Scales, topological reflections, and large cardinal issues

by Peter Nyikos

Reflection theorems in set-theoretic topology typically take the following form: if all "small" subspaces of a suitable kind of space X have a property $\mathcal{P}$, then so does the whole space. Here "small" often means "of cardinality $\aleph_1$" but in Sections 1 and 2 of this paper it will mean "separable". Usually, but not always, some sort of large cardinal axiom is needed, and the reflection theorem holds in some forcing extension. The following 1977 result of Shelah illustrates all this nicely:

Theorem A. [Sh] *Levy collapsing a supercompact cardinal to $\aleph_2$ produces a model in which every first countable space that is locally of cardinality $\leq \aleph_1$ is collectionwise Hausdorff (cwH) if every subspace of cardinality $\leq \aleph_1$ is cwH.*

In [T], Tall surveys a variety of similar results and open problems, usually stating things contrapositively: if X does not satisfy $\mathcal{P}$, then this is reflected by some small subspace of X. A number of reflection theorems involving cwH and variations on it will be given in Section 4, along with some new results showing the necessity of large cardinals.

In Section 1, a proof will be given of the following 1985 reflection theorem of the author, and it will be shown how to modify the proof for a number of variants.

Theorem 1. *Any Chang Conjecture variant $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ implies that if a locally compact Hausdorff space of density κ is of hereditary Lindelöf degree $> \kappa$, then it contains a separable non-Lindelöf subspace.*

This theorem was indirectly alluded to in [JSS], but has remained unpublished because the authors of [JSS] found a simpler proof of its main application: to put an upper bound on the size of certain kinds of spaces. (Corollary 1.)

In Section 2, it is shown how the axiom GS_κ(cofinality ω_1) is enough to give a counterexample to the statement in Theorem 1 and its variants. This is one of the weakest axioms surveyed in [F2] and [CFM], and very large cardinals are needed to negate it.

Section 3 is devoted to concepts generalizing sequentiality to higher cardinals in different ways. The example in Section 2 is shown to be a semiradial compact space which is not R-monolithic, and a simplification is used to give a ZFC example of a radial space which is not R-monolithic.

Section 4 gives applications of two axioms to the cwH property and variations on it, and recalls older applications of another axiom. All three axioms require very large cardinals to negate them; but various forcings starting with a supercompact cardinal can negate them all.

From now on, "space" will always mean "Hausdorff space."

Section 1. Reflections of cardinal invariants to separable subspaces

To state a generalization of Theorem 1, we recall some cardinal invariants.

1.1. Definition. The *density* of a space X, denoted $d(X)$, is the least cardinality of a dense set, *i.e.,* a subset D that meets every open set. A space is *separable* if it is of countable density.

The *spread* of X, denoted $s(X)$, is the supremum of the cardinalities of the discrete subspaces of X.

The *tightness* of X, denoted $t(X)$, is the least cardinal τ such that if a point x is in the closure of a subset A of X, there is a subset B of A such that $|B| \leq \tau$ and such that x is in the closure of B.

The *Lindelöf degree* of X, denoted $l(X)$, is the least cardinal κ such that every open cover of X has a subcover of cardinality $\leq \kappa$.

If P is a cardial invariant of spaces, then $hP(X) = sup\{P(Y) : Y \subset X\}$.

Theorem 1 is the hl case of the following theorem.

1.2. Theorem. *Let $P(X)$ denote either $s(X)$, or $hl(X)$ or $hd(X)$. The axiom $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ implies that every locally compact space X that satisfies $d(X) = \kappa$ and $P(X) > \kappa$ has a separable subspace Y such that $P(Y)$ is uncountable. If, in addition, X is compact, then this holds also for $P(X) = t(X)$.*

The $P(X) = t(X)$ case of Theorem 1.2 was essentially shown by A. Dow in [BD, Theorem 1]. It made use of the characterization of tightness in compact spaces as being the supremum of the cofinalities of all free sequences. A *free sequence* is a discrete subspace together with a well-ordering $\{x_\alpha : \alpha < \theta\}$ such that the closure of any initial segment is disjoint from the closure of the rest of the sequence.

Theorem 1.2 suggests the possibility of reflection theorems for all compact or locally compact spaces:

Problem 1. Let $P(X)$ denote either $s(X)$, or or $hl(X)$ or $hd(X)$. Is it consistent, modulo very large cardinals, that every locally compact space X satisfying $d(X) < P(X)$ contains a separable subspace Y such that $P(Y)$ is uncountable?

Problem 2. Is it consistent, modulo very large cardinals, that every compact space X satisfying $d(X) < t(X)$ contains a separable subspace of uncountable tightness?

Just how large the cardinals involved must be for affirmative answers to these problems, can be gleaned from Theorem 2.5 and Corollary 2.6. The axiom GS_κ(cofinality ω_1) used there is so weak that, if it fails, Determinacy holds in $L(\mathbb{R})$ and so there is an inner model with infinitely many Woodin cardinals. This is also true of the stronger axiom GS_κ, which is also used in 2.5 and 2.6, but unlike GS_κ and most related axioms, GS_κ(cofinality ω_1) can even hold for strong limit cardinals of cofinality ω above a supercompact [CFM]. Still, Magidor has shown that Martin's Maximum (MM) negates it [C], and MM is consistent if the existence of a supercompact cardinal is consistent.

On the other hand, the consistency of $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$ has thus far only been shown to follow from that of a 2-huge cardinal, and it may be too much to ask for a model where $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ for all κ simultaneously. In fact, the case of $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$ is one of the few known examples where a singular cardinal can be put in place of κ. So it would be good to find a weaker combinatorial principle that implies the reflection principle in Theorem 1.2.

Section 2. Negating reflection with good scales

In this section we kill the reflection statements in Theorem 1.2 by using an axiom to construct a unified counterexample to them all, and a slightly stronger axiom that negates some weaker reflection theorems. The axioms have to do with the following concept.

2.1 Definition. Let κ be a singular cardinal of cofinality ω. A *λ-scale for κ* is a pair $(\overrightarrow{\kappa}, F)$ where $\overrightarrow{\kappa} = \langle \kappa_i : i \in \mathbb{N} \rangle$ is a strictly increasing sequence of regular cardinals cofinal in κ, and F is a $<^*$-well-ordered family of functions that is of order type λ and cofinal in $(\prod_{i \in \mathbb{N}} \kappa_i, <^*)$. Here $<^*$ is the eventual domination order, whereby $f <^* g$ if $f(n) < g(n)$ for all but finitely many n.

A celebrated early result of PCF theory, due to Shelah, is that there is always a scale for κ when κ is as above. We will be using this result in Section 3 to produce the ZFC example mentioned in the introduction. It uses a scale in much the same way as we do in the proof of the Theorem 2.6 below, but without relying on the scale satisfying the extra properties which we now introduce.

2.2. Definition. Let $(\overrightarrow{\kappa}, F)$ be as in Definition 2.1. A point $\alpha < \kappa^+$ is *good for* $(\overrightarrow{\kappa}, F)$ if there exist $A_\alpha \subset \alpha$ unbounded in α and $i \in \mathbb{N}$ such that $f_\gamma(j) < f_\beta(j)$ whenever $\gamma, \beta \in A_\alpha$ and $\gamma < \beta$ and $i < j$.

A *good scale for* κ is a κ^+-scale $(\overrightarrow{\kappa}, F)$ for κ for which there is a club subset of points of κ^+ that are good for $(\overrightarrow{\kappa}, F)$.

2.3 Axioms. Let κ be a singular cardinal of countable cofinality. GS_κ is the axiom that there is a good scale for κ, while $GS_\kappa(cofinality\ \omega_1)$ is the axiom that there is a scale $(\overrightarrow{\kappa}, F)$ for which there is a club set of points of κ^+ such that the ones of cofinality ω_1 are good for $(\overrightarrow{\kappa}, F)$).

For more about GS_κ, including the definition when κ is of uncountable cofinality, and its relationships with other axioms, see [CFM] and [FM].

We begin with a ZFC construction using a λ-scale for a singular cardinal $\kappa < l$ of countable cofinality, and then show how the existence of a scale witnessing GS_κ(cofinality ω_1) can make it negate the statements in Theorem 1.2.

2.4. Example. Let $(\overrightarrow{\kappa}, F)$ be a scale for κ, $F = \{f_\alpha : \alpha < \lambda\}$. The underlying set for our space Z is $\Sigma \cup (\lambda + 1)$, where $\Sigma = \bigcup_{n\in\omega}\{n\} \times \kappa_n$. The relative topology on Σ is the relative topology from the usual product topology on $\mathbb{N} \times \kappa$, and the relative topology on $Z \setminus \Sigma = \lambda + 1$ is the usual order topology. A neighborhood base at $\alpha \in \lambda$ is $\{W(\beta, \alpha, n) : \beta < \alpha, n \in \mathbb{N}\}$, where

$$W(\beta, \alpha, n) = (\beta, \alpha] \cup (f_\alpha^\downarrow \setminus f_\beta^\downarrow) \setminus \bigcup_{i=1}^{n} \{i\} \times \kappa_i$$

Here $f^\downarrow = \{(i, \alpha) : \alpha \le f(i),\ i \in dom(f)\}$.

It is easy to show that these sets are compact and open in the resulting topology on $Z \setminus \{\lambda\}$, making this space locally compact, and we make Z its one-point compactification. The extra point λ thus has as a neighborhood base the collection of all sets of the form

$$W(\alpha, p) = (\alpha, \lambda] \cup (\Sigma \setminus f_\alpha^\downarrow) \setminus p^\downarrow$$

where p is a function whose domain is a finite initial segment of $\mathbb{N}$, such that $p(i) \in \kappa_i$ for all i. These sets are also open and compact.

Note that every column has the point l in the closure. In fact, the relative topology on $(\{n\} \times \kappa_n) \cup \{l\}$ makes it a copy of $\kappa_n + 1$.

The following construction obviously works for any singular κ of countable cofinality, but we let $\kappa = \aleph_\omega$ because of the published proof in [LMS] showing that the consistency of a 2-huge cardinal implies that of $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$.

2.5. Example. Let $G = \{g_\nu : \nu < \omega_{\omega+1}\}$ be a good scale for $\aleph_\omega$, and let C be a club set of points of $\omega_{\omega+1}$ that are good for $(\overrightarrow{\kappa}, G)$. Let $\{\nu_\alpha : \alpha < \omega_{\omega+1}\}$ list C in its natural order, let $f_\alpha = g_{\nu_\alpha}$ and let the scale $F = \{f_\alpha : \alpha < \omega_{\omega+1}\}$ determine the topology on $X = \Sigma \cup \omega_{\omega+1} + 1$ just as was done for Z in Example 2.4.

Generalizing this example to any singular κ gives:

Theorem 2.6. *Let κ be a singular cardinal of countable cofinality and assume the axiom GS_κ [respectively, GS_κ(cofinality ω_1)]. Then there is a compact space X of density κ, in which there is a closed subspace homeomorphic to $\kappa^+ + 1$, and such that $|Y| = |\overline{Y}|$ for all subsets Y of X of cardinality $< \kappa$ [respectively, for all countable $Y \subset X$].*

Proof for $\kappa = \aleph_\omega$. This is clearly true for subspaces of individual columns, including the last column $\omega_{\omega+1} + 1$; consequently, it is enough to show that if $A \subset \Sigma$ and $|A| = \aleph_n$ for some finite n, [respectively, if A is countable] then $|\overline{A} \cap \omega_{\omega+1}| \le |A|$.

Suppose not. Then there exists $\theta \in \omega_{\omega+1}$ such that $cf(\theta) > |A|$ and such that a cofinal subset L of θ is in the closure of A. Let A_θ and i be as in the definition of a good scale. Then $L \cup \theta$ is also in the closure of

$$H = A \cap \bigcup \{\{j\} \times \omega_{n_j} \mid j > i\}$$

but $\{H \cap g^\downarrow_\nu : \nu \in A_\theta\}$ is totally ordered by $\subset$, and so there can be at most $|A|$ sets of this form, none of which has θ in the closure, and this contradicts either $cf(\theta) > |A|$ or the cofinality of L in θ. $\square$

The special case of countable A in Theorem 2.6 is already enough to negate the reflection principle in Theorem 1.2.

2.7. Corollary. *Let notation be as in Theorem 2.6. If Y is a subspace of Z of density $< \kappa$ [respectively, a separable subspace of Z] then $d(Y) = hd(Y) = hl(Y) = s(Y) = t(Y) = |Y|$, but $d(Z) = \kappa$ while $hd(Z) = hl(Z) = s(Z) = t(Z) = |Z| = \kappa^+$.*

In the opposite direction, it is consistent to have $(\aleph_{n+1}, \aleph_n) \to (\aleph_1, \aleph_0)$ simultaneously for all finite n [F1], so that it is consistent (modulo large cardinals, but still smaller than the ones required to negate $GS_{\aleph_\omega}$) for every compact space X of density $< \aleph_\omega$ and $P(X)$ exceeding its density to have a separable subspace Y where $P(Y)$ is uncountable. It would enhance the significance of Example 2.5 if we could go one step higher, putting $\leq \aleph_\omega$ for $< \aleph_\omega$*above.*

Problem 3. Is it consistent (modulo very large cardinals) to have $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ for all $\kappa \leq \aleph_\omega$?

We close this section with an observation about scales on $\aleph_\omega$ which has been part of the folklore for some time but may not have appeared in print before. This is that $\overrightarrow{\kappa}$ can be all of $\langle \aleph_n : n \in \mathbb{N} \rangle$ if, and only if $cf([\aleph_\omega]^\omega) = \aleph_{\omega+1}$. On the one hand, a deep theorem of pcf theory is that there are λ-scales for $\aleph_\omega$ of each regular cardinal λ in the interval $[\aleph_{\omega+1}, \mu]$ where $\mu = min\{cf([\aleph_\omega]^\omega),\ \aleph_{\omega_1}\}$; clearly, different $\overrightarrow{\kappa}$ for different λ have to have almost disjoint ranges. So the only way we can have $\overrightarrow{\kappa} = \langle \aleph_n : n \in \mathbb{N} \rangle$ is for there to be only one cardinal, $\aleph_{\omega+1}$, in the interval.

On the other hand, if $cf([\aleph_\omega]^\omega) = \aleph_{\omega+1}$, there is a cofinal family $\{A_\alpha : \alpha < \omega_{\omega+1}\}$ in $([\Sigma]^\omega, \subset)$, where $\Sigma = \bigcup_{n \in \mathbb{N}} \{n\} \times \omega_n$, and we can define a scale by induction, making $A_\alpha \subset f^\downarrow_\alpha$, and also have $f_\beta <^* f_\alpha$ for all $\beta < \alpha < \omega_{\omega+1}$. Indeed, we can group the f_β ($\beta < \alpha$) into countably many subsets B_n, with $|B_n| \leq \omega_n$, and then have $f_\alpha(m) > f_\beta(m)$ for all $m > n$ and all $\beta \in B_n$.

Section 3. Radial and semiradial examples

Example 2.5, using a good scale, also gives an affirmative answer, barring very large cardinals, to the following question, posed in [BD]:

Problem 4. Is there a semiradial compact space which is not R-monolithic?

We still do not know whether there is a ZFC example of such a space, nor of a compact radial space that is not R-monolithic:

3.1. Definition. A space X is *radial* [resp. *Fréchet-Urysohn*] if every point in the closure of any subset A is the limit of a convergent well-ordered net [*resp.* of a convergent sequence] in A.

A space X is *pseudoradial* [resp. *sequential* if for every non-closed subset A there is a well-ordered net [*resp.* a sequence] converging to some point outside A.

If κ is a cardinal number, a subset A of X is *κ-closed* if $\overline{B} \subset A$ whenever B is a subset of A of cardinality $\leq \kappa$.

A space X is *semiradial* if for each cardinal κ, and each non-κ-closed subset A of X, there is a well-ordered net of order type $\leq \kappa$ in A converging to some point outside A.

The *radial character* $R_\chi(X)$ of a pseudoradial space X is the least cardinal κ such that well-ordered nets of order type $\leq \kappa$ suffice in the definition of pseudoradial. A pseudoradial space X is *R-monolithic* if $R_\chi(\overline{A}) \leq |A|$ for all $A \subset X$.

Clearly, sequential $\implies$ R-monolithic $\implies$ semi-radial $\implies$ pseudoradial, and a separable radial space is R-monolithic iff it is Fréchet-Urysohn.. It is easy to see that density $\geq$ tightness for all R-monolithic spaces; consequently, any compact semiradial space in which density $<$ tightness gives an affirmative answer to Problem 4. Similarly, if $(\aleph_{n+1}, \aleph_n) \to (\aleph_1, \aleph_0)$ holds simultaneously for all finite n, and every compact separable radial space is R-monolithic, then every compact radial space of density $< \aleph_\omega$ is R-monolithic. See [BD] and [D] for information on constructing such models.

If we drop the word "compact" from Problem 4, then there is a ZFC example which is radial to boot.

3.2. Example. Let $F = \{f_\alpha : \alpha < \lambda\}$, put the discrete topology on Σ, and let $\Phi = \Sigma \cup \lambda + 1$ with the following topology. A neighborhood base at $\alpha \in \lambda$ is all cofinite subsets of $f_\alpha \cup \{\alpha\}$. [As usual, we identify a function with its graph.] For the last point λ of $\lambda + 1$, let a local base be all sets of the form $W(\alpha, p)$ just as for Example 2.4.

The resulting space Φ is radial because every point except λ has a countable base, while the only way the point λ can be in the closure of a set A is for A to meet some column in a cofinal set; that is where the fact that we have a scale is essential. But it is not R-monolithic because $|\Sigma| = \kappa$ and Σ is dense in the whole space, and the only way to get at the point λ from the subspace λ is by a net of cardinality $\lambda > \kappa$.

Example 2.4, of which Example 2.5 is a special case, is of chain-net order 2, which means the closure of any set can be found by iterating the process of taking all limits of well-ordered nets a second time, but the second time is necessary for some subsets and some points in their closure (see below). Radial spaces are the spaces of chain net order 1.

3.3. Theorem. *Example 2.4 is pseudoradial.*

Proof. Clearly, the subspaces Σ and $\lambda + 1$ are radial. The fact that $\{f_\alpha : \alpha < \lambda\}$ is a scale ensures that a subset of $Z \setminus \{\lambda\}$ will have $\{\lambda\}$ in its closure iff it meets some column $\{n\} \times \kappa_n$ or λ in a cofinal set. Thus if A has this extra point λ in the closure, there will be a well-ordered net in A converging to the point. So we will be done when we show that if $\alpha < \lambda$ and α is in the closure of some subset T of Σ, then either there is a sequence from T converging to α, or there is a cofinal subset B of $[0, \alpha)$ such that every $\beta \in B$ is the limit of a convergent sequence from T.

To simplify the following demonstration, we say that a countably infinite set A converges to a point x if some (hence every) 1-1 sequence whose range is A converges to x. Let β_0 be the least ordinal β such that T meets $f_\beta^\downarrow$ in infinitely many columns. Then any subset of T that meets infinitely many columns in a finite subset of $f_{\beta_0}^\downarrow$ converges to β_0. If $\beta_0 = \alpha$ we are done, otherwise we continue the induction.

Assume β_η has been defined for all $\eta < \xi$ so that $\beta_\eta < \beta_\zeta < \alpha$ whenever $\eta < \zeta$, and so that there is a sequence in T converging to each β_η. Let $\gamma = sup\{\beta_\eta : \eta < \xi\}$. If $\gamma = \alpha$, then $\langle \beta_\eta : \eta < \xi \rangle$ converges to α and we are done. Otherwise, α is in the closure of $T \setminus f_\gamma^\downarrow$. Let β_ξ be the least ordinal β such that $T \setminus f_\gamma^\downarrow$ meets $f_\beta^\downarrow$ in infinitely many columns. Then any subset of $T \setminus f_\gamma^\downarrow$ that meets infinitely many columns in a finite subset of $f_{\beta_\xi}^\downarrow$ converges to β_ξ.

This induction can only end when either (1) $\beta_\xi = \alpha$ for some ξ, in which case there is a sequence from T converging to α, or (2) when $sup\{\beta_\eta : \eta < \xi\} = \alpha$ for some ξ, and then we let $B = \{\beta_\eta : \eta < \xi\}$ as desired. $\square$

3.4. Theorem. *Example 2.5 is semiradial, but not R-monolithic.*

Proof. X is not R-monolithic because $|\Sigma| = \aleph_\omega$ but there is a copy of $\omega_{\omega+1} + 1$ in the closure of Σ. To show X is semiradial, we use the following characterization of semiradiality: for each $A \subset X$, it is possible to reach every point in the closure of A by iterating the operation of taking limits of convergent well-ordered nets of cofinality $\leq |A|$. In fact, letting $\hat{A}$ stand for the set of limits of convergent well-ordered nets in A, we will show that every point of $\overline{A}$ is the limit of an ordinary convergent sequence in $\hat{A}$. [This is the opposite order from Theorem 2, where every point of $\overline{T}$ was reached by a well-ordered net in the set $A^{(1)}$ of all limits of convergent sequences in A.]

Clearly, if $A \subset \omega_{\omega+1}$, then every point in the closure of A is the limit of a convergent well-ordered net in A. Also, any point of Σ in the closure of $A \subset \Sigma$ is a limit of the same sort. If the extra point $\omega_{\omega+1}$ is in the closure of A, then it is in the closure of $A \cap \{i\} \times \omega_{n_i}$ for some i, because the f_α form a scale. So we will be done if we show that if $A \subset \Sigma$ is closed in the relative topology of Σ, then any point of $\omega_{\omega+1}$ in the closure of A is the limit of an ordinary convergent sequence in A.

Suppose α is in the closure of such a set A. Then α is also in the closure of $B = A \cap f_\alpha^\downarrow$. If $B \cap (\{n\} \times \omega_n) \neq \emptyset$, let $\xi_n = sup\{\xi : (n, \xi) \in B\}$. Since B is relatively closed in Σ, this supremum is actually a maximum. Note that if N is a basic neighborhood of α and $(n, \eta) \in N$ then $(n, \nu) \in N$ whenever $\eta \leq \nu \leq f_\alpha(n)$. Consequently, α is in the closure of the countable set $C = \{(n, \xi_n) : B \cap (\{n\} \times \omega_n) \neq \emptyset\} \subset B$. Now by Theorem 2, the closure of C is countable. It is also compact, hence metrizable in its relative topology. Clearly, then, there is a sequence in C converging to α. □

3.5. Corollary. *X is of chain net order 2.*

Proof. The proof of either Theorem 3.3 or Theorem 3.4 shows the chain net order ≤ 2. To see that it is not 1, let α be of uncountable cofinality γ. If $A \subset \Sigma \cap f_\alpha^\downarrow$, then any well-ordered net in A having α as its limit must be an ordinary convergent sequence in A. However, if $A \cap f_\alpha = \emptyset$, then no countable subset of A can have α in its closure: if A_α and $i \in \mathbb{N}$ are as in $GS_{\aleph_\omega}$, then

$$f_\alpha^\downarrow \cap \bigcup_{j>i} \{j\} \times \omega_{n_j} \quad = \quad \bigcup_{\nu \in A_\alpha} (g_\nu^\downarrow \cap \bigcup_{j>i} \{j\} \times \omega_{n_j})$$

and so if H is a countable subset of A, then the portion of H outside the first i columns is in some $g_\nu^\downarrow$ and so cannot have α in its closure; but then H cannot have α in its closure either.

An example of $A \subset \Sigma \cap f_\alpha^\downarrow \setminus f_\alpha$ which does have α in its closure is $\bigcup\{g_\nu : \nu \in A_\alpha\}$, which meets each column $\{j\} \times \aleph_{n_j}$ beyond the ith that goes with A_α in a set of cofinality γ, with supremum $(j, f_\alpha(j)$. □

Problem 5. Is there a compact radial space that is not R-monolithic, and which is obtainable from an axiom whose negation requires large cardinals? is there a ZFC example?

Section 4. Reflections of cwH and related properties, and counterexamples

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