p itself, but G is a neighborhood of p missing A, contradicting compactness of $\overline{G}$. □

Note that $\overline{G} \cap \overline{X_\alpha} \setminus X_\nu = \{p\} \cup (\overline{G} \cap X_\alpha) \setminus X_\nu$ for all ν as in Lemma 2.1. In the case of Y as constructed in Section 1, the structure of the "thorn" $\overline{G} \cap \overline{Y_\alpha} \setminus Y_\alpha$ can be analyzed further.

Theorem 2.2. *Let α be a countable limit ordinal. If p is isolated in the relative topology of $\overline{Y_\alpha} \setminus Y_\alpha$, there exists $\xi < \alpha$ and an open neighborhood G of p with compact closure, such that*

(a) *$G \cap \overline{Y_\alpha} \setminus Y_\alpha = \{p\}$;*

(b) $G \cap \overline{Y_\alpha} \setminus Y_\xi = \overline{G} \cap \overline{Y_\alpha} \setminus Y_\xi$ *is clopen in the relative topology of* $\overline{Y_\alpha} \setminus Y_\xi$*;*

(c) $V_\xi \cap \overline{Y_\alpha} \subset G$*; and*

(d) *If* $\xi \leq \nu < \alpha$*, and* V_ν *is the component of* $Y \setminus \overline{Y_\nu}$ *containing* p*, then* $V_\nu \cap \overline{Y_\alpha} \setminus Y_\alpha = \{p\}$*, and* V_ν *is the only component of* $V_\xi \setminus \overline{Y_\nu}$ *that meets* $Y \setminus Y_{\nu+1}$*.*

Proof. Let G be an open neighborhood of p satisfying (a). Then (b) is immediate from Lemma 2.1 and the fact that p is isolated in the relative topology of $\overline{Y_\alpha} \setminus Y_\alpha$. To show (c), use Lemma 1.7 (with ξ for α and α for β) to conclude that $V_\xi \cap Y_\alpha$ is connected. It is also a subset of $Y_\alpha \setminus \overline{Y_\xi}$, and has p in its closure, so it meets G, and is thus a subset of the component of G that contains p and hence of G itself.

This also implies $V_\xi \cap \overline{Y_\alpha} = \{p\}$. Of course, $V_\nu \cap \overline{Y_\alpha} \subset G$ also for $\xi \leq \nu < \alpha$ and so $V_\nu \cap \overline{Y_\alpha} = \{p\}$. Any component of $V_\xi \setminus \overline{Y_\nu}$ besides V_ν is a subset of G and so has compact closure. By the construction of Y in Section 1, it must therefore be a subset of $Y_{\nu+1}$. $\square$

It is clear from (d) in Theorem 2.2 that $\{p\} \cup (V_\nu \cap Y_\alpha) = V_\nu \cap \overline{Y_\alpha}$ is a neighborhood of p in the relative topology of $\overline{Y_\alpha}$ and that the set of all such neighborhoods is a local base at p in the relative topology. An interesting consequence of (d) is that $V_\nu \setminus Y_\alpha = V_\xi \setminus Y_\alpha (= \{p\} \cup V_\xi \setminus \overline{Y_\alpha})$ whenever $\xi \leq \nu < \alpha$. Informally, the part of V_ξ above $\{p\}$ is funneled through p, while the part below p is like a thorn, perhaps with barbs represented by the Lindelöf components of $V_\xi \setminus \overline{Y_\nu}$ for various ν above ξ; and the part above p does not change as we move from one ν to another. This implies that $\{p\}$ is the boundary of a non-Lindelöf component of $Y \setminus \overline{Y_\alpha}$.

The key lemma for our next theorem has as a corollary the well known fact that every locally compact, totally disconnected space is 0-dimensional [E, Theorem 6.2.9]. Its proof is virtually the same as that of this corollary.

Lemma 2.3. *In a locally compact space, every compact component has a neighborhood base consisting of clopen sets.*

The proof of the next theorem utilizes not only Lemma 2.3 but the proof of Lemma 2.1. Of course, its conclusion feeds directly into 2.1:

Theorem 2.4. *Let* α *be a countable limit ordinal. If* $\{p\}$ *is a component of* $\overline{Y_\alpha} \setminus Y_\alpha$*, then* p *is isolated in the relative topology of* $\overline{Y_\alpha} \setminus Y_\alpha$*.*

Proof. Let $B_\alpha = \overline{Y_\alpha} \setminus Y_\alpha$. Since B_α is locally compact and first countable, there is a sequence $\langle K_n : n \in \omega \rangle$ of compact clopen neighborhoods of p in the relative topology of B_α, that form a base for the neighborhoods of p there. We assume $K_{n+1} \subset K_n$ for all n, with the containment proper for all n if p is not isolated in B_α.

Let $\langle \alpha_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals whose supremum is α. For each $n \in \omega$, let G_n be a connected open neighborhood of p in Y such that $\overline{G_n}$ is compact and $G_n \cap B_\alpha = \overline{G_n} \cap B_\alpha = K_n$.

For each n, as in the proof of the Thorn Lemma (2.1), there exists $k(n)$ such that $G_n \cap Y_\alpha \setminus Y_\nu$ is clopen in the relative topology of $Y_\alpha \setminus Y_\nu$ whenever $\nu \geq \alpha_{k(n)}$. Let $\langle \nu_n : n \in \omega \rangle$ be a subsequence of $\langle \alpha_n : n \in \omega \rangle$ such that $\nu_n \geq \alpha_{k(n)}$.

Let V_n be the component of $Y \setminus \overline{Y_{\nu_n}}$ that contains p. Then $V_n \cap Y_\alpha$ is a subset of $Y_\alpha \setminus Y_{\nu_n}$ and is connected by Lemma 1.7, and meets G_n. Therefore, it is a subset of $G_n \cap Y_\alpha \setminus Y_{\nu_n}$. Then $\overline{V_n} \cap \overline{Y_\alpha} \subset \overline{G_n}$. Again by Lemma 1.7, $\overline{V_n} \cap \overline{Y_\alpha} = \overline{V_n \cap Y_\alpha}$, so that $\bigcap_{n=0}^\infty \overline{V_n}$ is the union of $\{p\}$ with a (perhaps empty) family of (open) components of $Y \setminus \overline{Y_\alpha}$, all of which have boundary $\{p\}$.

Again by Lemma 1.7, each "closed wing" of the form $\overline{V_n} \setminus V_{n+1}$ is connected. If p is not isolated in B_α, then infinitely many of these "wings" meet $B_\alpha = \overline{Y_\alpha} \setminus Y_\alpha$.

Case 1. All but finitely many "closed wings" $\overline{V_n} \setminus V_{n+1}$ are Lindelöf. Say $\overline{V_n} \setminus V_{n+1}$ is Lindelöf for all $n \geq m$. By Lemma 1.7, none of the components of $(\overline{V_n} \setminus V_{n+1}) \setminus \overline{Y_{\nu_{n+1}}}$ reaches beyond $Y_{\nu_{n+2}}$; but $Y_{\nu_{n+2}} \cap \overline{Y_\alpha} \setminus Y_\alpha = \emptyset$, a contradiction.

Case 2. There is a subsequence $\langle n_i : i \in \omega \rangle$ such that $\overline{V_{n_i}} \setminus V_{n_{i+1}}$ is not Lindelöf for all i.

<u>*Claim.*</u> *There is a point z_{n_i} on the boundary of G_{n_i} in $V_{n_i} \setminus V_{n_{i+1}}$ for all i.* Once this is proven, any limit point of the z_{n_i}'s is in $\bigcap_{n=0}^\infty \overline{V_n}$, but since none of the z_{n_i}'s is in any component of $Y \setminus \overline{Y_\alpha}$ that has $\{p\}$ as its boundary, the only possible limit point of the z_{n_i}'s is p. However, G is a neighborhood of p that excludes every z_{n_i}, and we get a contradiction as we did at the end of the proof of Lemma 2.1.

Proof of Claim. Note that $V_{n_i} \setminus V_{n_{i+1}}$ is connected and that $\overline{V_{n_i}} \cap \overline{Y_\alpha} \subset G_{n_i}$. Compactness of $\overline{G_{n_i}}$ implies that $(V_{n_i} \setminus V_{n_{i+1}}) \setminus \overline{G_{n_i}}$ is nonempty. If there were no such z_{n_i}, then $(V_{n_i} \setminus V_{n_{i+1}}) \setminus \overline{G_{n_i}}$ and $(V_{n_i} \setminus V_{n_{i+1}}) \cap G_{n_i}$ would thus be nonempty, disjoint open sets whose union is connected. $\square$

All our efforts up to now have had the aim of extending the following key theorem in [Ny3] to a carefully selected subspace of Y:

Theorem 2.5. *If M is a connected Hausdorff n-manifold with $n > 1$ and $\Sigma = \langle Y_\alpha : \alpha < \omega_1 \rangle$ is a canonical sequence for M with each Y_α a proper subset of M, then there is a canonical sequence $\Sigma_1 = \langle M_\alpha : \alpha < \omega_1 \rangle$ for M such that every point of $B_\alpha = \overline{M_\alpha} \setminus M_\alpha$ is contained in a connected infinite subset K_α of B_α.*

This theorem is the version of Lemma 2.6 of [Ny3] which was needed for the final proof of the manifold metrization Theorem M [Ny5]. As actually stated, Lemma 2.6 of [Ny3] had the additional condition of compactness for K_α. This can easily be obtained by cutting an infinite connected K_α down if necessary, but it was not needed in [Ny5]. Our extension of 2.5, for which we need one more section, will play a similarly crucial role in the proof of the Main Theorem.

The remainder of this section is about the special case of Hausdorff manifolds and can be skipped without loss of continuity. Readers interested in manifolds, on the other hand, can read to the end of this section and then skip to Section 4 without loss of understanding.

As part of the proof of Theorem 2.5 and hence of Theorem M in [Ny3] and [Ny5], the following theorem was given a proof, due to David Gauld, using algebraic topology.

Theorem S. *Suppose that D is an open, connected subset of $\mathbb{S}^n$, and $n > 1$. Then for every component C of the complement of D, the frontier of C is connected.*

We can now give a short proof of Theorem 2.5 using elementary general topology, which includes the fact that $\mathbb{R}^n \setminus \{p\}$ is connected for all $p \in \mathbb{R}^n$ if $n > 1$.

Proof of Theorem 2.5. By following the construction of $\Sigma(Y)$ in Sections 1 and 2, we may assume that all the theorems and lemmas of these sections apply to Σ.

Claim. If α is a limit ordinal, then K_α exists for each point of B_α.

Once this claim is proved, let $l(\alpha)$ be the αth limit ordinal and let $M_0 = Y_0$ and $M_\alpha = Y_{l(a)}$ whenever $\alpha > 0$. Then Σ_1 is as desired.

Proof of Claim. Suppose B_α has a finite component, which must be a singleton $\{p\}$. By Theorem 2.3, p is isolated in B_α. Let G be as in Lemma 2.1. We may take G to be an open n-disk D in M with p in its interior. If $D \setminus \overline{Y_\alpha} \neq \emptyset$, then $D \setminus \overline{Y_\alpha}$ and $D \cap Y_\alpha$ are disjoint open subsets of D, making p a cut point of D, which is impossible. So $D \subset \overline{Y_\alpha}$, and $D \setminus \{p\}$ is a connected subset of Y_α.

By the proof of Lemma 2.1, there exists $\xi < \alpha$ such that $D \cap Y_\alpha \setminus Y_\xi$ is clopen in the relative topology of $Y_\alpha \setminus Y_\xi$; therefore, $\overline{D} \setminus Y_\xi$ is clopen in the relative topology of $\overline{Y_\alpha} \setminus Y_\xi$. Let V be the component of $M \setminus Y_\xi$ containing p. Since $V \cap Y_\alpha$ is connected by 1.7 and meets $D \cap Y_\alpha \setminus Y_\xi$, it follows that $V \cap Y_\alpha$ is a subset of D.

From the fact that $D \setminus Y_\xi$ is clopen in $\overline{Y_\alpha} \setminus Y_\xi$, it follows that $D \cap V$ is clopen in V. Hence $V \subset D$, and so V has Lindelöf closure, but then $V \subset Y_{\xi+1}$, contradicting $p \in V$. $\square$

Those interested in seeing how Statement M was shown consistent can now skip to Section 4 and see the key role played by Theorem 2.5.

3. Dealing with bead strings in the ω_1-Lindelöf case

The "tree" introduced in Section 1 converts into a tree in the formal set-theoretic sense if we make the "limbs" (*i.e.,* the components of the various $X \setminus X_\alpha$) into *elements* ordered by reverse inclusion. Trivially, the set of $\supset$-predecessors of any element is well-ordered.

A closely related tree was introduced in [Ny1] for canonical sequences Σ on locally connected spaces Y of Lindelöf degree ω_1. We now extend this to arbitrary spaces X with canonical Σ as in Definition 2.1.

Definition 3.1. Let X be a space with a canonical sequence Σ indexed by an ordinal θ of uncountable cofinality and any club $C \subset \theta$. *The tree* $\Upsilon(\Sigma)$ (or $\Upsilon(X)$ or simply Υ if the context is clear) has as elements all boundaries $bd(V) = \overline{V} \setminus V$ of components V of some $X \setminus \overline{X_\alpha}$ whose closure $\overline{V}$ is not Lindelöf.

The order on Υ is from "bottom" to "top", *i.e.,* if V_0 is a component of $X \setminus \overline{X_\alpha}$ and V_1 is a component of $X \setminus \overline{X_\beta}$ for some $\beta > \alpha$, then we put $bd(V_0) < bd(V_1)$ iff $V_1 \subset V_0$.

It is easy to see that $bd(V_0) < bd(V_1)$ iff $V_1 \cap V_0 \neq \emptyset$ iff $bd(V_1) \subset V_0$ iff $bd(V_1) \cap V_0 \neq \emptyset$. We use the notation $\Upsilon(\alpha)$ to denote the αth level of Υ (with the 0th level as its first level), i.e., the set of members of Υ whose set of predecessors is of order type α. Note that $\bigcup \Upsilon(\alpha)$ is a (perhaps proper) subset of $\overline{X_\alpha} \setminus X_\alpha$.

The way Υ is defined, some members may form boundaries for more than one component of $X \setminus \overline{X_\alpha}$, including perhaps uncountably many components with compact closures. Also, distinct members of Υ may overlap, but not those from different levels.

A chain (that is, a totally ordered subset) of Υ that is bounded above need not have a (unique) supremum. In particular, if V_ξ is a component of $X \setminus \overline{X_\xi}$ for all $\xi < \eta < \gamma$, and $V_\eta \subset V_\xi$ whenever $\xi < \eta < \gamma$, and $\bigcap\{V_\xi : \xi < \gamma\} \setminus \overline{X_\gamma}$ has components with non-Lindelöf closures, then the boundary of each one of these components has $\{bd(V_\xi) : \xi < \gamma\}$ as its chain of predecessors in Υ.

If p is on the boundary of one of these components (call it V_γ), then (as noted above) $p \in \overline{X_\gamma} \setminus X_\gamma$. Moreover, if G is a connected open neighborhood of p, and $G \cap X \setminus V_\xi \neq \emptyset$, then G meets the boundary of V_ξ (otherwise G would be the union of the disjoint nonempty open sets $G \cap V_\xi$ and $G \setminus \overline{V_\xi}$). Hence, G meets the boundary of every V_η for which $\xi < \eta < \gamma$. By local connectedness, p is in the closure of $\bigcup\{bd(V_\xi) : \xi < \gamma\}$ if γ is a limit ordinal.

There is one important case where a chain that is bounded above in Υ has a unique supremum; it is the first step in identifying the bead strings of the Main Theorem.

Lemma 3.2. *Any bounded chain of singletons in $\Upsilon = \Upsilon(Y)$ has a singleton supremum. Moreover, if $\{p\} \in \Upsilon(\gamma)$ is the supremum of a chain $\mathcal{C}$ of singletons without a greatest element in the Υ order, then p is isolated in the relative topology of $\overline{Y_\gamma} \setminus Y_\gamma$.*

Proof. If a bounded chain $\mathcal{C}$ has a greatest element we are done with the first statement, so suppose not. For each $x \in \bigcup \mathcal{C}$ define $\xi(x)$ so that $x \in \overline{Y_{\xi(x)}} \setminus Y_{\xi(x)}$. Let V_x be the unique component of $Y \setminus \overline{Y_{\xi(x)}}$ such that $\{x\} = bd(V_x)$ and such that if $\{x\} < \{y\} \in \mathcal{C}$ then $y \in V_x$. Let $\gamma = sup\{\xi(x) : \{x\} \in \mathcal{C}\}$.

Let $p \in bd(V)$, where V is a component of $Y \setminus \overline{Y_\gamma}$ contained in $\bigcap\{V_x : \{x\} \in \mathcal{C}\}$. By the remarks preceding this lemma, $p \in \overline{Y_\gamma} \setminus Y_\gamma$, and every neighborhood of p contains a terminal segment ("tail") of $\bigcup \mathcal{C}$.

Since X is Hausdorff, p is the only point of $\overline{Y_\gamma} \setminus Y_\gamma$ in the closure of $\bigcup \mathcal{C}$. Hence $\{p\}$ is the unique supremum of $\mathcal{C}$.

For the "moreover" part, it is enough to show that $\{p\}$ is the component of p in $\overline{Y_\gamma} \setminus Y_\gamma$, in view of Theorem 2.4. And this follows quickly from the fact that this component is a subset of $\bigcap\{V_x : \{x\} \in \mathcal{C}\}$. [See also the next to last paragraph preceding Lemma 3.2.] $\square$

Beads are here formally defined with the above lemmas in mind, but for general X.

Definition 3.3. Let Υ be associated with a C-canonical sequence Σ on a locally compact, locally connected, connected space X. An Υ-*bead* (or Σ-*bead* or *bead* if Υ or Σ is clear) is a Lindelöf set of the form $\overline{V_p} \setminus W_q$, where $\{p\}$ and $\{q\}$ are in Υ; $\{p\} < \{q\}$; V_p is the component of some $X \setminus \overline{X_\alpha}$ such that the boundary of V_p is $\{p\}$; q is isolated in the relative topology of $\overline{X_\beta} \setminus X_\beta$; and W_q is the union of those components of $X \setminus \overline{X_\beta}$ whose boundary is $\{q\}$.

In $\Sigma(Y)$, the Lindelöf property is enough to ensure that an Υ-bead as in 3.3 is a subset of $\overline{Y_\beta} \setminus Y_\alpha$. If p is isolated in the relative topology of $\overline{Y_\alpha} \setminus Y_\alpha$, then the bead is clopen in the relative topology of $\overline{Y_\beta} \setminus Y_\alpha$, and in any case, removing p makes the rest of the bead clopen in the relative topology of $\overline{Y_\beta} \setminus \overline{Y_\alpha}$.

The following corollary of Lemma 3.2 is the key to locating the "bead strings" of the Main Theorem as applied to Y.

Corollary 3.4. *If a branch of* $\Upsilon(Y)$ *has uncountably many singletons, it has singletons on a club set of levels of* $\Upsilon(Y)$.

Proof. This is immediate from the first sentence of Lemma 3.2 and the way the levels of $\Upsilon(Y)$ are indexed by ordinals. $\square$

Let $\sigma = \{\{p_\xi\} : \xi \in C\}$ be a chain of singletons of $\Upsilon(Y)$, with p_ξ isolated in the relative topology of $\overline{Y_\xi} \setminus Y_\xi$, such that C is a club in ω_1. For each limit ordinal $\alpha \in C$ let p_α be associated with a $\xi = \xi(\alpha) < \alpha$ as in Theorem 2.2. If α is not the least element of C we may assume $\xi(\alpha) \in C$. Let $H(\alpha) = V_{\xi(\alpha)} \cap \overline{Y_\alpha}$.

By the Pressing-down Lemma there is an uncountable subset S of C and $\xi \in C$ such that $\xi = \xi(\alpha)$ all $\alpha \in S$. Connectedness of the sets H_α ensures that $p_\nu \in H_\alpha$ whenever $\xi < \nu \leq \alpha$ and $\alpha \in S$. Compactness of the sets $\overline{H_\alpha}$ ensures compactness of all the sets $(\overline{Y_{\beta^*}} \cap \overline{H_\alpha}) \setminus Y_\beta$, where $\xi \leq \beta < \alpha$, $\beta \in C$, and β^* is the immediate successor of β in C. With p_β playing the role of p and p_{β^*} playing the role of q in 3.3, it is easy to see that each of these sets $(\overline{Y_{\beta^*}} \cap \overline{H_\alpha}) \setminus Y_\beta)$ is a compact Υ-bead. And so we have a bead string exactly like in the Main Theorem for Y.

In preparation for the next section, we clip the bead strings near their bases, in the following way. Given an uncountable branch $\mathcal{B}$ of $\Upsilon(Y)$ in which the set $\mathcal{C}$ of singleton boundaries is cofinal, let $m(\mathcal{B})$ be any p_ξ for which ξ is as in the preceding paragraph. Let

$$T = Y \setminus \bigcup\{V_p : p = m(\mathcal{B}) \text{ for some } \mathcal{B} \text{ as described}\}.$$

where V_p is the unique component of $Y \setminus \overline{Y_\xi}$ whose boundary is $\{p\}$.

This trims a collection of open subspaces off Y, so that T is a locally compact subspace of Y, and normal because it is closed. It is also connected and locally connected, because of the following elementary theorem.

Theorem 3.5. *Let P be a subspace of a connected and locally connected topological space X, and let $\mathcal{V} = \{V_p : p \in P\}$ be a collection of disjoint open sets such that $\overline{V_p} \setminus V_p = \{p\} = \overline{V_p} \cap P$ for all $p \in P$. Then $X \setminus \bigcup \mathcal{V}$ is connected and locally connected.*

Proof. If $X \setminus \bigcup \mathcal{V}$ is disconnected, let K_0 and K_1 be disjoint subsets of $X \setminus \bigcup \mathcal{V}$ that are closed there, hence in X, such that $K_0 \cup K_1 = X \setminus \bigcup \mathcal{V}$. For each point x of K_i let U_x be a connected open neighborhood such that $U_x \cap K_{1-i} = \emptyset$. Then $U_x \setminus K_i$ is a subset of $\bigcup \mathcal{V}$. Since U_x is connected, it cannot meet any V_p without also containing p. Indeed, the absence of p from $U_x = (U_x \cap \overline{V_p}) \cup (U_x \setminus \overline{V_p})$ would make U_x the disjoint union of nonempty open sets. Let

$$U_i = \bigcup\{U_x : x \in K_i\} \text{ and let } W_i = \bigcup\{V_p : p \in P \cap K_i\}.$$

If $x \in K_i$, then $U_x \subset K_i \cup W_i$, so that $U_i \cup W_i = K_i \cup W_i$. Clearly, $W_0 \cap W_1 = \emptyset$, and $K_i \cap W_j = \emptyset$ for all i, j. Hence, if we let $G_i = K_i \cup W_i$, then G_0 and G_1 are disjoint nonempty open subsets of X. Every point of P is in either K_0 or K_1, so that $\bigcup \mathcal{V} = W_0 \cup W_1$, whence $G_0 \cup G_1 = X$, and X is disconnected. This contradiction shows $X \setminus \bigcup \mathcal{V}$ is connected.

Local connectedness of $X \setminus \bigcup \mathcal{V}$ follows easily. Let $x \in X \setminus \bigcup \mathcal{V}$. It is enough to show that if Z is a connected open neighborhoood of x in X, then $Z \setminus \bigcup \mathcal{V}$ is connected. If we substitute Z for X, $P \cap Z$ for P, and $\{V_p \cap Z : p \in P \cap Z\}$ for $\mathcal{V}$ in the argument for connectedness of $X \setminus \bigcup \mathcal{V}$, we obtain connectedness of $Z \setminus \bigcup\{V_p \cap Z : p \in P \cap Z\} = Z \setminus \bigcup \mathcal{V}$. This last equality follows from the fact that Z cannot meet V_p without also containing p. $\square$

Let $T_\xi = Y_\xi \cap T$ for all $\xi \in \omega_1$, let $\Sigma(T) = \langle T_\xi : \xi \in \omega_1 \rangle$ and let $\Upsilon(T) = \Upsilon(\Sigma(T))$. We will refer informally to T as the “core” of Y.

There might still be chains of singletons in $\Upsilon(T)$ of any countable order type, but eventually each one must terminate in some $\{p\} \in \Upsilon(T)$ above which all members of $\Upsilon(T)$ are infinite. We can go further, giving the extension of Theorem 2.5 that allows us to generalize the proof of Statement M in [Ny5], and subsequently prove the Main Theorem in the case of Lindelöf degree ω_1.

Theorem 3.6. *There is a club set C_T of limit ordinals in ω_1 such that, for each $\alpha \in C_T$, every component of $B_\alpha = \overline{T_\alpha} \setminus T_\alpha$ is infinite.*

Proof. Suppose not, then there is a stationary set S of limit ordinals in ω_1 for which there exists a singleton (quasi)component $\{p_\alpha\}$ of B_α for all $\alpha \in S$. By Theorem 2.2 and the following comments, there is for each $\alpha \in S$ an ordinal $\xi < \alpha$ such that, if V_ξ is the component of $X \setminus \overline{Y_\xi}$ to which p_α belongs (so that $V_\xi \cap \overline{Y_\alpha}$ is a neighborhood of p_α in the relative topology of $\overline{Y_\alpha}$), then $V_\xi \cap \overline{Y_\alpha} \setminus \overline{Y_\xi}$ is clopen in the relative topology of $\overline{Y_\alpha} \setminus \overline{Y_\xi}$.

By the Pressing Down Lemma, there is a fixed ξ which works for a stationary $E \subset S$, and so there is an uncountable $A \subset E$ such that all $p_\alpha, \alpha \in A$ share the same component V_ξ of $Y \setminus \overline{Y_\xi}$. But this means any two sets of the form $V_\xi \cap \overline{Y_\alpha}$ $(\alpha \in A)$ are $\subset$-comparable, because of the final clause in Theorem 2.2 (d). Therefore, $V_\xi \cap \overline{Y_{\alpha_1}} \setminus Y_{\alpha_2}$ is a compact bead whenever $\alpha_1 < \alpha_2$ in A. But this contradicts the assumption that all these p_α are in T. $\square$

4. The core of Y collapses under the main axioms

The stage is set for two applications of either of our main axioms (PFA and PFA(S)[S]) and one ZFC theorem, and of "normal" and "hereditarily ω_1-cwH." The argument is essentially like that beginning with Lemma A in [Ny5], but instead of M standing for a manifold of dimension > 1, we will let $M = \{B_\alpha : \alpha \in C_T\}$ (where $B_\alpha = \overline{T_\alpha} \setminus T_\alpha$).

Our strategy is to assume that the core T of Y is not Lindelöf and to get a proof by contradiction. One outcome is that the only reason Y itself is not Lindelöf is that there is at least one long bead string in Y. On the other hand, this also implies that there are only countably many long bead strings in Y. This, together with a few routine details, will finish the Lindelöf degree ω_1 case of the Main Theorem.

Remark. To those familiar with the argument following Lemma A in [Ny5], the only real difference beyond the choice of M is to let the set C_W defined thereby be a subset of C_T as in Theorem 3.6.

We begin by recalling a concept introduced in [Ny3].

Definition 4.1. A subset S of a poset P is *downward closed* if $\hat{s} \subset S$ for all $s \in S$, where $\hat{s} = \{p \in P : p \leq s\}$. A collection of subsets of a set X is an *ideal* if it is downward closed with respect to $\subset$, and closed under finite union. An ideal $\mathcal{J}$ of countable subsets of X is *countable-covering* if $\mathcal{J} \restriction Q$ is countably generated for each countable $Q \subset X$. That is, for each countable subset Q of X, there is a countable subcollection $\{J_n^Q : n \in \omega\}$ of $\mathcal{J}$ such that every member J of $\mathcal{J}$ that is a subset of Q satisfies $J \subset J_n^Q$ for some n.

Lemma 4.2. [EN, Lemma 2.1] *Let X be a locally compact Hausdorff space and let $\mathcal{J}$ be the ideal of all countable subsets of X with compact closure. Then $\mathcal{J}$ is countable-covering if, and only if, every countable subset of X has Lindelöf closure.* $\square$

Definition 4.3. *Axiom CC_{22}* is the axiom that for each countable-covering ideal $\mathcal{J}$ on a stationary subset S of ω_1, either:

(i) there is a stationary subset A of S such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there is a stationary subset B of S such that $[B]^\omega \cap J = \emptyset$.

In other words, either every countable subset of A is in $\mathcal{J}$ or $B \cap J$ is finite for all $J \in \mathcal{J}$. As part of the proof of Theorem T [DT], it was shown that Axiom CC_{22} follows from PFA(S)[S]. It was shown to be a consequence of the PFA in [EN].

The following Lemma from [Ny5] is implicit in the proof of Theorem 2.7 in [Ny3], but we will isolate its proof here.

Lemma 4.4. *If CC_{22} holds and X is a locally compact space in which every countable subset has Lindelöf closure, and S is a stationary subset of ω_1 and $\{x_\alpha : \alpha \in S\}$ is a subset of X, then there is a stationary subset E of S such that either:*

(1) *$\{x_\alpha : \alpha \in E\}$ is a closed discrete subspace of X, or*

(2) *every countable subset of $\{x_\alpha : \alpha \in E\}$ has compact closure in X.*

Proof. Let $\mathcal{J}$ be as in Lemma 4.2. If B is any subset of ω_1 such that $B \cap J$ is finite for all $J \in \mathcal{J}$, then any compact neighborhood of any point can only contain finitely members of $\{x_\alpha : \alpha \in B\}$; consequently, if alternative (ii) of CC_{22} holds, then (1) will hold. Otherwise, (2) obviously holds. □

We will apply Lemma 4.4 to $M = \{B_\alpha : \alpha \in C_T\}$. The projection map $\pi : M \to \omega_1$ that takes B_α to α is clearly continuous, and surjective if T is not Lindelöf, which is what we are assuming here.

Lemma 4.5. *Assuming CC_{22}, M contains a perfect preimage, wrt π, of a copy of ω_1.*

Proof. Choose $x_\alpha \in B_\alpha$ for all $\alpha \in C_T$, and apply Lemma 4.4. If (1) were to hold, there would be a closed discrete subspace of $\{x_\alpha : \alpha \in C_T\}$ meeting the B_α's indexed by a stationary set. Then the ω_1-cwH property of M would give a family of ω_1 disjoint neighborhoods, with the neighborhood of x_α reaching back to some $B_\xi, \xi < \alpha$. Now the Pressing Down Lemma gives one ξ that works for uncountably many α, but B_ξ is hereditarily Lindelöf, a contradiction.

So there is an uncountable subspace $H = \{x_\alpha : \alpha \in E\}$ such that each countable subset has compact closure. Therefore, $\overline{H}$ is countably compact and meets each B_α for which $\alpha \in \overline{E}$ in a closed, hence compact set. The restriction of π to $\overline{H}$ is closed, because if C is a closed subset of $\overline{H}$, it is countably compact and hence has countably compact image; but ω_1 is first countable and so every countably compact subset is closed. Finally, we use the elementary fact that every club subset of ω_1 is homeomorphic to ω_1. □

Now we use the following axiom, which Balogh showed to follow from the PFA (see [D] for a proof) and which Dow showed [DT] to hold under PFA(S)[S]:

Axiom B. *Every first countable closed preimage of ω_1 contains a copy of ω_1.*

Let W be such a copy in $\bigcup\{B_\alpha : \alpha \in C_T\}$, with $\pi^{\rightarrow}W = C_W \subset C_T$. Let $p_\alpha \in W \cap B_\alpha$ for all $\alpha \in C_W$. Using Theorem 3.6, let q_α be in the same component of B_α as p_α. Apply Lemma 4.4 to the open, hence locally compact space $M \setminus W$ to produce a stationary subset E of C_W such that the closure F of $q_\alpha : \alpha \in E$ is countably compact, hence has projection $\overline{E} \subset C_W$. Since F is disjoint from W we can apply Urysohn's Lemma to obtain a continuous function $f : M \to [0,1]$ taking W to 0 and F to 1. Because p_α and q_α are in a connected set K_α and $q_\alpha \in F$ whenever α in E, each fiber $f^{\leftarrow}\{r\}$ meets $\pi^{\leftarrow}\alpha$ for each $\alpha \in E$. The time is ripe to invoke:

Theorem 4.6. [Ny4, Theorem 2.3, in effect] *Let Z be a space which is either T_5 or hereditarily ω_1-scwH, for which there are a continuous $\pi : Z \to \omega_1$ and a stationary subset S of ω_1 such that the fiber $\pi^{\leftarrow}\{\eta\}$ is countably compact for all $\eta \in S$. Then Z cannot contain an infinite family of disjoint closed countably compact subspaces with uncountable π-images.* $\square$

Although Theorem 2.3 of [Ny4] had "scwH" rather than "ω_1-scwH," the proof goes through without change. We apply this theorem using the sets K_α, $\alpha \in E$. Let $x_\alpha \in K_\alpha$ be chosen by induction so that $f(x_\alpha) \neq f(x_\beta)$ for all $\beta < \alpha$. As in the proof of Lemma 4.5, use the ω_1-cwH property and the Pressing-Down Lemma to eliminate alternative (1) of Lemma 4.4. Alternative (2) then gives us a stationary subset S of E such that every countable subset of $\{x_\eta : \eta \in S\}$ has compact closure in M.

Let Z be the closure of $\{x_\eta : \eta \in S\}$ in M. Clearly Z is countably compact, and so $Z \cap B_\alpha$ is compact for each $\alpha \in \overline{S}$, again because B_α is hereditarily Lindelöf for all $\alpha \in \omega_1$. Now the image under f of $\{x_\eta : \eta \in S\}$ is an uncountable subset of $[0,1]$ and so has $\mathfrak{c}$-many condensation points. For each condensation point p and each countable ordinal α_0, there is a strictly ascending sequence of ordinals $\langle \alpha_n : n \in \omega \rangle$ and points $x_{\eta_n} \in K_{\eta_n}$ for $n > 0$ such that $|p - f(x_{\eta_n})| < 1/n$.

Let $\alpha = sup\{\alpha_n : n \in \omega\}$. Since Z is countably compact, there is a point of $Z \cap B_\alpha$ which is sent to p by f. Thus the sets $Z \cap f^{\leftarrow}\{p\}$ are a family of $\mathfrak{c}$-many disjoint closed countably compact sets with uncountable π-range.

This contradiction to Theorem 4.6 finishes the overall proof by contradiction by showing $B_\beta = \emptyset$ for all but countably many β, giving:

Theorem 4.7. *The subspace T of Y is Lindelöf.* $\square$

And so, Y as a whole is the union of T with a set of unbounded strings of compact beads, and since these strings all emanate from T, there are only countably many of them. The proof of the Main Theorem will now be complete if we go a bit up each unbounded bead string B. This string is associated with a branch $\mathcal{B}$ of Υ with uncountably many singletons. Now $m(\mathcal{B})$ remains in T, but all of B "above" this point is in $Y \setminus T$. Let $B_0 \subset B$ be a bead which conforms to Definition 3.3, with $p = m(\mathcal{B})$. Let $q = q_B$ then be as in Definition 3.3, with $\{q\}$ on a limit level in Υ, above that of p.

Now let $L = T \cup \bigcup\{B_0 \setminus \{q_B\} : B \in \mathfrak{B}\}$ where $\mathfrak{B}$ is the set of countably many bead strings that were clipped to produce T. Each $B_0 \setminus \{q_B\}$ is Lindelöf, because q_B is on a limit level, and so L is also Lindelöf, and it is clearly open.

What remains of each $B \in \mathfrak{B}$ is countably compact and noncompact, and together they constitute a discrete collection of closed sets in Y. This completes the proof of the Main Theorem for the case of Lindelöf degree ω_1.

In preparation for the general case, we let $X_\alpha = Y_\alpha$ for all $\alpha < \omega_1$, and $X_{\omega_1} = Y$, and let $B_{\omega_1} = \overline{X_{\omega_1}} \setminus X_{\omega_1}$. Each singleton $\{p\} \subset B_{\omega_1}$ is the boundary of some component of $X \setminus \overline{X_{\omega_1}}$: otherwise, by local compactness, p and most of the bead string leading up to it would have been in some Y_α $(\alpha < \omega_1)$. This argument can be extended to show that $\{p\}$ must be the boundary of at least one component of $X \setminus \overline{X_{\omega_1}}$ with non-Lindelöf closure, but no more than finitely many. Indeed, by local compactness, all but finitely many components of $X \setminus \overline{X_{\omega_1}}$ with $\{p\}$ as boundary are subsets of each neighborhood of p, and are thus have compact closure. For the same reason, if there are no components with non-Lindelöf closure, then the union of all these components with $\{p\}$, along with the bead string leading up to p is Lindelöf. So here, too, this union is part of some Y_α $(\alpha < \omega_1)$.

5. Maneuver at stage ω_1 when Lindelöf degree $> \omega_1$

The subspace X_{ω_1} defined just now is not closed in X if the Lindelöf degree of X is greater than ω_1. This is not a problem if we assume our space is T_5, and we can proceed with Section 6.

However, if we assume (2) in the Main Theorem, this puts a hurdle in the way of further progress: the argument in Section 4 will not go through without more careful preparation if X_{ω_1} is not normal. This preparation consists mostly of showing that $\overline{X_{\omega_1}}$ has a canonical ω_1-sequence in which all uncountable bead strings are closed in X. We begin by recalling the following concept:

Definition 5.1. A space X satisfies *Property wD* if every infinite closed discrete subspace D of X has an infinite subspace D' that expands [as explained in the introduction following the Main Theorem] to a discrete collection of open sets.

Of course, every ω_1-scwH space satisfies Property wD. So does every normal space, because normal spaces are "$\aleph_0$-collectionwise normal," meaning that every countable discrete family of closed sets has an expansion to open sets. The following is a special case of Lemma 1.6 of [Ny4]:

Lemma 5.2. *Let X be a locally compact space satisfying Property wD hereditarily, and let $X = \bigcup\{X_\alpha : \alpha < \theta\}$, with $\overline{X_\alpha} \subset X_\beta$ whenever $\alpha < \beta$, and with $X_\alpha = \bigcup\{X_\xi : \xi < \alpha\}$ whenever α is a limit ordinal. For each limit ordinal γ of uncountable cofinality, the boundary of X_γ in X is a closed discrete subspace.*

Corollary 5.3. $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ *is a (closed) discrete subspace of* X.

Corollary 5.4. $|\overline{X_{\omega_1}} \setminus X_{\omega_1}| \leq \omega_1$.

Proof. Otherwise, by a proof like that for Theorem 2.2, there is for each point p of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ an $\alpha < \omega_1$ such that one of the components of $X \setminus X_\alpha$ meets $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ in $\{p\}$. But there cannot be more than ω_1 such components altogether. $\square$

Now we look at $\overline{X_{\omega_1}}$ as though it were all of X, and go through the procedure of building $\langle Y_\alpha : \alpha < \omega_1 \rangle$ exactly as was done before Lemma 1.7, except that we label it $\Sigma^* = \langle Y^*_\alpha : \alpha < \omega_1 \rangle$ and make sure $Y_\alpha \subset Y^*_\alpha$ for all α. However, now that each point of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ has associated with it a component of some $X \setminus \overline{Y^*_\alpha}$ whose closure is a compact neighborhood of the point, the associated component now becomes a subset of $Y^*_{\alpha+1}$. And thus, $\overline{X_{\omega_1}} = \bigcup \{Y^*_\alpha : \alpha < \omega_1\}$. Since $\overline{X_{\omega_1}}$ is closed in X, it is normal, etc. and now we can collapse its core as in the preceding section.

Once the core is collapsed, it turns out that $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ is actually countable. This was obvious in the T_5 case since the points had to be at the end of uncountably long bead strings, of which there were countably many. Here, with those points pulled back into the various Y^*_α, it is because the points of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ were in non-Lindelöf components of $X \setminus \overline{Y_\alpha}$ for some α, and there can be only countably many such components emanating from each $\overline{Y_\alpha}$, hence only countably many from each $\overline{Y^*_\alpha}$. And so there exists $\alpha \in \omega_1$ such that $\overline{X_{\omega_1}} \setminus Y^*_\alpha$ consists of countably many bead strings, none of which have points of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ in their closure.

Now that Σ^* has served its purpose, we revert to $\Sigma = \langle Y_\alpha : \alpha < \omega_1 \rangle$ as in the earlier sections, so that now the points of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ are at the end of long strings of compact beads.

6. The dash to the finish

The analysis of X of Lindelöf degree $> \omega_1$ can now be completed in accelerated fashion. With $\overline{X_{\omega_1}}$ as our foundation, we can jump $\omega_1 + 1$ levels up in $\Upsilon(X) = \Upsilon$ with a construction like that in Sections 1 through 5. We repeat this as often as necessary (at least up to ω_2), with each jump beginning with some $\overline{X_\gamma}$ with closed discrete boundary and ending in $\overline{X_{\gamma+\omega_1}}$.

It is helpful to go along one branch $\mathcal{B}$ of $\Upsilon(X)$ at a time, pausing at each level $\Upsilon(\gamma)$ for which γ is either an ordinal of uncountable cofinality or the supremum of a sequence of such ordinals. On such a level, there is exactly one isolated $\{p\}$ in $\Upsilon(\gamma) \cap \mathcal{B}$. By a slight abuse of language, we call both p and $\{p\}$ "jump points," treating them as prescribed in the preceding paragraph. After each sequence of jumps indexed by a limit ordinal in its natural order, we have a single jump point on the branch $\mathcal{B}$ (as in Lemma 3.2) unless the sequence is cofinal in $\mathcal{B}$.

Call a jump point $\{p\} \in \Upsilon(\gamma)$ "trivial" if the analogue of $\overline{X_{\omega_1}}$ immediately above it and below level $\gamma+\omega_1$ is just a long string of compact beads, whose endpoints have a supremum in $\bigcup \Upsilon(\gamma+\omega_1)$. The following lemma makes it easy to identify these points.

Lemma 6.1. *Let p be a jump point in B_γ and let W be the union of all components of $X \setminus \overline{X_\gamma}$ with p in their closure. Then p is trivial if, and only if,*

(1) *$A =: \overline{W} \cap \overline{X_{\gamma+\omega_1}}$ is compact and*
(2) *$W \cap B_{\gamma+\omega_1} (= W \cap \overline{X_{\gamma+\omega_1}} \setminus X_{\gamma+\omega_1})$ is a singleton.*

Proof. Necessity is clear, so we show sufficiency. Compactness of A implies that all long bead strings in A meet $B_{\gamma+\omega_1}$. Condition (2) ensures that exactly one component V of $X \setminus X_\gamma$, [and indeed of $X \setminus X_{\gamma+\alpha}$ for all countable α], has non-Lindelöf closure. Finally, compactness of A implies that the Lindelöf space L_γ that corresponds to L has compact closure. There is a countable limit α such that $\overline{L_\gamma} \subset X_{\gamma+\alpha}$ and $W \cap X_{\gamma+\alpha}$ has discrete boundary. Exactly one point on this boundary is in the closure of the one component of $X \setminus \overline{X_{\gamma+\alpha}}$ that has non-Lindelöf closure. The closure of this component meets $X_{\gamma+\omega_1}$ in a compact bead string, and $\overline{W} \cap X_{\gamma+\alpha}$ meets all the qualifications of a compact bead. □

Lemma 6.2. *Any ω-sequence of nontrivial jump points on a branch $\mathcal{B}$ of Υ is cofinal in $\mathcal{B}$.*

Proof. Otherwise, by Lemma 3.2, the sequence $\langle p_n : n \in \omega \rangle$ converges on a point p for which $\{p\}$ is on a limit level α of countable cofinality. For each $n \in \omega$ let ν_n satisfy $p_n \in B_{\nu_n}$. By an easy exension of Theorem 2.2 to $\Sigma(X)$ and ordinals of countable cofinality, all but finitely many p_n are in a compact clopen neighborhood of p, of the form $\overline{V} \cap X_\alpha$ in the relative topology of X_α, where V is a non-Lindelöf component of $X \setminus X_\xi$ for some $\xi < \alpha$, such that Υ has no branches containing $bd(V)$ that branch off above ξ but before α.

If $p_n \in V$ then $V \cap \overline{X_{\nu_{n+1}}} \setminus X_{\nu_n}$ is easily seen to satisfy the properties of a compact bead with endpoints p_n and p_{n+1}. And now, by Lemma 6.1, the part of $\mathcal{B}$ between p_n and p_{n+1} represents a string of compact beads, no matter how many trivial jumps there are between p_n and p_{n+1}. This contradicts nontriviality of p_n. □

Theorem 6.3. *Υ has at most countably many nontrivial jump points and at most countably many branches with no side branches and with terminal segments that are copies of uncountable limit ordinals.*

Proof. Let Ψ be the subtree $\{\{p\} : p \text{ is a nontrivial jump point}\} \cup \Upsilon(0)$ of Υ. Then by Lemma 6.2, Ψ is of height $\leq \omega$; that is, every member of Ψ has at most finitely many predecessors. Also, $\Upsilon(0)$ is countable, and each element $\{p\}$ of Ψ has at most countably many immediate successors in Ψ, and so Ψ is countable.

The terminal segments described in the statement of this theorem go with the sets C in the "Moreover" part of the Main Theorem. Each one emanates either within

X_{ω_1} itself, or within an analogue of X_{ω_1} immediately above a nontrivial jump point, or from a trivial jump point such that all jump points above it (if any) are trivial, and is minimal among the trivial jump points in this category. Let J be the set of these minimal points. If $x \in J$, then $\{x\} \in \Upsilon(\gamma+\omega_1) \cap V$ where V is a component of $X \setminus \overline{X_\gamma}$ with a member $\{p\}$ of Ψ as its boundary. There can only be countably many such x associated with a given $\{p\} \in \Psi$, so J is countable, and it quickly follows that the set of all branches described in the theorem is countable. $\square$

Finally, we clip the bead strings represented by the branches described in 6.3. In $Y = X_{\omega_1}$ we do it as we did when defining T, and follow a similar process for the countably many analogues of X_{ω_1} with nontrivial jump points on their boundaries. Finally, for each $x \in J$, we remove $B(x) \setminus \{x\}$ where $B(x)$ is the unbounded string of compact beads that has $\{x\}$ as its boundary, except for x itself. Let L_0 be what remains.

Lemma 6.4. *L_0 is Lindelöf.*

Proof. Let Q be the set of points on the boundaries of the excised bead strings. Each $\{q\} \subset Q$ is above at most finitely many members of Ψ, including a unique member of $\Upsilon(0)$. Between successive members of Ψ below $\{q\}$, and between the last member of Ψ below q and q itself, there is one analogue of X_{ω_1} and at most one compact set, a bead string. What remains of these analogues in L_0 is Lindelöf. Now an easy induction, jumping ω_1 levels at a time all across Υ, shows that we have accounted for every point of $L_0 \setminus X_0$, which is thus Lindelöf, as is L_0 itself. $\square$

The proof of the Main Theorem will be completed once L_0 is embedded in an open Lindelöf subspace L which fits the description there. The only points of L_0 that are not in its interior are the points of Q. For each $q \in Q$ let B_q be a compact bead for which q plays the role of p in Definition 3.3, and for which the point q' that plays the role of q there is in $\overline{X_\alpha} \setminus X_\alpha$ for some limit ordinal α. Since the union of an increasing ω-sequence of such beads, together with the one point on the boundary of the union, also fits Definitiion 3.3, there is no problem with doing this. Now $L = L_0 \cup \bigcup\{B_q \setminus \{q'\} : q \in Q\}$ is as desired. In particular, its boundary is $\{q' : q \in Q\}$, which is closed discrete; and each q' is in a unique bead string that was clipped to produce L_0.

7. Corollaries of the Main Theorem, and a preview with some major open problems

A very quick corollary of the main theorem is that every component of every space X as described σ-countably compact: the subspace L is σ-compact, and "σ-[countably] compact" means "the union of countably many [countably] compact subspaces." This in turn trivially implies X is "ω_1-compact." That is, every closed discrete subspace is countable. [Another expression for this concept is "countable extent."]

A far more significant corollary of the Main Theorem has the same conclusion as a corollary, for locally compact spaces, of a celebrated theorem of Balogh and Rudin on monotonically normal spaces [BR]. Monotone normality is a very specialized property, and replacing it with hereditary normality (T_5) is a huge generalization, even at the cost of adding "locally connected" to the hypothesis:

Corollary 7.1. *Assume* PFA(S)[S] *and let X be a T_5, locally compact, locally connected space. If $\mathcal{U}$ is an open cover of X, then $X = V \cup \bigcup \mathcal{W}$, where $\mathcal{W}$ is a discrete family of copies of regular uncountable cardinals, and V is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.*

Proof. It is enough to show 7.1 for the individual components since they are (cl)open by local connectedness: the discrete families and partial refinements can be done on each component separately and the union taken over all components. So we work directly with an arbitrary component X.

As remarked at the end of Section 6, the complement of the open Lindelöf subspace X_ω is a discrete collection of countably many strings $\mathfrak{B}_n$ of compact beads. The endpoints of the beads in each $\mathfrak{B}_n$, taken together, form a closed copy θ_n of an ordinal of uncountable cofinality.

For each bead $B \in \mathfrak{B}_n$, let $B' = B \setminus \theta_n$. Then B' is open, and finitely many members of $\mathcal{U}$ cover it because its closure is compact. Let $\mathcal{V}(B)$ be the set of intersections of B' with these finitely many members of $\mathcal{U}$. It is an elementary exercise to divide

$$\bigcup_{n=0}^{\infty} \bigcup \{\mathcal{V}(B) : B \in \mathfrak{B}_n\}$$

into a countable family of disjoint open sets. Then a countable subcover of L as in the Main Theorem can be trivially added. $\square$

For easy reference, the conclusion of Corollary 7.1 will be referred to as "$\mathcal{U}$ has a strong Balogh-Rudin refinement". The reason for the word "strong" is that the property they showed for monotonically normal spaces in general has copies of stationary subsets of regular uncountable cardinals, whereas Corollary 7.1 has copies of the cardinals themselves.

The property that every open cover has a strong Balogh-Rudin refinement is so powerful that, when it is added to the topological properties of X in the Main Theorem, the relatively modest Axiom Sz is enough to give the conclusion for X. The argument for this also gives the conclusion of the Main Theorem for monotonically normal, locally compact, locally connected spaces without recourse to anything beyond the usual (ZFC) axioms. See Corollaries 7.12 and 7.13 of Theorem 7.11 at the end.

In Corollary 7.1, we can substitute "normal and hereditarily ω_1-scwH" for "T_5" by using (2) in the Main Theorem. With this substitution, PFA can be substituted for PFA(S)[S], again by (2). The same substitution works for the following corollary [recall T_5 = hereditarily normal].

Corollary 7.2. *Assuming* PFA(S)[S]*, every locally compact, locally connected, [hereditarily] normal space is [hereditarily] collectionwise normal (CWN) and [hereditarily] countably paracompact.*

Proof. This too can be proven just by using the components. By the Main Theorem, each component is the union of countably many closed countably compact subsets; for example, the subspace L is the union of countably many compact ones. So every discrete collection of closed subsets is countable, and CWN follows from the elementary exercise that every normal space is "$\aleph_0$-collectionwise normal."

Countable paracompactness follows quickly from the facts that regular Lindelöf spaces are paracompact; that the $\overline{X_\omega}$ described at the end of Section 6 is Lindelöf; and that each of the bead strings in the discrete family that covers $X \setminus X_\omega$ is countably compact.

Hereditary normality in the hypothesis is enough to produce the "hereditarily" in the conclusions: because every open subspace of every locally compact [*resp.* locally connected] space is locally compact [*resp.* locally connected], so the first two paragraphs apply to make every open subspace CWN and countably paracompact. And now, use the well-known fact that a space is hereditarily {normal, CWN, [countably] paracompact} iff every open subspace has the listed property. □

Corollary 7.3. [PFA] *There are no locally compact, locally connected, hereditarily ω_1-scwH Dowker spaces.*

Proof. A Dowker space is a normal space that is not countably paracompact, so this is immediate from 7.2. □

Of course, PFA(S)[S] could have been substituted for PFA, and this newer axiom also implies there are no locally compact, locally connected T_5 Dowker spaces.

A different sort of corollary of the Main Theorem is a strengthening of the theorem itself for the case of countably tight spaces. [Recall that a space X is *countably tight* if for each $A \subset X$ and each $p \in \overline{A}$, there is a countable subset $B \subset A$ such that $p \in \overline{B}$.]

Corollary 7.4. *Under the assumptions of either* (1) *or* (2) *of the Main Theorem, each component of every locally compact, locally connected, T_5, countably tight space is of Lindelöf degree $\leq \omega_1$ and is the union of a discrete collection of at most countably many closed copies of ω_1 and a disjoint union of open Lindelöf subspaces.*

In particular, if the space has a countably tight compactification, then it is hereditarily paracompact.

Proof. Let X be a component of a space as described. If X is Lindelöf, we are done; otherwise, $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ must be empty: any point of it would be in the closure of a copy W of ω_1, yet not in the closure of any countable subset of W. Hence, $X_{\omega_1} = X$, and X is therefore of Lindelöf degree ω_1. The countably many closed subspaces are the copies of ω_1, each of which answers to the description of C in the Main Theorem.

The beads are compact and so they cannot contain copies of ω_1 either, since a point in the closure of such a copy cannot be in the closure of a countable subset of it. When the endpoints of any bead are removed, the components of the resulting open subspace are Lindelöf under Case (1) because they satisfy all the other conditions of the Main Theorem. As for Case (2), normality was never invoked until Section 4, and by the time it was invoked, a copy of ω_1 had already been produced. But this is a contradiction, so the process of analyzing a component of the interior of a bead has to terminate with $Y = Y_\alpha$, which is Lindelöf.

The foregoing argument can be readily adapted to show the "in particular" statement, by showing that the compactification of the whole space cannot contain a copy of ω_1 and hence that every open subset of is the topological direct sum of open Lindelöf subspaces, and hence is paracompact; and now we use the well-known fact in the proof of Corollary 7.2. $\square$

A forthcoming paper [Ny7] will begin the analysis of what happens if countable tightness is dropped. In it, the additional assumption of hereditary normality seems unavoidable because we do not have the control over open Lindelöf subspaces that we had over X_{ω_1}. At one stage in the process we arrive at something like Corollary 7.4, but with a somewhat more complicated subspace replacing the union of copies of ω_1.

Theorem 7.5. *Under* PFA(S)[S], *every locally compact, locally connected, T_5 space is the disjoint union of a closed, rim-finite, monotonically normal subspace and a family of disjoint open Lindelöf subspaces.*

If a component is not Lindelöf, the rim-finite subspace incorporates the union of the sets C described in the Main Theorem, but that is just the beginning. The interiors of the beads described there not only do not have to have Lindelöf components; each can be any locally compact, locally connected T_5 space with any number of components. In fact, the process that results in Theorem 7.5 is reminiscent of doing repeated magnifications of the Mandelbrot set. However, this process does not stop after ω steps, but could continue for an arbitrarily long time.

Further ongoing research leads naturally to a pair of conjectures involving the following concepts:

Definition 7.6. *A point p of a space X is a local cut point if p has a connected neighborhoood N of which it is a cut point. That is, $N \setminus \{p\}$ is not connected. A point p is a rim-finite-based point if it has a base of neighborhoods with finite boundaries.*

Conjecture 7.7 [*resp.* **7.8**]. *Under PFA(S)[S], if X is a locally compact, locally connected, T_5 space, then $X = R \cup S$ where R and S are disjoint, R is monotonically normal and consists of local cut points and rim-finite-based points, and S is first countable* [resp. *perfectly normal*] *and locally compact, and is the union of a family of disjoint, relatively open, Lindelöf subspaces.*

Recall that a space is *perfectly normal* if it is a normal (including Hausdorff) space in which every closed set is a G_δ. Perfect normality and monotone normality are well known to be hereditary properties, so such spaces are T_5.

A much more speculative possibility is putting "metrizable" in place of "perfectly normal," although this is far from having been ruled out. The following problem is even more ambitious.

Problem 3. *Is it consistent (perhaps modulo large cardinals) that every locally compact, locally connected, T_5 space is monotonically normal?*

An answer of Yes would have such far-reaching consequences that I have nicknamed it "The Holy Grail". For one thing, it would immediately imply that every compact, locally connected, T_5 space is the continuous image of a *"generalized arc," i.e.,* a compact connected linearly orderable space. This is due to the exceptionally deep theorem that every compact, locally connected, monotonically normal space is the continuous image of such a space.

This theorem is a beautiful generalization of the Hahn-Mazurkiewicz theorem, which had "metrizable" instead of "monotonically normal" and the closed unit interval in place of "generalized arc." The generalization was the culmination of well over four decades of intensive research by many topologists, most relevantly Mardešic, Treybig, Nikiel, and Mary Ellen Rudin, recounted in [Ma] in detail. The contribution of these four is briefly outlined in [Ny6] [Treybig's name was omitted due to an oversight]. A large part of that research consisted of some exceptionally deep probing of the structure of continuous images of generalized arcs by several topologists. It uncovered a wealth of information about these images and played a pivotal role in the proof of the generalization.

Much of the structure theory of Treybig and Nikiel finds its echo in the work leading up to Theorems 7.4 and 7.5, and towards Conjectures 7.7 and 7.8. So, even if Problem 3 has a negative answer, the two kinds of spaces have a lot in common under PFA(S)[S].

A Yes answer to Problem 3 would also provide one to the following question posed, in effect, by Mary Ellen Rudin. [She specifically asked whether $MA + \neg CH$ would give a Yes answer.] It is also Problem 4 in [GM], where a fascinating trichotomy problem due to Gruenhage, and a mapping problem due to Fremlin, give two possible avenues towards answering it.

Problem 4. *Is it consistent that every perfectly normal, locally compact, locally connected space is metrizable?*

A Yes answer to Problem 4 would in turn solve a 1935 problem of Alexandroff:

Problem 5. *Is every generalized manifold in the sense of Čech metrizable?*

The one-point compactification of a Souslin line is a consistent counterexample (also to Problem 4, of course). At one time it was generally believed that Mary Ellen Rudin had shown the consistency of a Yes answer to Problem 5, but that was due to a mistaken idea of what a generalized manifold in the sense of Čech was. Similarly, a 1949 problem of Wilder, asking whether a perfectly normal "generalized manifold" was metrizable, remains open. [See [Ny2] for a discussion.] Both kinds of generalized manifolds are perfectly normal, locally compact and locally connected, and so a Yes answer to Problem 4 would imply ZFC-independence for both of these old problems.

To show why a Yes answer to Problem 4 is implied by one to Problem 3, we will show that the following formal weakening of Problem 4, for which this same implication is obvious, is actually equivalent to Problem 3.

Problem 4^-. *Is it consistent that every perfectly normal, locally compact, locally connected space is monotonically normal?*

That this is a weakening of Problem 4 is clear from the easy fact that every metrizable space is monotonically normal. The following theorem takes us a good part of the way to showing that this is only a formal weakening:

Theorem 7.10. *The following are equivalent:*

(1) *There is a Souslin line.*
(2) *There is a monotonically normal, locally compact, locally connected, perfectly normal space that is not metrizable.*

Proof. A Souslin line is monotonically normal, as is every linearly orderable space, and it is locally compact and locally connected. But it is not metrizable.

To show (2) $\implies$ (1), we begin with the fact [BR] that every perfectly normal, monotonically normal space is paracompact. Every locally compact, paracompact space is the topological direct sum of Lindelöf (clopen) subspaces [E, 5.1.27], which in a locally connected space can be taken to be the components. It is easy to see that the one-point compactification of a Lindelöf, locally compact, locally connected, perfectly normal space has all these properties. So by the generalization of the Hahn-Mazurkiewicz theorem, each component is the continuous image of a "generalized arc."

Now, every perfectly normal Lindelöf space is hereditarily Lindelöf, hence it has countable cellularity. This is referred to in [MP] as "the Suslin property," and Corollary 6 in that paper is that the nonexistence of a Souslin line is equivalent to every countable cellularity continuous image of a generalized arc being metrizable. □

The proof of equivalence of Problems 4 and 4^- is completed by showing that, if there is a Souslin line, and X is its one-point compactification, (which is also perfectly normal, and locally connected) then $X \times [0,1]$ is not monotonically normal, but is locally compact, locally connected, and perfectly normal. Local compactness and local connectedness are obvious. Perfect normality of the product follows from the general theorem of Morita [Mo] that the product of a perfectly normal space and a metrizable space is perfectly normal. Finally, Treybig showed [Tr] that if the product of two infinite compact spaces is monotonically normal, then both are metrizable. [Treybig had "the continuous image of an ordered compact space" rather than "[compact and] monotonically normal," but Mary Ellen Rudin showed that the two are equivalent in an extraordinarily deep paper.]

Incidentally, these two old results of Treybig and Morita give us an easy example of a perfectly normal, compact space that is not monotonically normal: take the product of the "double arrow" space (*i.e.,* the lexicographically ordered product $[0,1] \times \{0,1\}$ with the closed unit interval (or indeed any infinite compact metric space). This shows how essential local connectedness is to Problems 3, 4 and 4^-. It is also essential to a ZFC theorem which will be shown in [Ny7]: the one-point compactification of every locally compact, locally connected, monotonically normal space is monotonically normal. In contrast, Mary Ellen Rudin gave an example of a locally compact, monotonically normal space whose one-point compactification is not monotonically normal [M]. To complete the contrast, it is an easy exercise to show that the one-point compactification of a locally compact T_5 space is likewise T_5.

There is no analogue of Morita's theorem for T_5 spaces, so that, *e.g.,* the attempt to use the lexicogaphically ordered unit square as above fails to produce a counterexample to Problem 3. In fact, Katětov showed (see [E, hint to Problem 2.7.16]) that if $X \times Y$ is T_5, then either X is perfectly normal or all countable subsets of Y are closed. So if both are locally compact and locally connected, then we are back to the situation of Theorem 7.10, and have made no progress on Problem 3.

An exciting possibility is that Problem 3 may reduce to Problem 4 if Conjecture 7.8 is correct. If PFA(S)[S] (or PFA with the added hypothesis of ω_1-scwh) were to validate 7.8, then the problem would boil down to how the monotonically normal subspace and the locally compact, perfectly normal subspaces relate to each other. Given a structure like that given in the proof of Theorem 7.5, this is easily handled, but work on Conjecture 7.7 and especially on 7.8 has not progressed that far yet. As for Problem 4 itself, both PFA and PFA(S)[S] remain plausible candidates for showing that Gruenhage's conjectured trichotomy and Fremlin's mapping statement are consistent. As outlined in [GM], either result would give a Yes answer to Problem 4, with "The Holy Grail" no longer a remote possibility.

One easy reduction of Problem 3 is its restriction to compact, connected spaces: if X is locally compact, locally connected and T_5, then each component is connected and has the same properties, and the one-point compactification of a connected, locally connected T_5 space X likewise has these properties. [For local connectedness, take an arbitrary compact subset K of X and show that $(X \setminus K) \cup \{\infty\}$ has the property that every neighborhood of K in X will contain of all but finitely many components of $X \setminus K$ that have compact closure in X. This can be shown by an argument like that for the Thorn Lemma 2.1.]

We now give a theorem and two corollaries illustrating the power of the strong

Balogh-Rudin refinement property. Note the lack of any set-theoretic hypotheses (except, as usual, ZFC).

Theorem 7.11. *If X is a locally compact, locally connected, hereditarily ω_1-scwH space in which every open cover has a strong Balogh-Rudin refinement, and if*

(†) *every locally compact subspace S of X that has countable spread is hereditarily Lindelöf,*

then the conclusion of the Main Theorem holds for X.

Proof. As remarked after Theorem 1.3, the only place where MA(ω_1) came into its proof is with Axiom Sz, which states that (†) holds for all spaces. Now the conclusion of Theorem 1.3 holds: every open Lindelöf subset of X has hereditarily Lindelöf boundary, and therefore Lindelöf closure.

Armed with this, we can establish everything about X in Sections 1, 2, 3, and 5, with only Section 4 to be compensated for by the strong Balogh-Rudin property, after which Section 6 goes through without any trouble. But it is not necessary to go through all the details of Sections 1, 2, and 3. The necessary ones will be given below.

Begin with the special case where the Lindelöf degree $\ell(X)$ of X is ω_1. Let Σ be a canonical ω_1-sequence $\langle X_\alpha : \alpha \in \omega_1 \rangle$ as in 1.4. Let $\mathcal{U}$ be a cover of $X = \bigcup \Sigma$ by open sets with compact closures, so that each is a subset of some X_α. Let $X = V \cup \bigcup \mathcal{W}$ as in 7.1, so that $V = \bigcup \mathcal{V}$, where $\mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_n$, and each $\mathcal{V}_n$ is a disjoint collection of open sets. By taking components we may assume each member of each $\mathcal{V}_n$ is connected.

Each member of $\mathcal{W}$ is a copy of ω_1—there is no other possibility since each W in $\mathcal{W}$ is countably compact and noncompact and so its intersection with each X_α must be countable. Hence, each W in $\mathcal{W}$ meets each $X \setminus X_\alpha$, and also each X_α beyond some $\alpha_W < \omega_1$.

For ease of comparison with Sections 2 and 3, we make the change of notation $Y = X$, $Y_\alpha = X_\alpha$ for all $\alpha \in \omega_1$. Let

$$C = \{\alpha : \text{ for all} V \in \mathcal{V}, (V \cap Y_\alpha \neq \emptyset \implies V \subset Y_\alpha)\}$$

Then C is a club. Closedness is trivial, and we use a standard leapfrog argument to show that it is unbounded. Let α_0 be any countable ordinal. The members of each $\mathcal{V}_n$ are connected, and $\overline{Y_{\alpha_0+1}} \setminus Y_{\alpha_0+1}$ is hereditarily Lindelöf and disconnects Y, and $\mathcal{V}_n$ is a disjoint collection for each n. Therefore, at most countably many members of each $\mathcal{V}_n$ (and hence of $\mathcal{V}$) that meet Y_{α_0} will also meet $Y \setminus Y_{\alpha_0+1}$. Since these members of $\mathcal{V}$ have compact closure, they are all contained in some Y_β, $(\beta < \omega_1)$. Let the least such $\beta > \alpha_0$ be α_1. Now by induction we get a strictly increasing sequence $\langle \alpha_n : n \in \omega \rangle$ with α_{n+1} defined from α_n in the same way that α_1 was defined from α_0. Let $\alpha = sup_n \alpha_n$. Then $\alpha \in C$, because $Y_\alpha = \bigcup \{Y_{\alpha_n} : n \in \omega\}$.

Simply because they are open, the members of $\mathcal{V}$ miss all the boundaries $B_\alpha = \overline{Y_\alpha} \setminus Y_\alpha$ such that $\alpha \in C$. The only sets in the cover of Y given by the Balogh-Rudin property that meet these boundaries are the members of $\mathcal{W}$, and they are the only things in Y that are holding it together. Since they are a discrete family, each point of B_α is isolated in the relative topology of B_α whenever $\alpha \in C$.

Now we can skip over the rest of Section 1 and go to the Thorn Lemma 2.1. For each limit ordinal $\alpha \in C$ and each $p \in B_\alpha$ there is $\xi < \alpha$ and a compact, connected neighborhood D_p of p such that $D_p \cap \overline{Y_\alpha} \setminus Y_\xi$ is clopen in the relative topology of $\overline{Y_\alpha} \setminus Y_\xi$. In each $W \in \mathcal{W}$, the Pressing-down Lemma gives a uniform ξ_W that works for an unbounded collection of points of W. We may assume, since $\mathcal{W}$ is discrete, that D_p misses all members of $\mathcal{W}$ besides the member in which p is located. And this means that if q is the unique member of $B_\alpha \cap W$, D_q must meet every B_ν, $\xi_W \leq \nu \leq \alpha$, $\nu \in C$, in the singleton $B_\nu \cap W$. It is now easy to see that

$$S_W = \bigcup \{D_q : q \in W \cap B_\alpha \, (\alpha \geq \xi_W)\}$$

is a bead string as in the Main Theorem. Moreover, since S_W is connected, it must meet every B_α such that $\alpha \geq \xi_W$.

By connectedness of the $V \in \mathcal{V}$, every member that meets some S_W must be a subset of S_W. What is left of $\mathcal{V}$ after these members are accounted for is a collection of open sets whose union misses $M = \bigcup\{B_\alpha : \alpha \in C\}$. And now the connectedness of Y implies that these remaining members of $\mathcal{V}$ are all in Y_{α_0} where $\alpha_0 = min(C)$. Letting $L = Y_{\alpha_0}$ makes L a witness of the Main Theorem for Y, along with the discrete set $\{S_W \setminus Y_{\alpha_0} : W \in \mathcal{W}\}$ of closed countably compact subspaces.

When Y is a proper subspace of X, where $\ell(X) > \omega_1$, we go through the maneuvers of Section 5, where $\Sigma^* = \{Y^*_\alpha : \alpha < \omega_1$ is constructed. Then we apply the strong Balogh-Rudin refining property to $c\ell_X(Y)$; this refining property is clearly inherited by closed subspaces, and Σ^* makes all unbounded copies of ω_1 in Y with compact closure in $c\ell_X(Y) \setminus Y$ into subsets of some Y^*_α. So now $c\ell_X(Y)$ is as in the conclusion of the Main Theorem, and we can go through Section 6 after taking due note of the last paragraph of Section 4. □

Corollary 7.12. *Every monotonically normal, locally compact, locally connected space is as in the conclusion of the Main Theorem.*

Proof. Every monotonically normal space X satisfies (†) [O] and is hereditarily CWN, and if X is locally compact, every open cover has a strong Balogh-Rudin refinement. □

Corollary 7.13. *If Axiom Sz holds, and X is a space satisfying the hypotheses of the Main Theorem, and every open cover has a strong Balogh-Rudin refinement, then the conclusion of the Main Theore holds for X.*

Proof. Axiom Sz simply says that (†) holds for all locally compact spaces X. Now use Theorem 7.11. □

Finally, it may be worth remarking that neither Theorem 1.3 or any other result in this paper used the full force of "hereditarily ω_1-scwH." Its weakening "hereditarily wD" was enough for Section 5; and elsewhere, weakening to "hereditarily cwH" and/or "hereditarily weakly ω_1-scwH" was adequate:

Definition 7.14. A space X is *weakly ω_1-scwH* if every closed discrete subspace D of cardinality ω_1 has an uncountable subset $A \subset D$ with an expansion to a discrete collection of open sets.

With considerable extra work, it is even possible to eliminate the reliance on "hereditarily wD" in Section 5, by using ideas in the proof of Theorem 2.4, and applying Urysohn's Lemma to the components of $\overline{X_{\omega_1}} \setminus X_{\omega_1}$ to show that they are singletons. Details are left to the interested reader.

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