Function rings, ultrafilters, and nonstandard analysis

Let X be a set. The set ${}^X\mathbb{R}$ of all functions from X into $\mathbb{R}$ is a ring by the usual operations of addition and multiplication of real-valued functions: $(f+g)(x) = f(x) + g(x)$, etc. An important subring is the collection of all bounded functions from X to $\mathbb{R}$, best known as the Banach algebra $\ell_\infty(X)$, with ℓ_∞ the case $X = \omega$.

This ring is one of a class of examples produced from topological spaces; given a topological space X, the real-valued continuous functions on X form a ring just as above, with the bounded ones forming a subring important in Banach space theory. ${}^X\mathbb{R}$ is the special case where X has the discrete topology, and it is the only case dealt with below.

Ideals in this ring are intimately connected with filters. Recall that a *filter on* a set X is a nonempty collection $\mathcal{F}$ of subsets of X such that

(1) if $F \in \mathcal{F}$ and $F \subset G$ then $G \in \mathcal{F}$;

(2) if F_0 and F_1 are both members of $\mathcal{F}$, then so is $F_0 \cap F_1$.

We take all filters to be *proper*: a filter is not the whole power set. Equivalently, in the light of (1):
(3) $\emptyset \notin \mathcal{F}$.

Similarly, "ideal" will mean "proper ideal," — not the whole ring. The connection between filters on X and ideals of ${}^X\mathbb{R}$ is given by the following concept.

Given $f : X \to \mathbb{R}$ we define *the zero-set of* f to be $Z(f) = f^{-1}(0) = \{x \in X : f(x) = 0\}$. Since the product of real numbers is 0 iff one of them is 0, we have:

$Z(fg) = Z(f) \cup Z(g)$

It is also easy to see:

$Z(f^2 + g^2) = Z(f) \cap Z(g) \subset Z(f+g)$

If $r \in \mathbb{R} \setminus \{0\}$, then $Z(rf) = Z(f)$.

Consequently, if I is an ideal of ${}^X\mathbb{R}$ then $\mathfrak{F}(I) = \{Z(f) : f \in I\}$ is a filter on X, while if $\mathcal{F}$ is a filter on X, then $J(\mathcal{F}) = \{f : Z(f) \in \mathcal{F}\}$ is an ideal of ${}^X\mathbb{R}$.

Theorem 1. (a) *If I is an ideal of ${}^X\mathbb{R}$, then $I = J(\mathfrak{F}(I))$.*

(b) *If $\mathcal{F}$ is a filter on X then $\mathcal{F} = \mathfrak{F}(I(\mathcal{F}))$.*

Example 1. The *Fréchet filter* on an infinite set X is the collection of all subsets of X whose complement is finite. To show (2) in the definition of a filter, we use the fact that the union of two finite sets is finite, along with de Morgan's laws: letting A^c denote the complement $X \setminus A$ of A, we have $(A \cap B)^c = A^c \cup B^c$.

Definition 1. A filter $\mathcal{F}$ is *free* if $\bigcap F = \emptyset$, otherwise it is *fixed.* A filter $\mathcal{U}$ on X is an *ultrafilter* if it is not properly contained in any other filter on X. Equivalently: given any subset A of X, exactly one of A, A^c is a member of $\mathcal{U}$.

It is easy to see that the Fréchet filter is free and is a subcollection of every free filter on X. An easy application of Zorn's Lemma is that every filter can be extended to an ultrafilter, and hence that there are free ultrafilters. The only ultrafilters on X for which we have explicit formulas are the fixed ultrafilters; these are of the form $\mathcal{U}_x = \{A \subset X : x \in A\}$.

Theorem 2. *An ideal of* ${}^X\mathbb{R}$ *is maximal iff* $\mathfrak{F}(I)$ *is an ultrafilter.*

Theorem 3. *If* $\mathcal{U}$ *is an ultrafilter on* X*, then* $\mathbb{R}$ *has a natural injection into the quotient field* ${}^X\mathbb{R}/J(\mathcal{U})$*, which is onto iff* $\mathcal{U}$ *has the countable intersection property.*

Theorem 4. *There is a free ultrafilter with the countable intersection property on* X *iff* $|X|$ *is* $\geq$ *the first measurable cardinal.*