THE STRUCTURE OF LOCALLY COMPACT NORMAL SPACES: SOME QUASI-PERFECT PREIMAGES

PETER NYIKOS

Abstract. A *quasi-perfect map* is a continuous, closed function such that the preimage of every point is countably compact. An ambitious old problem due to van Douwen [1] is whether every first countable regular space of cardinality $\leq \mathfrak{c}$ is a quasi-perfect image of a locally compact space. Here we construct locally compact, normal, quasi-perfect preimages for all stationary, co-stationary subsets of ω_1. These subsets are ω_1-compact but not σ-countably compact. Quasi-perfect preimages preserve these properties, and the preimages constructed here are all of cardinality $\mathfrak{b}$. This provides a new lower bound for the following problem:

What is the least cardinality of a ZFC example of a locally compact, ω_1-compact space that is not σ-countably compact?

1. Introduction

This paper continues the exploration of the theme, begun in [4], of the structure of locally compact spaces satisfying additional properties. In [5] this focused on the question of when normal, locally compact, ω_1-compact spaces are σ-countably compact.

Definition 1.1. A space is ω_1-*compact* if it is *of countable extent,* by which is meant that every closed discrete subspace is countable. A space is σ-*countably compact* if it is the union of countably many countably compact spaces.

The following was one of the main results of [5]:

Theorem 1.2. *It is consistent, modulo large cardinals, that every locally compact, hereditarily normal, ω_1-compact space is σ-countably compact.*

[All through this paper, "space" means "Hausdorff topological space."]

In this theorem, "hereditarily" cannot be eliminated. In [2], Eric van Douwen provided ZFC examples of locally compact, locally countable refinements of the usual topology on the real line. No such refinement can be σ-countably compact, as explained in [5]; but some of the ones in [2] are (collectionwise) normal, and ω_1-compact.

The reason for the parenthetical "collectionwise" is that every normal, ω_1-compact space is collectionwise normal; that is, every discrete collection of closed sets can be expanded in bijective fashion to a disjoint collection of open sets. (In an ω_1-compact space, every discrete collection of closed sets is countable, and every countable discrete collection of closed sets in a normal space can be thus expanded by a simple inductive argument.) However, to

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extend this to subspaces, one needs not only hereditary normality but also "hereditarily ω_1-compact," which is the very strong property of countable spread. The examples in this paper are neither hereditarily normal nor of countable spread, and ZFC is not enough to make any of van Douwen's examples satisfy either property.

It is also shown in [5] how an affirmative answer to an old problem of van Douwen [1] would provide examples of locally compact, ω_1-compact spaces that are not σ-countably compact:

Problem 1. Is every first countable space of cardinality $\leq \mathfrak{c}$ the quasi-perfect image of a locally compact space?

Definition 1.3. A map $f : X \to Y$ of topological spaces is *quasi-perfect* if it is continuous and closed, and each fiber $f^{\leftarrow}\{y\}$ is countably compact.

Using the axiom $\mathfrak{b} = \mathfrak{c}$, which will be explained below, van Douwen gets every first countable space of cardinality $\leq \mathfrak{c}$ to be the quasi-perfect image of a locally compact, locally countable (hence first countable) space [1]. However, the space thus constructed can fail to be normal, as shown in [5].

The constructions in this paper are done without any axioms beyond the usual (ZFC) axioms. They provide normal, locally compact, quasi-perfect preimages for every stationary, co-stationary subset of ω_1 with the relative topology from the usual order topology on ω_1.

Definition 1.4. A *stationary subset* of a cardinal κ is a subset E of κ that meets every *club* (closed unbounded subset of κ). A *co-stationary subset of* κ is a subset whose complement is stationary in κ.

Example 1.5. Let E be any stationary, co-stationary subset of ω_1. Then E is locally countable and first countable, and normal (indeed, hereditarily normal). But no co-stationary, uncountable subset of ω_1 can be σ-countably compact because every uncountable, countably compact subset of ω_1 is a club. On the other hand, E is ω_1-compact for almost the same reason.

Of course, E not locally compact: if x is a point of E that is a limit of points of $\omega_1 \setminus E$ which are, themselves, limits of points in E, then x fails to have a compact neighborhood. But our examples are locally compact, and in [5] it is shown how both the property of ω_1-compactness and failure of σ-countable compactness are inversely preserved by quasi-perfect preimages. They are also normal, and of cardinality $\mathfrak{b}$, where $\mathfrak{b}$ is one of the "small uncountable cardinals" whose notation was stanardized by van Douwen and Vaughan [1], [8] . Since they are locally compact and normal, our main examples are thus relevant to both parts of the following problem:

Problem 2. **(a)** *What is the least cardinality of a locally compact, ω_1-compact space that is not σ-countably compact?* **(b)** *one that is also normal?*

Specifically:

Problem 3. **(a)** *Is there a ZFC example of a locally compact, ω_1-compact space of cardinality $\aleph_1$ that is not σ-countably compact?* **(b)** *one that is also normal?*

Our examples provide the lowest known cardinality, $\mathfrak{b}$, for Problem 2 (both parts). Each point-inverse is either a singleton or naturally homeomorphic to $\mathfrak{b}$ with the order topology.

The best earlier lower bounds were $\mathfrak{c}$ in the case of van Douwen's examples and others constructed along the same lines, and $\mathfrak{b}$ under the extra condition $\mathfrak{b} = \aleph_1$. The latter examples [7] are even perfectly normal and of countable spread, hence hereditarily collectionwise normal.

These long-familiar spaces involve an inductive process of defining a basic compact open neighborhood of each point p, beginning with isolated points and choosing a denumerable closed discrete subspace $D(p)$ in the space $X(p)$ defined up to the point where p is added. Then $D(p)$ gets expanded in bijective fashion to a discrete-in-$X(p)$ collection $\{U_n : n \in \omega\}$ of compact open sets. Then, if $G_n = \bigcup\{U_i : i \geq n\}$, a base of neighborhoods of p is given by $\{G_n \cup \{p\} : n \in \omega\}$.

The construction of our spaces can be looked upon as an elaboration of this process, but the means of choosing the analogues of the U_n involves a technique of self-similarity that may be new here. Instead of a phenomenon like copies of the Mandelbrot set appearing infinitely many times under repeated magnifications, we have the reverse phenomenon of the same general pattern appearing infinitely many times as we step further and further back. Also, the techniques described above gave neighborhood bases to at most countably many points at a time. There is a rough and ready technique that quickly winds up handling only one point-inverse $\{\alpha\} \times \mathfrak{b}$ at a time, given in Section 7. However, it seemed worthwhile to treat a special class of stationary sets in Sections 2 through 6 which enable a much more structured construction, taking advantage of our geometric intuitions, in which point-inverses are given bases for uncountably many $\alpha \in \omega_1$ at a time.

Our spaces are also the first ZFC examples of locally compact, ω_1-compact spaces that are not σ-countably compact, but in which every separable subset is σ-countably compact.

Notation 1.6. In this paper, we use E. K. van Douwen's almost self-explanatory notations $f^{\rightarrow}A$ and $f^{\leftarrow}B$ to designate the image of A under f and the preimage of B under f respectively. We also use the suggestive notation $\alpha_n \uparrow \alpha$ to designate a strictly ascending sequence of ordinals α_n whose supremum is α. Overhead bars denote closure in all of ω_1, not just in E, except above natural numbers, where $\overline{n}$ denotes the (constant) function from ω to ω with range $\{n\}$.

2. Setting the stage

In Section 7, it will be explained how every stationary, co-stationary subset of ω_1 is the quasi-perfect image of a normal, locally compact space. However, to improve readability, we make our first example E satisfy some simplifying assumptions. Recall that the space ω_1 is scattered. This is equivalent to one of its Cantor-Bendixson derivatives being empty:

Definition 2.1. Let X be a space. The *α-th Cantor-Bendixson (C-B) derivative* $X^{(\alpha)}$ *of* X is defined by induction as follows. $X^{(0)} = X$; if $X^{(\alpha)}$ has been defined, $X^{(\alpha+1)}$ is the derived set $(X^{(\alpha)})'$ of $X^{(\alpha)}$; if λ is a limit ordinal and $X^{(\alpha)}$ has been defined for all $\alpha < \lambda$, then $X^{(\lambda)} = \bigcap\{X^{(\alpha)} : \alpha < \lambda\}$.

The *αth C-B level of* X is $X^{(\alpha)} \setminus X^{(\alpha+1)}$.

For instance, the 0-th C-B level of any space is its set of isolated points, and for any countable ordinal α, the $\alpha + 1$st C-B level of ω_1 is the set of points of the form $\xi \cdot \omega$, where ξ is on the αth C-B level of X.

Notation. All through this paper, W_α will denote the αth C-B level of ω_1 with the usual (interval) topology, and, until Section 7, E will be the set described in the following example.

Example 2.2. Let S be a stationary, co-stationary set of limit ordinals in ω_1, such that $\omega \in S$, and let $E = E_N \cup E_S$, where

$$E_N = \bigcup\{W_{2n} : n \in \omega\} \cup \bigcup\{W_{\delta+2n+1} : \delta \notin S \text{ and } n \in \omega\}.$$

and

$$E_S = \bigcup\{W_{\delta+2n} : \delta \in S \text{ and } n \in \omega\}.$$

As the notation suggests, E_N is nonstationary while E_S is stationary. In fact, even if E_S had only consisted of the first elements of the various W_δ, $\delta \in S$, it would still be stationary. This is clear from a well known fact involving the following concept.

Definition 2.3. *A continuous rapidly descending ω_1-sequence of clubs* is an ω_1-indexed family of club subsets K_α of ω_1 such that $K_{\alpha+1} \subset K'_\alpha$ (= the derived set of K_α) for all $\alpha \in \omega_1$, and $K_\lambda = \bigcap\{K_\alpha : \alpha < \lambda\}$ for all limit ordinals λ.

An obvious example of such a sequence is given by $K_\alpha = (\omega_1)^{(\alpha)}$, and the following lemma clearly implies that E_S is stationary and that E_N is nonstationary.

Lemma 2.4. *Let $\mathcal{K} = \{K_\xi : \xi \in \omega_1\}$ be a continuous rapidly descending ω_1-sequence of clubs. Let α_ξ be the least element of K_ξ. Then $I(\mathcal{K}) = \{\alpha_\xi : \xi \in \omega_1\}$ is a club subset of ω_1.*

Proof. "Unbounded" is immediate from the fact that α_ξ is isolated in the relative topology of K_ξ, and hence $\alpha_{\xi+1} > \alpha_\xi$ for all $\xi \in \omega_1$.

To show "closed," suppose $\xi(n) \uparrow \xi$. Let $\nu_n \uparrow \alpha_\xi$. Then $\nu_n \in K_{\xi(k_n)} \setminus K_\xi$ for some k_n, with $k_n \to \infty$. But then $\alpha_{\xi(k_n)} \leq \nu_n$ and so $\alpha_{\xi(n)} \to \alpha_\xi$. □

The underlying set of our space X is $W_0 \cup [(E \setminus W_0) \times \mathfrak{b}]$, where $\mathfrak{b}$ is one of the "small uncountable cardinals" whose notation was stanardized by van Douwen [1].

Definition 2.5. Given functions f and g from ω to ω, the notation $f <^* g$ means g is eventually above f, meaning that there exists $n \in \omega$ such that $f(m) < g(m)$ whenever $n \leq m$. The least cardinality of a $<^*$-unbounded family of functions from ω to ω is denoted $\mathfrak{b}$.

A well-known fact, established by a simple transfinite induction, is that there is a $<^*$-unbounded family $\{f_\alpha : \alpha < \mathfrak{b}\}$ of increasing functions that is well-ordered by $f_\eta <^* f_\xi$ iff $\eta < \xi$.

Notation. For the rest of this paper, $\mathcal{F}$ denotes a fixed family of functions $f_\xi : \omega \to \omega$ such that f_n is the constant function with range $\{n\}$ if $n \in \omega$, while $\{f_\alpha : \omega \leq \alpha < \mathfrak{b}\}$ is a $<^*$-unbounded family of increasing functions that is well-ordered as above. Note that $f_n <^* f_\omega$ for all $n \in \omega$.

The projection map $\pi : X \to E$ is the identity on W_0, the set of isolated points of ω_1 and also of E. The topology is defined so that π is a homeomorphism when restricted to W_0, which is open and dense in X just as it is in ω_1. The restriction of π to the subset of nonisolated points, $(E \setminus W_0) \times \mathfrak{b}$, is the usual projection map, and the topology on the nonisolated points will be defined by induction, in such a way that π is quasi-perfect, and each fiber $\pi^{\leftarrow}\{\alpha\}$, where $\alpha \in E \setminus W_0$, is homeomorphic to $\mathfrak{b}$ with its usual topology. Also, X will be locally countable (and hence first countable) $\iff$ $\mathfrak{b} = \omega_1$. [If $\mathfrak{b} > \omega_1$ then the point that is the copy of ω_1 in any fiber is in the closure of the copy of the countable ordinals, but not of any countable set in this copy.]

The foregoing description of X, together with the continuity of π, is enough to show that every separable subspace of X is σ-countably compact: let $A \in [X]^\omega$, and let $\alpha = \pi(supA)$. Then by continuity, $\overline{A} \subset \pi^{\leftarrow}[0, \alpha]$, which is the union of countably many fibers, each of which is countably compact.

The next three sections are devoted to defining the topology on X by induction. Then in Section 6, the key properties of X will be shown, and Section 7 explains how to modify the construction and proofs for arbitrary stationary, co-stationary subsets $E \in \omega_1$.

3. The neighborhood bases of points in $W_2 \times \mathfrak{b}$

Recall that $E \cap W_1 = \emptyset$ while $W_2 \subset E$, and that W_2 (which thus $= E^{(1)}$) is the collection of all countable ordinals of the form $\alpha = \beta + \omega^2$. Given $\alpha \in W_2$, there is a least such β, either 0 or a limit ordinal in $(\omega_1)^{(2)}$. Let $\alpha_n = \beta + \omega \cdot n$ if $n \in \omega \setminus \{0\}$. These are all the ordinals of W_1 in $(\beta, \alpha]$, and together with $\alpha_0 = \beta$, they are a set of order type ω. Think of the set of (isolated) points of E in (α_n, α_{n+1}) as a vertical copy of ω, and of the union of these vertical copies — which is (β, α) — as a copy of $\omega \times \omega$. Below, we define a topology on $(\beta, \alpha) \cup (\{\alpha\} \times \mathfrak{b})$ that is homeomorphic to the following well-known (largely folklore) space in the obvious way:

Example 3.1. Let F be any $<^*$-unbounded family $\{f_\xi : \xi < \mathfrak{b}\}$ of increasing functions $f_\xi : \omega \to \omega$ that is faithfully $<^*$-well-ordered. In the space $Y = (\omega \times \omega) \cup \mathfrak{b}$, points of ω are isolated. For each $\xi \in \mathfrak{b}$ and each $\eta < \xi$ let

$$V_\xi = [0, \alpha] \cup f_\alpha^\downarrow, \text{ let } V_\xi(n) = (V_\xi) \setminus (n \times \omega) \text{ and let } V_{\xi,\eta}(n) = V_\xi(n) \setminus V_\eta.$$

where, in general, $f^\downarrow$ means the part of $\omega \times \omega$ on or below the graph of f. Explicitly: $f^\downarrow = \{(n, i) \in \omega \times \omega : i \leq f(n)\}$. Let $\{V_{\xi,\eta}(n) : n \in \omega, \eta < \xi\}$ be a base of neighborhoods of each $\xi \in Y$.

If Y is as in Example 3.1, then Y is a locally compact space in which the relative topology on $\mathfrak{b}$ is the usual topology, and every sequence that meets infinitely many columns has a cluster point in $\mathfrak{b}$. This is essentially shown in the proof of Theorem 3.3 below.

Use the family $\mathcal{F}$ in a way analogous to the use of F in 3.1, letting $\nu^\alpha_{ni} = \beta + \omega \cdot n + i + 1$ be the ith entry in column n of an array like the above. For each $\alpha \in W_2$ and each $\xi \in \mathfrak{b}$, let

$$(f^\alpha_\xi)^\downarrow = \{\nu^\alpha_{ni} : i \leq f_\xi(n), n \in \omega\}.$$

Next, for each $\eta < \xi < \mathfrak{b}$, and each $n \in \omega$, let

$$V^\alpha_\xi = \{\langle \alpha, \nu \rangle : \nu \leq \xi\} \cup (f^\alpha_\xi)^\downarrow, \text{ let } V^\alpha_\xi(n) = V^\alpha_\xi \cap \pi^{\leftarrow}(\alpha_n, \alpha] \text{ and let } V^\alpha_{\xi,\eta}(n) = V^\alpha_\xi(n) \setminus V^\alpha_\eta.$$

Of course, $V^\alpha_\xi(n) \setminus V^\alpha_\eta = V^\alpha_\xi(n) \setminus V^\alpha_\eta(n)$.

For each $\xi < \mathfrak{b}$, let $\mathcal{V}^\alpha_\xi = \{V^\alpha_{\xi,\eta}(n) : \eta < \xi,\ n \in \omega\}$ be a base of neighborhoods of $\langle \alpha, \xi \rangle$ in $\pi^\leftarrow(\beta, \alpha]$. Also let $\mathfrak{V}_\alpha = \{\mathcal{V}^\alpha_\xi : \xi < \mathfrak{b}\}$. If $x \in W_0$, let $\mathfrak{V}_x = \{\{x\}\}$.

Lemma 3.2. *The system of neighborhood bases of the form $\mathfrak{V}_\alpha$ and $\mathfrak{V}_x$ is well-defined; in particular, if $x \in V^\alpha_{\xi,\eta}(n)$, then there is a subset of $V^\alpha_{\xi,\eta}(n)$ that is a basic neighborhood of x.*

Proof. If $x \in W_0$, this is obvious. The nonisolated points of $V^\alpha_{\xi,\eta}(n)$ are of the form $\langle \alpha, \nu \rangle$ where $\eta < \nu \leq \xi$ [and we may assume $\nu < \xi$]. Let $m = min\{i : f_\nu(j) < f_\xi(j) \text{ for all } j \geq i\}$ and let $k = max\{m, n\}$. Then $V^\alpha_{\nu,\eta}(k) \subset V^\alpha_{\xi,\eta}(n)$. □

A simple modification of this proof shows that these basic neighborhoods form an actual base for a topology on $\pi^\leftarrow(\beta, \alpha]$. This amounts to showing that if $\langle \alpha, \nu \rangle$ is in the intersection of basic neighborhoods of $\langle \alpha, \xi \rangle$ and $\langle \alpha, \eta \rangle$ in $\{\alpha\} \times \mathfrak{b}$, then there is a basic neighborhood of $\langle \alpha, \nu \rangle$ in the intersection. These basic (open) neighborhoods are closed, and in fact, compact, as the proof of the following theorem shows.

Theorem 3.3. *With the topology defined in Lemma 3.2, the subspace $\pi^\leftarrow(\beta, \alpha]$ is a locally compact space in which the relative topology on $\{\alpha\} \times \mathfrak{b}$ is the usual topology.*

Proof. Local compactness is obvious for $x \in W_0$, and this transfers to $\pi^\leftarrow(\beta, \alpha]$. The other points in $\pi^\leftarrow(\beta, \alpha]$ are of the form $\langle \alpha, \nu \rangle$, and we will show that each cover $\mathcal{U}$ of a basic clopen neighborhood $V^\alpha_{\gamma,\eta}(n)$ of $\langle \alpha, \nu \rangle$ by basic open sets has a finite subcover.

Let U_0 be a member of $\mathcal{U}$ containing $\langle \alpha, \xi \rangle$. Then $U_0 = V_{\gamma_0,\eta_0}(n_0)$ where $\gamma_0 \geq \xi$ and $\eta_0 < \xi$. Then $\langle \alpha, \nu \rangle \in U_0$ whenever $\eta_0 < \nu \leq \xi$. If $\eta < \eta_0$, let $\langle \alpha, \eta_0 \rangle \in U_1 \in \mathcal{U}$, so that $U_1 = V_{\gamma_1,\eta_1}(n_1)$ and $\eta_1 < \eta_0$.

In this way, we get a descending sequence $\eta_0 > \eta_1 > \dots$ which terminates after finitely many steps with $\eta_k \leq \eta$. Now $f_{\eta_k} <^* \dots <^* f_{\eta_0}$, so there exists $N \in \omega$ such that $N \geq n_i$ for all $i \leq k$, and such that $f_{\eta_{i+1}}(m) < f_{\eta_i}(m)$ for all $i < k$ and all $m \geq N$.

So $U_0 \cup \dots U_k$ covers $V^\alpha_{\gamma,\eta}(N)$. If $N \leq n$ we are done; otherwise, $V^\alpha_{\gamma,\eta}(n) \setminus V^\alpha_{\gamma,\eta}(N)$ meets finitely many columns in finitely many compact sets, and so finitely many members of $\mathcal{U}$ are enough to cover it.

The basic open sets $V^\alpha_{\gamma_i,\eta_i}(N)$ trace compact intervals on $\{\alpha\} \times \mathfrak{b}$, so its relative topology is the interval topology when $\{\alpha\} \times \mathfrak{b}$ is given its natural order. □

Theorem 3.4. *Every sequence in $W_0 \cup W_2$ whose projection converges to α has a cluster point in $\{\alpha\} \times \mathfrak{b}$.*

Proof. Let $\pi(x_n) \uparrow \alpha$, with $x_n = \nu^\alpha_{mi}$. Let $i = i(n)$ and $m = m(n)$. Because $\mathcal{F}$ is $<^*$-unbounded, there exists ξ such that $f_\xi(m(n)) > i(n)$ for infinitely many n. Then $x_n \in (f^\alpha_\xi)^\downarrow$ for these same n. And $(f^\alpha_\xi)^\downarrow$ is compact. □

Theorem 3.5. *Let $B = [(\beta, \alpha) \cap E] \cup (\{\alpha\} \times \mathfrak{b})$. The projection $\pi : B \to (\beta, \alpha] \cap E$ is quasi-perfect.*

Proof. Continuity is obvious, and the only non-singleton fiber is is $\pi^\leftarrow\{\alpha\}$. This is countably compact because the topology on $\mathfrak{b}$ is the order topology as is clear from the proof of Theorem 3.3, and because of the fact that $\mathfrak{b}$ is a regular uncountable cardinal, its uncountability being established by a simple diagonal argument.

Finally, the map is closed because of Theorem 3.4: every closed subset of B is either bounded above in (β, α) or it meets $\{\alpha\} \times \mathfrak{b}$. In the first case we use the fact that $\pi \restriction (\beta, \alpha)$ is a homeomorphism; in the second, the fact that α is the only nonisolated point of $(\beta, \alpha] \cap E$ in E. $\square$

Definition 3.6. Let $\alpha \in W_2$. The *block at* α is $B(\alpha) = \pi^{\leftarrow}(\beta, \alpha] = [(\beta, \alpha) \cap W_0] \cup (\{\alpha\} \times \mathfrak{b})$, also termed *a rank* 1 *hyperblock of the first kind,* and also denoted $H(\alpha)$. Sets of the form $\pi^{\leftarrow}(\gamma, \alpha]$, where $\beta < \gamma < \alpha$, are termed *rank* 1 *hyperblocks of the second kind* and denoted $H(\alpha, \gamma)$.

For example, $H(\alpha, \alpha_n)$ is a hyperblock of the second kind if $n > 0$. Note that $\pi^{\rightarrow} H(\alpha) = (\beta, \alpha] \cap E$, but that $\pi^{\rightarrow} V_\xi^\alpha$ is a proper subset of $(\beta, \alpha] \cap E$ for all ξ, while $\bigcup\{V_{\overline{n}}^\alpha : n \in \omega\} = (\beta, \alpha] \cap E$. In the next two sections, hyperblocks of all ranks and neighborhoods of all $\alpha \in E$ will be defined, and the same equations will hold.

In this section, $W_0 \cup W_2$ has been partitioned into sets of the form $\pi^{\leftarrow}(\beta, \alpha]$, each of which is a block. In the next section, $W_0 \cup W_2 \cup W_4$ will be partitioned into rank 2 hyperblocks of the first kind, setting up a pattern for later W_ξ where ξ is not a limit ordinal.

4. The neighborhood bases of points in $W_4 \times \mathfrak{b}$

Much of what was done for W_2 in the preceding section will be done for $W_4 = E^{(2)}$ in this section, but there are a few key differences.

Given $\alpha \in W_4$, there is a greatest $\beta < \alpha$ in $(\omega_1)^{(4)} \cup \{0\}$. In the interval (β, α), the ordinals in W_3 ($\subset \omega_1 \setminus E$) form a set of order type ω. Let α_n be the nth ordinal in this sequence, and let $\alpha_0 = \beta$. In each interval (α_n, α_{n+1}), let α_{ni} be the $i+1$st ordinal in W_2; then α_{n+1} is the supremum of the α_{ni}.

In defining the basic neighborhoods of $\langle \alpha, \xi \rangle$ when $\alpha \in W_4$, the role of the individual points ν_{ni}^α in W_0 is taken by blocks $B(\alpha_{ni})$, with $\{B(\alpha_{ni}) : i \in \omega\}$ like the $n+1$st column in a square array. This time, the functions f_ξ in $\mathcal{F}$ have three roles. First, they pick out which members of the array meet V_ξ^α: in order for V_ξ^α to be compact, it can only be allowed to meet finitely many blocks in each column in the array. These are the $B(\alpha_{ni})$ where $i \leq f_\xi(n)$. Second, the intersection of each block $B(\alpha_{ni})$ with $V_\xi^\alpha \cap W_2$ is either $\{\langle \alpha_{ni}, \nu \rangle : \nu \leq \xi\}$ or the empty set. And third, we add an extra piece of each $B(\alpha_{ni})$ consisting of finitely many of its rows, the ones indexed by $i \leq f_\xi(n)$. That is , $V_\xi^\alpha \cap B(\alpha_{ni}) = V_\xi^{\alpha_{ni}} \cup V_{f_\xi(n)}^{\alpha_{ni}}$ where $f_\xi(n)$ stands for the constant function $\omega \times \{f_\xi(n)\}$.

This third role is there to ensure that all sequences with ascending projections have limit points as in Theorem 4.2 below. For all $\xi \geq \omega$, this actually entails adding only finitely many elements to each row of $V_\xi^\alpha \cap W_0$, because f_ξ is strictly increasing.

And thus,

$$V_\xi^\alpha = \{\langle \alpha, \nu \rangle : \nu \leq \xi\} \cup \bigcup V_\xi^{\alpha_{ni}} \cup V_{f_\xi(n)}^{\alpha_{ni}} : i \leq f_\xi(n),\ n \in \omega\}.$$

As before, $V_\xi^\alpha(n) = V_\xi^\alpha \cap \pi^{\leftarrow}(\alpha_n, \alpha]$ and $V_{\xi,\eta}^\alpha = V_\xi^\alpha(n) \setminus V_\eta^\alpha = V_\xi^\alpha(n) \setminus V_\eta^\alpha(n)$.

In the special case where $\xi = k \in \omega$, we have $f_\xi(n) = k$ for all n, so that

$$V_k^\alpha = \{\langle \alpha, j \rangle : j \leq k\} \cup \{\langle \alpha_{ni}, j \rangle : i, j \leq k\} \cup \{\nu_{ni}^\alpha : i \leq k\}.$$

The proofs of Lemma 3.2 and Theorem 3.3 have routine modifications for W_4; these will be subsumed in the general proofs for all $x \in X$.

The topology on the set of points in $(W_2 \cup W_4) \times \mathfrak{b}$ is very different from the product topology.

Lemma 4.1. *Let $\alpha \in W_4$ and let $\nu \leq \xi < \mathfrak{b}$. For each $n \in \omega$ let $i_n = f_\xi(n)$. Then $\langle \alpha_{ni_n}, \nu \rangle \to \langle \alpha, \xi \rangle$.*

Proof. Obviously, $\langle \alpha_{ni_n}, \nu \rangle \in V_\xi^\alpha$ for all n. So there is a limit point by compactness of V_ξ^α, and clearly all limit points are of the form $\langle \alpha, \mu \rangle$ for some $\mu \leq \xi$. But also, $V_\eta \cap H(\alpha_{ni_n}) = \emptyset$ for $\eta < \xi$, so $\mu = \xi$. □

Of course, the product topology would make the sequence converge to $\langle \alpha, \nu \rangle$.

The following theorem plays the same role for W_4 that Theorem 3.4 did for W_2.

Theorem 4.2. *Let $\alpha \in W_2 \cup W_4$. Every sequence whose projection converges to α has a cluster point in $\{\alpha\} \times \mathfrak{b}$.*

Proof. The case of $\alpha \in W_2$ is covered by Lemma 3.3. If $\pi(x_k) \to \alpha$ and $\alpha = W_4$, we may assume either that $x_k = \langle \gamma_k, \nu_k \rangle \in W_2$ for all k or that $x_k \in W_0$ for all k, and assume that there is at most one x_k in each interval $(\alpha_{n-1}, \alpha_n]$. In either case, $x_k \in B(\alpha_{n_k j_k})$ for unique n_k and j_k.

Let μ be the least ordinal such that $f_\mu(n_k) \geq j_n$ for infinitely many k for which j_k is defined. If $x_k = \langle \gamma_k, \nu_k \rangle$ for all k let $\nu = sup_k \nu_k$ and let $\xi = max\{\mu, \nu\}$. Then $x_k \in V_\xi^\alpha$ for infinitely many k, and we use compactness of V_ξ^α.

If $x_k \in W_0$ for all k, let m_k be the greater of j_k and the i_{n_k} defined in the proof of Lemma 4.2, and then the rest of that proof goes through with m_n replacing i_n. □

Definition 4.3. Let $\alpha \in W_4$. The *hyperblock at α* is $H(\alpha) = \pi^{\leftarrow}(\beta, \alpha]$, where β is defined in the second parpagraph of this section. $H(\alpha)$ also termed *a rank 2 hyperblock of the first kind.* Sets of the form $\pi^{\leftarrow}(\gamma, \alpha]$, where $\beta < \gamma < \alpha$, are termed *rank 2 hyperblocks of the second kind.*

In general, rank θ hyperblocks will be associated with α on the θth C-B level of E. The rank 2 hyperblocks are easily seen to partition $W_0 \cup W_2 \cup W_4$.

5. Neighborhoods of all points and hyperblocks of all ranks

The inductive steps in defining rank n hyperblocks, where $1 < n < \omega$, are routine generalizations of the steps for rank 1 and rank 2 hyperblocks. In particular, they are the disjoint union of rank $n-1$ hyperblocks, and are associated with points of W_{2n}. To define the basic neighborhoods of these points, we introduce the following notation;

Definition 5.1. *Let $\varepsilon \in W_{2h}$, where $2 \leq h \in \omega$. Given $\xi \in \mathfrak{b}$, we define:*

$$V_\xi^\varepsilon = \{\langle \varepsilon, \nu \rangle : \nu \leq \xi\} \cup \bigcup \{V_{\xi \vee f_\xi(n)}^{\varepsilon_{ni}} : i \leq f_\xi(n),\ n \in \omega\}.$$

and, if $k > 0$, we also define:

$$V_{\xi \vee k}^\varepsilon = \{\langle \varepsilon, \nu \rangle : \nu \leq max\{k, \xi\}\} \cup \bigcup \{V_{\xi \vee k}^{\varepsilon_{ni}} : i \leq max\{k, f_\xi(n)\},\ n \in \omega\}.$$

If $\varepsilon \in W_2$, we define

$$V^{\varepsilon}_{\xi\vee k} = V^{\varepsilon}_{\xi} \cup V^{\varepsilon}_{k} = \{\langle \varepsilon, \nu \rangle : \nu \leq max\{k, \xi\}\} \cup \{\nu^{\varepsilon}_{ni} : i \leq max\{k, f_\xi(n)\}, n \in \omega\}\,.$$

The first equality shows that this formula agrees with our earlier formula for V^{α}_{ξ} when $\alpha \in W_4$, with $k = f_\xi(n)$ for $\varepsilon = \alpha_{ni}$.

This equality, $V^{\varepsilon}_{\xi\vee k} = V^{\varepsilon}_{\xi} \cup V^{\varepsilon}_{k}$ for $\varepsilon \in W_2$, does not hold in general. For instance, if $\beta = \alpha_{ni}$ and if $k < f_\xi(n)$, then $\langle \beta_{nf_\xi(n)}, \xi\rangle \in V^{\alpha}_{\xi} \setminus V^{\alpha}_{\xi\vee k}$; on the other hand, if $f_\xi(0) < k < \xi$, then $\langle \alpha_{0k}, \xi\rangle \in V^{\alpha_{0k}}_{\xi} \subset V^{\alpha}_{\xi\vee k}$, but $\langle \alpha_{0k}, \eta\rangle \in V^{\alpha}_{k}$ iff $\eta \leq k$, and $V^{\alpha}_{\xi} \cap V^{\alpha_{0k}}_{\xi} = \emptyset$.

The importance of having $\langle \alpha_{0k}, \xi\rangle$ in $V^{\alpha}_{\xi\vee k}$ comes into play in the proof of Theorem 6.3 below, of which the following example is a special case.

Example 5.2. Let $\beta \in W_6$ and $\xi \geq \omega$. For all $n \in \omega$, let $f_\xi(n) = k_n$ and let $\alpha^n = \beta_{nk_n}$. Then $\alpha^n \in W_4$, $\langle \alpha^n, \xi\rangle \in V^{\beta}_{\xi}$ and

$$\langle \alpha^n_{0k_n}, \xi\rangle \in V^{\alpha^n}_{\xi\vee f_\xi(n)} = V^{\beta_{nk_n}}_{\xi\vee f_\xi(n)}$$

for all n, and it is routine to show that $\langle \alpha^n_{0k_n}, \xi\rangle \to \langle \beta, \xi\rangle$.

On the other hand, $\langle \alpha^n_{0k_n}, \xi\rangle \notin V^{\alpha^n}_{\xi}$ for $n > 0$ because $k_n > f_\xi(0)$, and $\langle \alpha^n_{0k_n}, \eta\rangle \in V^{\alpha^n}_{f_\xi(n)}$ iff $\eta \leq k_n$ for any n.

For $\alpha \in W_{2n}$, $2 < n \in \omega$, the formula for V^{α}_{ξ} propagates downwards in the following fashion. For each choice of n_0 and i_0,

$$V^{\alpha_{n_0 i_0}}_{\xi\vee f_\xi(n_0)} = \{\langle \alpha_{n_0 i_0}, \nu\rangle : \nu \leq max\{n_0, \xi\} \cup \bigcup\{V^{\beta_{ni}}_{\xi\vee f_\xi(n_0)} : i \leq f_\xi(max\{n_0, n\}), n \in \omega\},$$

where $\beta = \alpha_{n_0 i_0}$; and $V^{\beta_{n_1 i_1}}_{\xi\vee f_\xi(n_0)}$ has the same formula, but with $\beta_{n_1 i_1}$ in place of $\alpha_{n_0 i_0}$ and γ_{ni} in place of β_{ni}, where $\gamma = \beta_{n_1 i_1}$. Of course, the last displayed formula uses the fact that $max\{f_\xi(n_0), f_\xi(n)\} = f_\xi(max\{n_0, n\})$ because f_ξ is nowhere decreasing.

One effect of this downwards propagation is to expand the first role of f_ξ, by adding finitely many sub-hyperblocks to be met by V^{α}_{ξ} over and above the ones associated with $V^{\alpha_{ni}}_{\xi}$ (and so on down the line) when $\xi \geq \omega$. [When $\xi = n \in \omega$ it sometimes replaces n with a larger integer.] The second role remains unchanged within each of the sub-hyperblocks met by V^{α}_{ξ}. The third role comes into play at the end of the propagation, when new rows $\{\nu^{\varepsilon}_{ni} : n \in \omega\}$ may be added at the bottom of the rank 1 hyperblock. Role 2 already comes into play in Theorem 5.3 while Roles 1 and 3 play a decisive role in making π closed.

Hyperblocks where δ is a limit ordinal need to be handled differently. This is already evident in the case of ω^ω, the first ordinal in W_ω. The part of X that precedes it is the union of an ascending sequence of hyperblocks, each of which reaches all the way back to 0, instead of being a union of a disjoint sequence of hyperblocks headed by ordinals in order type ω. Moreover, although there is a canonical ω-sequence of ordinals in $\omega_1 \setminus E$ leading up to ω^ω, there is no such thing for ordinals in W_δ for limit δ beyond a certain point. Yet it is ω-sequences that are needed for delimiting hyperblocks at each stage of the induction.

This need is taken care of by defining an ω-sequence $\langle \alpha_n : n \in \omega\rangle$ of ordinals converging to each point α of W_δ ($\delta \in S$), in the following way. First, let $\langle \delta_n : n \in \omega\rangle$ be an increasing sequence in E of ordinals > 0 converging to δ. Next, let β be the least ordinal such that $(\beta, \alpha) \cap E$ contains no ordinals of W_γ for $\gamma \geq \delta$. [In particular, if $\alpha = min(W_\delta)$ then $\beta = 0$.] Let $\alpha_0 = \beta$, and let α_n be the least ordinal in $W_{\delta_n+1} \cap (\beta, \alpha)$ when $n > 0$. Then $\alpha_n = \beta_n \cdot \omega$ where β_n is the least ordinal in $W_{\delta_n} \cap (\beta, \alpha)$.

Let $\alpha_{ni} = \beta_n \cdot (i+1)$ for all i. The associated hyperblocks are all of the first kind, $H(\alpha_{ni})$, except perhaps for $\pi^{\leftarrow}(\alpha_{n-1}, \alpha_{n0})$. Then, for each n, these hyperblocks will be treated as the $(n+1)$st column in a $\omega \times \omega$ array, as in the case of W_γ where γ is a successor.

Special treatment is also required for ordinals in $W_{\delta+1}$ where δ is a limit ordinal which is not in S. Each point $\alpha \in W_{\delta+1}$ has a canonical increasing ω-sequence $\{\alpha_n : n \in \omega\}$ in $W_\delta \cup \{0\}$ converging to it, where α_i is the ith ordinal in $(\beta, \alpha) \cap W_\delta$ if $i > 0$, and $\alpha_0 = \beta =$ the least ordinal such that $(\beta, \alpha) \cap E$ contains no ordinals of W_γ for $\gamma > \delta$. [In particular, if $\alpha = min(W_{\delta+1})$ then $\beta = 0$.]

This time, it is the α_{ni} that generally do not have a simple formula. Let $\{\delta_{ni} : i \in \omega\}$ be an increasing sequence of ordinals with supremum δ such that $W_{\delta_{ni}} \subset E$ for all i, and let α_{ni} be the least ordinal in $(\alpha_{n-1}, \alpha_n) \cap W_{\delta_{ni}}$.

Our basic formula now becomes:

$$(*) \qquad V^\alpha_\xi = \{\langle \alpha, \nu \rangle : \nu \leq \xi\} \cup \bigcup \{(V^{\alpha_{ni}}_{\xi \vee f_\xi(n)} \cap \pi^{\leftarrow}(\zeta_{ni}, \alpha_{ni}] : i \leq f_\xi(n), n \in \omega\}.$$

where $\zeta_{ni} = \alpha_{n-1}$ if $i = 0$, while if $i > 0$ then $\zeta_{ni} = \alpha_{nj}$, where $j = i - 1$.

Formula $(*)$ is valid for all $\alpha \in E \setminus (W_0 \cup W_2)$, but much of it is redundant if $\alpha \notin W_{\delta+1}$ where δ is a limit ordinal outside E. In fact, the only other case where the extra $\cap \pi^{\leftarrow}(\zeta_{ni}, \alpha_{ni}]$ is not redundant is for various limit $\delta \in E$. Even there, it is redundant when $i > 0$, when $(\zeta_{ni}, \alpha_{ni}] = (\beta_n \cdot i, \beta_n \cdot (i+1)]$. These intervals are order-isomorphic for all $i \neq 0$, and the earlier stages of the induction had $H(\beta_n \cdot (i+1)) = \pi^{\leftarrow}(\beta_n \cdot i, \beta_n \cdot (i+1)]$.

The other formulas remain as before, except for modifying $V_{\xi \vee k}$ with $\cap \pi^{\leftarrow}(\zeta_{ni}, \alpha_{ni}]$; otherwise we have $V^\alpha_\xi(n) = V^\alpha_\xi \cap (\alpha_n, \alpha]$ and $V^\alpha_{\xi,\eta} = V^\alpha_\xi(n) \setminus V^\alpha_\eta = V^\alpha_\xi(n) \setminus V^\alpha_\eta(n)$.

Theorem 5.3. *The sets of the form $V^\alpha_{\xi,\eta}(n)(\alpha \in E \setminus W_0,\ \xi, \eta \in \mathfrak{b}, n \in \omega)$, together with singletons of W_0, form a base for the topology on X, and the ones with a superscript α and first subscript ι form a base for the neighborhoods of $\langle \alpha, \iota \rangle$.*

Proof. Obviously, the union of these sets is X, and it is enough to show that if $\alpha \in E \setminus W_0$, and

$$\langle \alpha, \iota \rangle \in V^\gamma_{\xi,\eta}(n) \cap V^\varepsilon_{\mu,\nu}(m)$$

then there exist k and ζ such that

$$V^\alpha_{\iota,\zeta}(k) \subset V^\gamma_{\xi,\eta}(n) \cap V^\varepsilon_{\mu,\nu}(m).$$

The way the formulas propagate downwards, the second role entails that, for the intersection on the right to be nonempty, it is necessary and sufficient that $min\{\xi, \mu\} > max\{\eta, \nu\}$. If $\langle \alpha, \iota \rangle$ is in the intersection, then $\iota \leq min\{\xi, \mu\}$. Then if $\zeta = max\{\eta, \nu\}$ then we also have $\zeta < \iota$, and then

$$(V^\gamma_{\xi,\eta} \cap V^\varepsilon_{\mu,\nu}) \setminus W_0 = \{\langle \psi, \rho \rangle : \psi \leq min\,\{\gamma, \varepsilon\}, \zeta < \rho \leq min\{\xi, \mu\}\}\,.$$

If $\chi = max\left\{min\left(\pi^{\rightarrow}V^{\gamma}_{\xi,\eta}(n)\right), min\left(\pi^{\rightarrow}V^{\varepsilon}_{\mu,\nu}(m)\right)\right\}$, then $\alpha > \chi$, and there exists k such that $min\left(\pi^{\rightarrow}V^{\alpha}_{\iota,\upsilon}(k)\right) > \chi$ since these minima converge to α. So k and ζ are as desired. □

6. The main properties of X

The proof of local compactness works more smoothly if we use the sets $V^{\alpha}_{\xi\vee k}$ rather than the basic V^{α}_{ξ} in the inductive proof.

Lemma 6.1. *Let $\alpha \in E \setminus W_0$, $\xi \in \mathfrak{b}$, and $k \in \omega$. The open set $V^{\alpha}_{\xi\vee k}$ is compact.*

Proof. For $\alpha \in W_2$ this follows easily from the proof of Theorem 3.3, where it is shown that both V^{α}_{ξ} and V^{α}_{k} are compact, and from the fact that $V^{\alpha}_{\xi\vee k} = V^{\alpha}_{\xi} \cup V^{\alpha}_{k}$ when $\alpha \in W_2$.

So suppose that $V^{\beta}_{\eta\vee j}$ is compact for all $\beta < \alpha$ and all $\eta \in \mathfrak{b}$, $j \in \omega$. If $\xi \in \omega$, then $V^{\alpha}_{\xi\vee k}$ is simply V^{α}_{m} where $m = max\{\xi, k\}$. And we can refine any open cover of V^{α}_{m} to sets of the form $V^{\beta}_{i+1,i}(n)$ $(\beta \leq \alpha, i < m, n \in \omega)$. For each $i < m$, one set $V^{\alpha}_{i+1,i}(n_i)$ can be chosen to cover $\langle \alpha, i+1\rangle$, where $i \leq m$. The rest of V^{α}_{m} is covered by finitely many compact sets, $V^{a_{ji}}_{m}$, where $j \leq n_i, i \leq m$.

The case where $\xi \geq \omega$ requires only minor modifications. Let $\mathcal{U}$ be an open cover of $V^{\alpha}_{\xi\vee k}$ by basic neighborhoods $V^{\varepsilon}_{\mu,\eta}(n)$, where $\varepsilon \leq \alpha$ and $\mu \leq \xi$. Then $\alpha \times \{\mathfrak{b}\}$ is covered by the sets of the form $V^{\alpha}_{\mu,\eta}(n)$, and these can be handled as in the proof of Theorem 3.3, resulting in a finite cover of a set $V^{\alpha}_{\xi}(N)$. What is left over of $V^{\alpha}_{\xi\vee k}$ is covered by finitely many compact sets of the form $V^{\alpha_{ni}}_{\xi\vee k}$ where $n \leq N$ and $i \leq max\{k, f_{\xi}(n)\}$. □

The following corollary completes the proof that X is locally compact.

Corollary 6.2. *Each basic open set $V^{\alpha}_{\xi,\eta}(n)$ is compact.*

Proof. These basic open sets are clopen in the relative topology of V^{α}_{ξ}, which is compact by Lemma 6.1. □

Theorem 6.3. *Let $\alpha \in E \setminus W_0$. Every sequence whose projection converges to α has a cluster point in $\{\alpha\} \times \mathfrak{b}$.*

Proof. Let $\pi(x_k) \uparrow \alpha$ and assume, without loss of generality, that $x_k < \alpha$ for all k and that either $x_k \in W_0$ for all k or $x_k \notin W_0$ for all k.

For each $k \in \omega$, there are unique $n(k)$ and $i(k)$ such that $x_k \in H(\alpha_{n(k)i(k)})$; we may assume $n(k) < n(k+1)$ for all k. Let $\alpha^1 = \alpha_{n(k)i(k)}$; if α^j has been defined so that $x_k \in H(\alpha^j)$ but $\pi(x_k) \neq \alpha^j$, and $\alpha^j \notin W_2$, let $\alpha^{j+1} = \alpha^j_{ni}$ for the unique n and i such that $x_k \in H(\alpha^j_{ni})$. But if $\alpha^j \in W_2$ and $x_k \in W_0$ or $\pi(x_k) = \alpha^j$, let $\alpha^j = \alpha^{J(k)}$. We must arrive at $J(k)$ after finitely many steps because $\alpha^i > \alpha^{i+1}$ and every descending sequence of ordinals is finite.

Fix k and refine the notation as follows: $\alpha^{j+1} = \alpha^j_{m(j)o(j)}$. If $x_k = \nu_{ni} \in W_0$ let $O(k) = max\left\{i, max\{o(j) : 0 < j < J(k)\}\right\}$. Using $<^*$-unboundedness of $\{f_\xi : \xi < \mathfrak{b}\}$, choose $\xi \in \mathfrak{b}$ so that $f_\xi(n(k)) \geq O(k)$ for infinitely many k. Then the first role of f_ξ, and the downward propagation of ξ, ensures that $V^{\alpha}_{\xi} \cap H(\alpha^j) \neq \emptyset$ for all j, and the third ensures that $x_k = \nu_{ni} \in V^{\alpha}_{\xi}$.

If $x_k = \langle \beta, \eta \rangle$, then $\beta = \alpha^{J(k)}$. Let $O(k) = max\{o(j) : 0 < j < J(k)\}$ and let ξ be as before. But this time, let $\mu = max\{\xi, \eta\}$ to ensure that $\langle \beta, \eta \rangle \in V^\alpha_\mu$ according to the second role of f_μ. $\square$

Theorem 6.4. *The map $\pi : X \to E$ is quasi-perfect.*

Proof. Continuity is due to the fact, easily shown by induction, that the preimage of each interval $(\beta, \alpha]$ is open. The proof of the countable compactness of each fiber is essentially as for Theorem 3.5. The closedness of the map also follows as in that proof, this time from Theorem 6.3 rather than Theorem 3.4, except that the last clause in that proof needs to be replaced by "in the second, the fact that α is the only point $(\beta, \alpha] \cap E$ that is on the boundary of $\pi^{\rightarrow}B$." $\square$

Lemma 6.5. *If $A \subset X$ and $\pi^{\rightarrow}A$ is uncountable, the following subset $C(A)$ of ω_1 is a club.*

$$\{\gamma \in E \cap \overline{\pi^{\rightarrow}A} : c\ell_X(A) \cap (\{\gamma\} \times \mathfrak{b}) \textit{ is cofinal in } \{\gamma\} \times \mathfrak{b}\}.$$

Proof. Since $\overline{\pi^{\rightarrow}A}$ is a club, there exist, for each $\gamma \in \omega_1$, an ordinal $\alpha \in E \setminus [0, \gamma)$ and a sequence $\{\delta_n : n \in \omega\} \subset \overline{\pi^{\rightarrow}A} \setminus E$ such that $\delta_n \uparrow \alpha$. The proof that $C(A)$ is unbounded will be complete once we show $\alpha \in C(A)$.

For each n, $\overline{\pi^{\rightarrow}A} \cap E \cap [\delta_n, \delta_{n+1}]$ is infinite, and there exists a sequence $\langle \beta_{ni} : i \in \omega \rangle \uparrow \delta_n$ such that $\beta_{ni} \in \overline{\pi^{\rightarrow}A} \cap E$ for all n, i. Let x_{ni} satisfy $\pi(x_{ni}) = \beta_{ni}$. Then $\{x_{ni} : i \in \omega\}$ is closed discrete, so that it meets each basic (compact) neighborhood $V^\alpha_\xi(\xi \in \mathfrak{b})$ in a finite set.

So then, for each $n \in \omega$, and each $\xi \in \mathfrak{b}$, there exists $i(n)$ such that $x_{ni(n)} \notin V^\alpha_\xi$. However, by Theorem 6.3, $\langle x_{ni(n)} : n \in \omega \rangle$ has a limit point $\langle \alpha, \nu \rangle$, and clearly $\nu > \xi$.

To show that $C(A)$ is closed, note that if $\pi(\gamma_n, \nu_n) = \gamma_n \uparrow \gamma$, and $\langle \gamma, \nu \rangle$ is a limit point as in Theorem 6.3, then $\nu \geq min\{\nu_n : n \in \omega\}$. $\square$

Corollary 6.6. *At least one of every pair of disjoint closed subsets of X has countable image under π.*

Proof. Let F_0 and F_1 be closed subsets of X, with uncountable images. Then $\pi^{\rightarrow}F_i$ is a club by Theorem 6.4 and contains $C(F_i)$ for $i \in 2$. If $\alpha \in C(F_0) \cap C(F_1)$, then F_0 and F_1 both meet $\{\alpha\} \times \mathfrak{b}$ in a club subset, and since $\mathfrak{b}$ is a regular uncountable cardinal, they meet in a club subset of $\{\alpha\} \times \mathfrak{b}$. $\square$

Theorem 6.7. *X is normal.*

Proof. Let F_0 and F_1 be disjoint closed subsets of X. By this last corollary, there exists $\alpha < \omega_1$ such that at least one of these sets (say, F_1) is in the clopen set $\pi^{\leftarrow}[0, \alpha]$, and so it suffices to separate $F_0 \cap \pi^{\leftarrow}[0, \alpha]$ from F_1.

The following induction is on α. Suppose that the sets $F_i \cap [0, \zeta]$ can be put into disjoint open sets for all $\zeta < \alpha$. If $F_i \cap (\{a\} \times \mathfrak{b}) = \emptyset$ for some i, then by Theorem 6.3 there exists $\gamma < \alpha$ such that $F_i \cap \pi^{\leftarrow}(\gamma, \alpha] = \emptyset$, and there exists $n \in \omega$ such that $V^\alpha_\xi(n) \cap \pi^{\leftarrow}[0, g] = \emptyset$ for all ξ. What is left uncovered of F_{1-i} can be separated from F_i by the induction hypothesis.

So suppose $F_i \cap (\{\alpha\} \times \mathfrak{b}) \neq \emptyset$ for both $i \in 2$. As an ordinal in the order topology, $\mathfrak{b}$ is almost compact and normal. This implies that of any two disjoint closed subsets, at least one is compact. Thus $\exists\delta < \mathfrak{b}$ such that $\{\alpha\} \times [0,\delta]$ contains (without loss of generality) $F_1 \cap (\{\alpha\} \times \mathfrak{b})$. There is a partition $\mathcal{P}$ of $\{\alpha\} \times [0,\delta]$ into finitely many clopen intervals, each of which meets exactly one F_i. For each $(\eta,\xi] \in \mathcal{P}$ (with the convention that $[0,\xi] = (-1,\xi]$) there exists $n_\xi \in \omega$ such that $V^\alpha_{\xi,\eta}(n_\xi)$ also meets only F_i. Let $n = max\{n_\xi : (\eta,\xi] \in \mathcal{P}\}$. The whole of $V^\alpha_\delta(n)$ is partitioned into sets of the form $V^\alpha_{\xi,\eta}(n) \subset V^\alpha_{\xi,\eta}(n_\xi)$, each of which meets exactly one F_i.

Then if $F_0 \cap (\delta, \mathfrak{b}) \neq \emptyset$, there exists $\gamma < \alpha$ such that $\pi^{\leftarrow}(\gamma,\alpha] \cap F_1 \subset V^\alpha_\xi(n)$, by Theorem 6.3. The induction hypothesis then says we can separate F_0 and F_1 on $[0, max\{\alpha_n, \gamma\}]$. $\square$

7. Arbitrary stationary, co-stationary subsets of ω_1

In extending the construction to arbitrary E, it simplifies matters considerably to assume that every successor ordinal is in E, as is 0. There is no loss of generality since the closure of any uncountable subset Z of ω_1 is order-isomorphic to ω_1 itself by the unique order-preserving bijection, which is also a homeomorphism. The isolated points of $\overline{Z}$ are also the isolated points of Z, and they map onto the isolated ordinals in ω_1 (in other words, onto the successor ordinals together with 0). The image of Z then behaves in exactly the way in ω_1 as Z itself did in $\overline{Z}$.

The most essential difference arises almost immediately: if β_1 and β_2 are successive (limit) ordinals of $\omega_1 \setminus E$, then we could have $\beta_2 = \beta_1 + \delta$ for any countable limit ordinal δ, whereas before we always had $\delta = \omega$. Equivalently, $(\beta_1, \beta_1 + \delta) \subset E$. Points in such intervals play the role that W_0 played in the spaces of the earlier sections. That is, let

$$S_0 = \{\xi \in E : \xi \notin \overline{\omega_1 \setminus E}\} \text{ and let } X = S_0 \cup [(E \setminus S_0) \times \mathfrak{b}].$$

The points of S_0 then form an open set whose topology is its order topology as a subset of ω_1. If β_1 and $\beta_2 = \beta_1 + \delta$ are as above, then there is a sequence $\delta_n \uparrow \delta$ with $\delta_0 = 0$, and the intervals $(\beta_1 + \delta_i, \beta_1 + \delta_{i+1}]$ form a partition of (β_1, β_2) into compact open sets.

To play the role of W_1 and W_2, we let S_1 [*resp.* S_2] be the set of points of $\omega_1 \setminus E$ [*resp.* $E \setminus S_0$] which have neighborhoods meeting E [*resp.* $\omega_1 \setminus E$] only in points of S_0 [*resp.* S_1].

As with S_1, successive members of S_2 might be spread far apart: the points of S_1 between successive points of S_2 can be of any countable order type. But we can define neighborhoods of points of S_2 by a procedure that substitutes clopen intervals for the points ν^α_{ni} used in defining the neighborhoods of points of W_2.

The technique is as follows. If α is the least element of S_2, let $\beta = 0$, otherwise let β be the greatest element of $E \setminus S_0$ that is $< \alpha$. Let $\alpha_n \uparrow \alpha$ be strictly increasing, with $\alpha_0 = \beta$ and $\alpha_n \notin E$ otherwise. Then $(\omega_1 \setminus E) \cap (\beta, \alpha)$ is relatively closed in (β, α), and each subinterval (α_n, α_{n+1}) is the union of a denumerable discrete family $\mathcal{C}_n$ of compact open intervals, obtained through use of successive members β_1, B_2 of $S_1 \cap (\alpha_n, \alpha_{n+1})$ as above. [In some or all subintervals, these may be singletons, as they always are in the earlier sections.] List the members of each $\mathcal{C}_n$ in an ω-sequence in no particular order. Make this sequence column n in an $\omega \times \omega$ array.

Use the family $\mathcal{F}$ in a way analogous to its use in Section 3, treating each member of $\mathcal{C}_n$ like a single point when defining the generalization of $(f_\xi^\alpha)^\downarrow$. In other words, let C_{ni}^α be the ith entry in the nth column of the above array. Let

$$(f_\xi^\alpha)^\downarrow = \bigcup\{C_{ni}^\alpha : i \leq f_\xi(n)\}.$$

The formulas for neighborhoods of points in S_2 are the same as for W_2 in Section 5, with C_{ni}^ε in place of ν_{ni}^ε where appropriate. Thus, $V_\xi^\varepsilon = \{\langle \varepsilon, \nu\rangle : \nu \leq \xi\} \cup (f_\xi^\varepsilon)^\downarrow$ and

$$V_{\xi\vee k}^\varepsilon = V_\xi^\varepsilon \cup V_k^\varepsilon = \{\langle \varepsilon, \nu\rangle : \nu \leq max\{k, \xi\}\} \cup \bigcup\{C_{ni}^\varepsilon : i \leq max\{k, f_\xi(n)\}, n \in \omega\}.$$

For nonisolated $x \in S_0$, a base of neighborhoods is given by

$$\mathfrak{V}_x = \{(\nu, x] : \nu < x \text{ and } (\nu, x] \subset E\}.$$

As before, if x is isolated, then $\mathfrak{V}_x = \{\{x\}\}$.

The various proofs of Section 3 only require routine modifications, through the substitutions of countable, compact open intervals for single points. For instance, in the proof of the generalization of Theorem 3.5, the last clause can be replaced as in that of Theorem 6.4: "in the second, the fact that α is the only point $(\beta, \alpha] \cap E$ that is on the boundary of $\pi^\rightarrow B$."

It is possible to go on to define generalizations of W_γ for all $\gamma \in \omega_1$, but there is a less explicit method that is adequate for obtaining a quasi-perfect preimage, and requires less detail at the price of losing some geometric intuition. The technique makes the use of two facts: (1) every point of $\omega_1 \setminus E$ has a sequence of points of E converging to it [if nothing else, there are points of W_0 that can do the job]; and (2) every point α of E outside S_0 has a sequence from $\omega_1 \setminus E$ converging to it.

Notation 7.1. The sequel will use some double subscripts in which one subscript involves addition or subtraction. Such a subscript is enclosed in parentheses: $\alpha_{n(i+1)}$, etc.

The construction is by induction on $E_2 = E \setminus (S_0 \cup S_2)$. The induction hypothesis at every ordinal $\alpha \in E_2$ is that a base of neighborhoods of the form $V_{\xi,\eta}^\varepsilon(n)$ has been defined for all $\varepsilon \in E \cap [0, \alpha)$; and also, if $\varepsilon \in E_2$, then $\pi^\rightarrow V_\xi^\varepsilon$ starts with $\alpha_0 + 1 = 1$, as in the case with all minimum points of each W_γ in the constructions of Sections 3 through 5.

The base case, $\alpha = min(E_2)$, has two subcases.

Subcase 1: α is a limit point of $\omega_1 \setminus (E \cup S_1)$.

This is similar to the case where $\alpha \in S_2$. Let $\alpha_0 = 0$, and let $\alpha_n \in \omega_1 \setminus (E \cup S_1)$, for each $n > 0$, such that $\alpha_n \uparrow \alpha$. When $n > 0$, each interval (α_{n-1}, α_n) breaks up into a discrete collection $\mathcal{C}_n$ of projections of basic blocks, defined as in Definition 3.6 . List their maxima as $\{\alpha_{ni} : i \in \omega\}$ in no particular order, and let

$$V_\xi^\alpha = \{\langle \alpha, \nu\rangle : \nu \leq \xi\} \cup \bigcup\{V_{\xi\vee f_\xi(n)}^{\alpha_{ni}} : i \leq f_\xi(n), n \in \omega\}.$$

In other words, this is as in Definition 5.1 with α in place of ε; $V_{\xi\vee k}$ is defined in the same way, and $V_\xi^\alpha(n)$ and $V_{\xi,\eta}^\alpha(n)$ are as in Definition 3.1.

Subcase 2: Otherwise. Equivalently, α has a neigborhood which only meets $\omega_1 \setminus E$ in S_1. The following involves a bit of scrambling of S_0 in comparison to the earlier placement of its points, but it is done in order to make the proof of normality of X generalize with minimal changes.

By minimality of α in E_2, $S_2 \cap [0, \alpha)$ is of order type ω with α as supremum; list it in natural order as $\{\beta_n : n \in \omega \setminus \{0\}\}$ and let $\beta_0 = 0$. Let $\alpha_0 = 0$ and let $\alpha_n = (\beta_n)_1$ and $\alpha_{n0} = \beta_n$ for $n > 0$. Let $\alpha_{ni} = (\beta_{n+1})_{1(i-1)}$ for $i > 0$ and all $n \in \omega$. Informally, the first entry in column 0 for α is the basic block $\pi^{\leftarrow}(0, \beta_n]$, while the rest of the entries are the C_{1i} of the first column associated with β_1; the same holds for column $n > 0$, except that the first entry is the rank 1 hyperblock $\pi^{\leftarrow}(\beta_{n-1}, \beta_n]$.

Since each basic neighborhood $V_\xi^{\beta_n}$ begins where the preceding one left off, the formula for Subcase 1 still applies, with the abuse of notation $V^{\alpha_{ni}}_{\xi \vee f_\xi(n)} = C_{1(i-1)}$, where $C_{1(i-1)}$ is defined with respect to β_{n+1}.

Now suppose the induction hypothesis has been satisfied for $\{\varepsilon \in E_2 : \varepsilon < \alpha\}$, where $\alpha > min(E_2)$. There are three subcases for showing it also holds for α.

<u>Subcase A: α is isolated in E_2.</u>

This is treated in a way almost identical to the base case. The only real change needed is to let $\alpha_0 = \alpha_{00}$ be the greatest ordinal in E_2 that is less than α, and to let α_{0i} as prescribed in either subcase be re-labeled as $\alpha_{0(i+1)}$ for all i; then the induction hypothesis continues to hold.

<u>Subcase B: α is in the closure of $\overline{E_2} \cap (\omega_1 \setminus (E \cup S_1))$.</u>

This subcase is similar to the case in Section 5 where $\delta \in S$ is a limit ordinal. Here, too, there may be no canonical sequence of ordinals to serve as $\langle \alpha_n : n \in \omega \rangle \uparrow \alpha$.

Let α_0 be any point of $E_2 \cap (0, \alpha)$. Let $\langle \alpha_n : n \in \omega \setminus \{0\} \rangle \uparrow \alpha$ be a sequence of ordinals in $\overline{E_2} \cap (\omega_1 \setminus (E \cup S_1))$ such that $\alpha_1 > \alpha_0$. Except for letting $\alpha_{00} = \alpha_0$, let $\langle \alpha_{ni} : i \in \omega \rangle$ be an increasing sequence from $(\alpha_n, \alpha_{n+1}) \cap E_2$ converging to α_{n+1}. The basic formula $(*)$ carries over:

$$(*) \qquad V^\alpha_\xi = \{\langle \alpha, \nu \rangle : \nu \leq \xi\} \cup \bigcup \{(V^{\alpha_{ni}}_{\xi \vee f_\xi(n)} \cap \pi^{\leftarrow}(\zeta_{ni}, \alpha_{ni}] : i \leq f_\xi(n), n \in \omega\}.$$

As before, $\zeta_{ni} = \alpha_{n-1}$ if $i = 0$, while if $i > 0$ then $\zeta_{ni} = \alpha_{n(i-1)}$. The formulas that followed the earlier display of $(*)$ continue to hold here.

This time, the extra $\cap\, \pi^{\leftarrow}(\zeta_{ni}, \alpha_{ni}]$ gets a lot of use, since we need to have the summands disjoint, and the basic neighborhoods V^ε_ξ overlap for any two $\varepsilon \in E_2$.

Remark 7.2. Except for the way $\alpha_0 = \alpha_{00}$ is shoehorned into the beginning of V^α_ξ, this can still serve as a generalization of the construction of Sections 3 through 5. But our final subcase, like Subcase 2, is not found at all in this earlier construction.

<u>Subcase C: Otherwise.</u> In other words, α is a limit point of E_2 but has a neigborhood which meets $\omega_1 \setminus E$ only in S_1.

This is similar to Subcase 2, but there may not be a canonical $\langle \beta_n : n \in \omega \rangle$. Instead, we let this be an arbitrary ascending sequence in E_2 converging to α, with β_0 a point in $E_2 \cap (0, \alpha)$. Now, because each hyperblock $H(\beta_n)$ reaches all the way back to 0, all but the first hyperblock needs to be replaced with $H(\beta_n, \beta_{n-1})$, a hyperblock of the second kind.

For each $n > 0$, there is a least ordinal above β_n in $\{(\beta_{n+1})_m : m \in \omega\}$; let this ordinal be $\alpha_n = (\beta_{n+1})_{m_n}$. Then there is a least $j \in \omega$ such that $(\beta_{n+1})_{m_n j} > \alpha_n$. Let $\alpha_{n0} = \beta_n$ and let $\alpha_{n(i+1)} = (\beta_{n+1})_{m_n(i+j)}$ for all $n, i \in \omega$. The basic formula still holds, but with a different definition of ζ_{ni}.

$$(*) \qquad V^\alpha_\xi = \{\langle \alpha, \nu \rangle : \nu \le \xi\} \cup \bigcup \{(V^{\alpha_{ni}}_{\xi \vee f_\xi(n)} \cap \pi^\leftarrow(\zeta_{ni}, \alpha_{ni}] : i \le f_\xi(n), n \in \omega\}.$$

This time, $\zeta_{00} = 0$, $\zeta_{n0} = sup\{(\beta_n)_{m_n k} : k \in \omega\}$ for all $n > 0$, while if $i > 0$ then $\zeta_{ni} = \alpha_{n(i-1)}$ for all i.

The proofs of Section 6 go through with routine changes. In particular, the proof of Lemma 6.5 is "coordinate-free," and the construction of Subcase 2 and Subcase C was specially designed to make any point of $\omega_1 \setminus E$ able to play the role of some δ_n. The only new restriction is that γ should not be isolated in E_2, but that does not significantly affect the proof.

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(Peter Nyikos) Department of Mathematics, University of South Carolina, Columbia, SC 29208

E-mail address: nyikos@math.sc.edu