NON-STRATIFIABILITY OF $C_k(X)$ FOR A CLASS OF SEPARABLE METRIZABLE X

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Abstract. If X is a separable 0-dimensional metrizable space in which every compact subset is countable, then $C(X)$ with the compact-open topology is stratifiable iff X is scattered. This answers a question of Gruenhage and lends credence to a conjecture of Gartside and Reznichenko.

A significant advance in our understanding of the compact-open topology was made by Gartside and Reznichenko when they showed:

Theorem A. [1] *$C_k(X)$ is stratifiable whenever X is a Polish (= separable and completely metrizable) space; in particular, $C_k(\mathbb{P})$ is stratifiable.*

Here $\mathbb{P}$ stands for the space of irrational numbers. As usual, $\mathbb{Q}$ and $\mathbb{R}$ will stand, respectively, for the rational and real numbers with the usual topology. Theorem A makes $C_k(\mathbb{P})$ a prime candidate for a negative solution to the 43-year-old problem of whether every stratifiable space is M_1. The converse of Theorem A is also of interest:

Problem 1. *Let X be separable metrizable. If $C_k(X)$ is stratifiable, must X be completely metrizable?*

Gartside and Reznichenko conjectured a positive solution to Problem 1, which easily reduces to the 0-dimensional case [1, Proposition 27 (3)]. Since every scattered metrizable space is completely metrizable, the only restriction on the following partial solution to Problem 1 is in the last clause in the hypothesis.

Theorem 1. *Let X be a 0-dimensional separable metrizable space which is not scattered, and has the property that every compact subset is countable. Then $C_k(X)$ is not stratifiable.*

Theorem 1 negatively answers the following question of Gary Gruenhage, which he posed at the 2004 Spring Topology Conference in Lubbock, Texas; see also [2]:

1991 *Mathematics Subject Classification.* [2000 update] 54C35, 54E20, 46E10; secondary 54E35, 54E50.

Key words and phrases. stratifiable, compact-open, scattered, 0-dimensional.

Problem 2. *Is $C_k(\mathbb{Q})$ stratifiable?*

The proof of Theorem 1 rests on the following theorem of Gartside and Reznichenko, [1] which dispenses with the need to define either the compact-open topology or stratifiability.

Theorem B. *Let X be a 0-dimensional separable metrizable space. Then $C_k(X)$ is stratifiable if, and only if, it is possible to assign to each clopen subset W of X a compact $F(W) \subset W$, and to each compact $K \subset X$ a compact $\phi(K) \supset K$ in such a way that, whenever $W \cap K \neq \emptyset$, it follows that $F(W) \cap \phi(K) \neq \emptyset$ also.*

For convenience, we say X has *the Gartside-Reznichenko property* if it has assignments $\phi(\cdot)$ and $F(\cdot)$ as above. It is by no means obvious that the Gartside-Reznichenko property is inherited by closed subspaces, but that follows from another theorem in [1]:

Theorem C. [1, Proposition 27 (1)] *Let X be separable metrizable. If $C_k(X)$ is stratifiable, and Y is a closed subspace of X, then $C_k(Y)$ is also stratifiable.*

An immediate corollary of Theorems 1 and C is that if a separable metrizable space X has a closed subspace satisfying the hypothesis of Theorem 1, then $C_k(X)$ is not stratifiable. I am indebted to Gary Gruenhage for pointing out the following corollary.

Theorem 2. *If X is a coanalytic subspace of $\mathbb{R}$, then $C_k(X)$ is stratifiable if, and only if, X is not a G_δ.*

Proof. A subspace of a separable metric space is *coanalytic* if its complement is analytic (that is, the continuous image of a Polish space). A coanalytic subspace of $\mathbb{R}$ is not a G_δ iff it contains a closed copy of $\mathbb{Q}$ [3], [4, Theorem 21.18] and this in turn implies $C_k(X)$ is not stratifiable. Conversely, a subspace of a complete metric space is completely metrizable iff it is a G_δ, and we have Theorem A. $\square$

Corollary. *A σ-compact metric space has a stratifiable C_k iff it can be given a complete metric.*

Proof. A σ-compact space is an F_σ in every Hausdorff space containing it, hence is coanalytic. Now use the equivalence at the end of the preceding proof. $\square$

To prove Theorem 1, we will show that if X satisfies its hypotheses, then no pair of assignments $\{\phi(\cdot), F(\cdot)\}$ can witness the Gartside-Reznichenko property. Our strategy will be to find a sequence of clopen sets W_n in X and a descending sequence of collections of compact sets $\mathcal{K}_n$ such that $\bigcup_{n=0}^{\infty} W_n$ is clopen, and such that $W_n \cap \phi(K) = \emptyset$ for all $K \in \mathcal{K}_n$ but $W_n \cap K \neq \emptyset$ for some $K \in \mathcal{K}_i$ when $i < n$.

Once this is done, we need only set $W = \bigcup_{n=0}^{\infty} W_n$: since $F(W)$ is compact, $F(W) \subset \bigcup_{i=0}^{n} W_i$ for some n; then $W_{n+1} \cap K \neq \emptyset$ for some $K \in \mathcal{K}_n$, but $W_i \cap \phi(K) =$

$\emptyset$ for $i \leq n$, so $W \cap K \neq \emptyset$ but $F(W) \cap \phi(K) = \emptyset$, and so X fails to have the Gartside-Reznichenko property.

In the special case of $\mathbb{Q}$, the sequences we seek can be found directly, but for the general case we construct a whole tree of sets W_σ and $\mathcal{K}_\sigma$ and then show that this tree must have some infinite branch which behaves as desired.

To carry out our strategy, we introduce the following concept. Call a collection of countable (hence scattered) compact subsets of a metrizable space M *large* if it has members of arbitrarily high countable scattered height. Clearly every large collection is uncountable. Also, if every compact subset of M is countable, then the union of every large collection of compact sets has noncompact closure, since every countable compact space is scattered, and height does not increase in going to subspaces. The following is also obvious:

Lemma 1. *If a large collection is expressed as a union of countably many subcollections, at least one of the subcollections must also be large.* $\square$

Similarly, we have:

Lemma 2. *If $\mathcal{K}$ is large and $\{V_n : n \in \omega\}$ is a descending sequence of clopen sets whose intersection is finite, then there exists n such that $\{K \setminus V_n : K \in \mathcal{K}\}$ is large.*

Proof. If V_n is as above and K is compact and $\alpha_n \in \omega_1$ is an upper bound for the heights of the points in $K \setminus V_n$ then $sup_n \alpha_n + 1$ is an upper bound for the heights of the points in K. A proof by contrapositive is now immediate. $\square$

Lemma 3. *If M is a nowhere locally compact, 0-dimensional metric space, $\mathcal{K}$ is a large collection of countable compact subsets of M, $\phi(K)$ is a compact set for each $K \in \mathcal{K}$, and C is a nonempty clopen subset of M, then there is a nonempty clopen subset B of C such that $\{K \in \mathcal{K} : B \cap \phi(K) = \emptyset\}$ is large.*

Proof. Let $\{C_n : n \in \omega\}$ be a descending sequence of nonempty clopen subsets of C whose intersection is empty. By Lemma 1, all but finitely many C_n will do for B. $\square$

Proof of Theorem 1. By a well-known classical result, we may assume $X \subset \mathfrak{C}$ where $\mathfrak{C}$ stands for ${}^{\omega}2$, a.k.a. the Cantor set. For each finite sequence σ of 0's and 1's, let $B[\sigma]$ be the basic clopen subset of $\mathfrak{C}$ consisting of all points that extend σ. Let $\mathcal{B} = \{B[\sigma] : \sigma \in {}^{<\omega}2\}$. As is well known, $\mathcal{B}$ is a base for $\mathfrak{C}$, each member of which is homeomorphic to $\mathfrak{C}$ itself, with $B[\emptyset] = \mathfrak{C}$.

Lemma 4. *If $\mathcal{K}$ is large, and $\bigcup \mathcal{K} \subset B[\sigma]$, then there are at least two sequences σ_0, σ_1 of the same length extending σ such that $\mathcal{K} \restriction B[\sigma_i] = \{K \cap B[\sigma_i] : K \in \mathcal{K}\}$ is large for $i = 0, 1$.*

$\vdash$ *Proof of Lemma 4:* Let $\sigma \in {}^{n}2$ and let $m > n$. For each $K \in \mathcal{K}$, some point of maximal height in K is in one of the $B[\tau] (\tau \in {}^{m}2)$, so there is at least one $\tau \in {}^{m}2$ for which $\mathcal{K} \restriction B[\tau]$ is large. Suppose there is only one for each m. Then the associated clopen sets close down on a single point of $\mathfrak{C}$, and this contradicts Lemma 2. $\dashv$

To continue the proof of Theorem 1, assume we are given a compact subset $\phi(K)$ of X for each compact subset K of X. Define finite sequences σ and associated points $y_\sigma \in B[\sigma]$, and sets $B_\sigma \in \mathcal{B}$ such that $B_\sigma \subset B[\sigma]$, and large collections $\mathcal{K}_\sigma$ of compact sets by repeated application of Lemmas 1 through 4, in the following way. Begin with $\sigma = \emptyset$ and let $y_\emptyset$ be any point of $\mathfrak{C} \setminus X$ in the closure of X. The argument for Lemma 3 shows that there is some $B_\emptyset$ in the neighborhood base of $y_\emptyset$ such that $\{K \in \mathcal{K} : \phi(K) \cap B_\emptyset = \emptyset\}$ $(= \mathcal{K}_\emptyset)$ is large.

Suppose y_σ, etc. have been defined, in such a way that $\mathcal{K}_\sigma \restriction B[\sigma]$ is large, and $B_\sigma \cap \phi(K) = \emptyset$ for each $K \in \mathcal{K}_\sigma$, and B_σ is a neighborhood of y_σ in $B[\sigma]$. Applying Lemma 4 to $B[\sigma]$, let σ_1 and σ_2 be distinct sequences of the same length, extending σ, for which $\mathcal{K}_\sigma \restriction B[\sigma_i]$ is large. Let y_{σ_i} be a point of $\overline{\bigcup \mathcal{K}_\sigma \cap B[\sigma_i]} \setminus X$ [overhead bars denote closure in $\mathfrak{C}$] and let B_{σ_i} be a neighborhood of y_{σ_i} in $B[\sigma_i]$ for which $\mathcal{K} = \{K \in \mathcal{K}_\sigma : \phi(K) \cap B_{\sigma_i} = \emptyset\}$ is large. Let $\mathcal{K}_{\sigma_i} = \mathcal{K}$.

Once this induction is complete, the set of all σ for which y_σ, etc. have been defined is a copy of the full binary tree of height ω, and each branch defines a unique point of $\mathfrak{C}$. Moreover, each such point is in $\overline{X}$, but not all of these points are in X, because the branches together define a copy of $\mathfrak{C}$.

Let y be one of these points in $\mathfrak{C} \setminus X$. The branch that runs to y defines a sequence of clopen subsets $B_\sigma \cap X$ of X. The union W of these sets is clopen since they converge on y. Re-index the B_σ and the $\mathcal{K}_\sigma$ by the natural numbers in order of the length of σ, and let $W_n = B_n \cap X$. These sets are exactly as required by the strategy explained above. $\square$

In the case of a countable space such as $\mathbb{Q}$, a direct construction of the sets W_n and $\mathcal{K}_n$ can be done as follows. List $\mathbb{Q}$ as $\{q_n : n \in \omega\}$. Let $\mathcal{B}$ be a countable base for $\mathbb{Q}$ consisting of proper clopen subsets. Let B_0 be a member of $\mathcal{B}$ for which there is a large subcollection $\mathcal{K}_0$ of $\mathcal{K}$ such that $B_0 \cap \phi(K) = \emptyset$ for all $K \in \mathcal{K}_0$. Using Lemma 2, let $V_0 \in \mathcal{B}$ be a neighborhood of q_0 such that $\{K \setminus V_0 : K \in \mathcal{K}_0\}$ is large.

Suppose $\mathcal{K}_i$, B_i and V_i have been defined for all $i \le n$ in such a way that $\{K \setminus (V_0 \cup \ldots \cup V_i) : K \in \mathcal{K}_i\}$ is large and $\phi(K) \cap B_i = \emptyset$. Let B be a member of $\mathcal{B}$ that meets the perfect core (that is, the union of the dense-in-itself subspaces) of $\bigcup \mathcal{K}_n$ and misses $V_0 \cup \cdots \cup V_n$, and for which the following is a large subcollection of $\mathcal{K}$:

$$\{K \setminus (V_0 \cup \cdots \cup V_n) : K \in \mathcal{K}_n \text{ and } B \cap \phi(K) = \emptyset\}$$

Let $B_{n+1} = B$ and let

$$\mathcal{K}_{n+1} = \{K \in \mathcal{K}_n : B_{n+1} \cap \phi(K) = \emptyset\}.$$

Let $V_{n+1} \in \mathcal{B}$ be a neighborhood of q_{n+1} such that $\{K \setminus (V_0 \cup \dots \cup V_{n+1}) : K \in \mathcal{K}_{n+1}\}$ is large. It is easy to show that the union of the sets $W_n = B_n$ is as desired.

In any 0-dimensional separable metric space not covered by Theorem 1, we may assume without loss of generality that $\phi(K)$ is always uncountable. So we need some other concept of "large" collections of compact sets. However, every concept of "large" I have considered to date runs into difficulties, even for special kinds of spaces. For example, if X is Baire, a natural concept for "large" is "having a union which is of second category in X." This makes Lemmas 1 through 4 easy to verify (with "countable" omitted from Lemma 3 and X used in place of M), but I have not been able to ensure that the binary tree of B_σ's does not give a compact space that is completely in X.

In the opposite case where X is a countable union of nowhere dense subsets, one can hope for a modification of the direct proof for $\mathbb{Q}$ which circumvents this last hurldle. The idea is to find a concept of "large" which allows us to replace q_n with a closed nowhere dense set C_n and to have Lemma 2 also handle the case when the clopen sets close down on some C_n.

A natural idea here is to take "$\mathcal{K}$ is large" to mean "$\mathcal{K}$ cannot be dominated by countably many sets which are the union of finitely many C_n and of compact sets" [meaning: there is no countable collection of sets F_n, each a union of finitely many C_i and of compacta, such that every member of $\mathcal{K}$ is contained in some F_n]. However, there are difficulties even with the unmodified Lemma 2. On the other hand, either form of Lemma 2 can be taken care of if we modify this choice of "large" to say that if A is the set of points p such that every neighborhood of p meets a subfamily of $\mathcal{K}$ which cannot be dominated by countably many F_n as above, then A has nonempty interior; but then Lemma 1 breaks down.

References

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