PROXIMAL AND SEMI-PROXIMAL SPACES

Peter Nyikos

Abstract. Proximal and semi-proximal spaces are defined with the help of a game played on uniform spaces. Proximal spaces are where the entourage-picker has a winning strategy, and semi-proximal spaces are where the opponent, the point-picker, does not have a winning strategy. The class of proximal spaces is closed under Σ-products and closed subspaces. In this paper it is shown that product of two semi-proximal spaces need not be semi-proximal; that every proximal space is strongly monolithic, (*i.e.,* every countable subspace has metrizable closure); and that every every semi-proximal space is Fréchet and α_2. Twenty problems are posed, all but one of them unsolved. Some of these relate to the concept of a uniform box product.

Introduction

This article has to do with two class of uniform spaces defined with the help of a game of length ω played on uniform spaces. One player (Player A) picks entourages, the other (Player B) picks points. The ones where Player A has a winning strategy are called proximal, while the ones where Player B does not have a winning strategy are called semi-proximal.

Proximal spaces have very strong preservation, convergence, covering and separation properties. In particular, the Σ-product of proximal spaces is proximal, and every proximal space is collectionwise normal, countably paracompact, [B3] and strongly monolithic, (*i.e.,* has the property that every separable subspace is metrizable) [Theorem 2.5 below].

We will list several open problems on the usual (Tychonoff) product of spaces, and also their uniform box product. This concept was introduced by Scott Williams in 2001, at the 9th Prague International Topological Symposium (Toposym). At the

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1991 *Mathematics Subject Classification.* (Updated to 2013.) Primary: 54B10, 54D15, 54D18, 54D20, 54D30, 54D55, 54E15 Secondary: 54A20, 54D55, 54E35.

Key words and phrases. proximal, strategy, semi-proximal, uniform space, uniform box product, normal, collectionwise normal, Fort space, Fréchet, W-space, w-space, metrizable, submetrizable, Corson compact, shrinking.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

time, Prague was perhaps the best place in the world for research on uniform spaces, but very little was done with the uniform box product until the recent (2010 and 2012) doctoral dissertations of Jocelyn Bell and Jeffrey Hankins [B1], [H1]. Their results had to do with two questions that Williams posed at the 2001 Toposym and then at the 2006 Toposym.

Problem 1. *Is the uniform box product of denumerably many compact spaces normal? collectionwise normal?*

Problem 2. *Is the uniform box product of denumerably many compact spaces paracompact?*

Jocelyn Bell introduced some new ideas on Problem 1 in [B1] that enabled her to show that the denumerable power of the one-point compactification of a discrete space of cardinality $\aleph_1$ is normal, countably paracompact, and collectionwise Hausdorff in the uniform box topology. Hankins showed it was also collectionwise normal by modifying Bell's proof of the collectionwise Hausdorff property, then answered Problem 2 by showing that this same product is not paracompact in the uniform box topology.

Problem 1 remains open as do all numbered problems given below. We do not even have consistency results for any of them, except for Problem 18.

All through this paper, "space" means "Hausdorff space," but since the topic is uniform spaces, this is equivalent to the spaces being Tychonoff. In particular, all uniformities are assumed to be separated.

1. Definitions

Definition 1.1. *A* **diagonal uniformity** *on a set X is a filter $\mathfrak{D}$ of relations on X, called* **entourages** *or* **surroundings** *or* **vicinities** *satisfying the following conditions:*

1. *$\Delta \subset D$ for all $D \in \mathfrak{D}$, where Δ is the diagonal $\{(x,x) : x \in X\}$*
2. *If $D \in \mathfrak{D}$, then $D^{-1} \in \mathfrak{D}$, where D^{-1} is the inverse of D, that is, $D^{-1} = \{(y,x) : (x,y) \in D\}$;*
3. *If $D \in \mathfrak{D}$, then there exists $E \in \mathfrak{D}$ such that $E \circ E \subset D$, where*
$$E \circ F = \{(x,z) : \exists y \in X \text{ such that } (x,y) \in F, (y,z) \in E\}$$

A uniform space $(X, \mathfrak{D})$ is **separated** *if $\bigcap \mathfrak{D} = \Delta$.*

Alternative ways of defining uniformities in general are by way of continuous pseudometrics [GJ], [W] and by uniform covers [I], [W]. Given an entourage $D \in \mathfrak{D}$, the associated uniform cover is

$$\mathcal{P}_D = \{D[x] : x \in X\} \text{ where } D[x] = \{y : \langle x, y\rangle \in D\}.$$

Given any uniform space $\langle X, \mathfrak{D}\rangle$, and $x \in X$, the associated topological space has the sets $\{D[x] : D \in \mathfrak{D}\}$ as a base for the neighborhoods of x. Part of the elementary theory of uniformities, used in the proof of 2.6 below, is that the interiors of the sets $D[x]$ cover X for each $D \in \mathfrak{D}$.

In the case where D is an equivalence relation ($D \circ D = D = D^{-1}$), $\mathcal{P}_D$ is trivially a partition of X into clopen sets, and the notation $\mathcal{P}_D$ is motivated by our examples, in which the uniformity has a base of equivalence relations.

At the Prague Toposyms and all completed papers referenced here, only the special case where all factors are the same uniform space are considered, so the following could be called the "uniform box power".

Definition 1.2. *Let $\mathfrak{D}$ be a diagonal uniformity on the space X, and let κ be a cardinal number. For each $D \in \mathfrak{D}$ let*

$$\overline{D} = \{\langle x, y\rangle \in X^\kappa \times X^\kappa : \langle x(\alpha), y(\alpha)\rangle \in D \text{ for all } \alpha \in \kappa\}.$$

The uniformity $\overline{\mathfrak{D}}$ on X^κ whose base is the collection of all $\overline{D}$ is called **the uniform box product**.

In particular, $\{\overline{D}[x] : D \in \mathfrak{D}\}$ is a base for the neighborhoods of $x \in X^\kappa$, and $\overline{D}[x] = \Pi_{\alpha\in\kappa} D[x(\alpha)] = \{y : y(\alpha) \in D[x(\alpha)] \text{ for all } \alpha < \kappa\}$.

Where many questions about topological properties and their preservation are concerned, this restriction to powers of a space is more general than may seem at first, because lots of topological properties are preserved by the topological direct sum, including, clearly, proximality. A paper in preparation [Ny4] will deal with the more general concept of a uniform box product.

Applying Definition 1.2 to the usual uniformity $\mathfrak{U}$ on $\mathbb{R}$, we have that $\overline{\mathfrak{U}}$ is an extension of $\ell_\infty(\kappa)$ to all of $\mathbb{R}^\kappa$; and ℓ_∞ itself is the component of $\overrightarrow{0}$ in $\mathbb{R}^\kappa$ with the uniform box product.

More generally, $(X^\kappa, \overline{\mathfrak{D}})$ is metrizable if the uniformity on X has a countable base (and thus X has a metrizable topology). These uniformities and many others are subsumed by the concept of a proximal uniform space, due to Jocelyn Bell [B3]. It uses a game of length ω, played by Player A and Player B on a uniform space $(X, \mathfrak{D})$. Player A chooses members of $\mathfrak{D}$ and Player B chooses points in the space X.

Definition 1.3. The proximal game on a uniform space $(X, \mathfrak{D})$ proceeds as follows:

(1) Player A chooses $D_1 \in \mathfrak{D}$
  Player B chooses $x_1 \in X$

(2) Player A chooses $D_2 \in \mathfrak{D}$ with $D_2 \subset D_1$
  Player B chooses $x_2 \in 4D_1[x_1]$, where in general $4D = 2D \circ 2D$ and $2D = D \circ D$.

In general, if $D_1, \dots D_{n-1}$ and $x_1, \dots x_{n-1}$ and have been chosen for $n > 1$, we have

(n) Player A chooses $D_n \in \mathfrak{D}$ with $D_n \subset D_{n-1}$
Player B chooses $x_n \in 4D_{n-1}[x_{n-1}]$.

Player A wins iff either
(I) there exists $z \in X$ such that $x_1, x_2 \dots$ converges to z or
(II) $\bigcap_{n=1}^{\infty} 4D_n[x_n] = \emptyset$.

Note that if D is an equivalence relation, then $4D_i$ can be replaced by D_i for all i, and $D_i[x]$ is the unique member of the partition $\mathcal{P}_{D_i}$ that contains x.

As usual, a strategy for a Player A is a function σ whose domain is all finite sequences of legal plays $x_1, \dots, x_n$ by Player B, and which sends such a sequence to a play D_{n+1} on Player A's next move. A strategy σ for Player A is **winning** if every sequence of legal moves in which $\sigma(x_1, \dots, x_n)$ is Player A's response to $x_1, \dots x_n$ results in a win for Player A. A winning strategy for Player B is defined similarly.

Definition 1.4. A uniform space $(X, \mathfrak{D})$ is **proximal** if Player A has a winning strategy in the proximal game on $(X, \mathfrak{D})$. A topological space X is **proximal** if there is a proximal uniformity $\mathfrak{D}$ compatible with the topology on X.

If $\mathfrak{D}$ has a base of equivalence relations, and Player A has a winning strategy, then Player A has one that uses only equivalence relations, since replacing D_i by a finer member of a given base is easily seen to force a win for Player A [B3, Lemma 3.2]. Similarly, if $(X, \mathfrak{D})$ is proximal and $\mathfrak{E}$ is a finer uniformity than $\mathfrak{D}$ that gives the same topology that $\mathfrak{D}$ does, then $(X, \mathfrak{E})$ is proximal also: a winning strategy for $(X, \mathfrak{D})$ is also one for $(X, \mathfrak{E})$.

2. Fundamental properties of proximal spaces

Easy examples of proximal spaces include metric spaces, the one-point compactifications of discrete spaces, and the space ω_1 of countable ordinals. In contrast, Player B has a winning strategy on $\omega_1 + 1$. It consists of alternating between the last point ω_1 and a countable ordinal inside $D_n[x_{n-1}]$ (not just inside $D_{n-1}[x_{n-1}]$) on move n whenever $x_{n-1} = \omega_1$. A generalization of this technique is used below in the proof of Theorem 3.1, which has the corollary that every proximal space is Fréchet. [Recall that a space is called **Fréchet** or **Fréchet-Urysohn** if for every subset A and every point x in the closure of A, there is a sequence from A converging to x.]

Jocelyn Bell originally introduced the concept of a proximal space to help unify a number of proofs which showed the denumerable uniform box powers of some spaces to be collectionwise normal and countably paracompact. Ironically, this concept has been more successful with the usual (Tychonoff) product topology so far:

Theorem 2.1. [B3] *The Σ-product of any number of copies of a proximal space is proximal, and every proximal space is collectionwise normal and countably paracompact.*

The strength of Theorem 2.1 can best be appreciated by noting that every metric space is proximal in its metric uniformity [B3, Lemma 4.1] and that it was an open problem for almost two decades whether the Σ-product of metric spaces is normal. It was solved by Sergei P. Gul'ko [Gu] and Mary Ellen Rudin [R2]; a proof can be found also in [P]. In contrast, the following problem is unsolved:

Problem 3. *Is the denumerable uniform box power of every proximal uniform space normal?*

The common weakening of Problems 1 and 3 is also unsolved:

Problem 4. *Is the denumerable uniform box power of every compact proximal space normal?*

There is no need to put "uniform" after "proximal" since all compact spaces have unique uniformities, provided by the filter of all neighborhoods of the diagonal [W]. More generally:

Problem 5. *Is the denumerable uniform box power of every proximal space normal with respect to some compatible proximal uniformity?*

Example 2.4 below shows how much this may depend on the uniformity.

If the answer to any of Problem 3, 4, or 5 is affirmative, then it also follows that this power is countably paracompact and collectionwise normal. This follows from a theorem of Ofelia Alas, whose proof also appears in [P], and a few simple observations:

Theorem 2.2. [A] *A topological space X is collectionwise normal and countably paracompact iff its product with every Fort space is normal.*

"Fort space" is a convenient shorthand for the one-point compactification of a discrete space. It is easy to see that every Fort space is proximal. Given a proximal space X, let Y_κ be the product of X with the Fort space A_κ of cardinality κ. By Theorem 2.1, Y_κ is proximal. The countable uniform box power of Y_κ has a closed subspace homeomorphic to $X^\omega \times A_\kappa$, which is thus normal if the whole box power is normal.

Similar reasoning gives us the following: every proximal space is a Morita P-space, and if the answer to 3, 4, or 5 is affirmative, then the uniform box power in question is one also. Normal Morita P-spaces can be defined "externally" as being the spaces whose product with every metric space is normal. Morita P-spaces in general can be defined "internally" by a game reminiscent of the proximal game. Players 1 and 2 alternate, with Player 1 defining a closed set F_n on the nth move that is a subset of F_{n+1} while Player 2 defines an open set U_n such that $F_n \subset U_n$. Player 2 wins if either $\bigcap_{n=1}^\infty F_n \neq \emptyset$ or $\bigcap_{n=1}^\infty U_n = \emptyset$. A Morita P-space is a space in which Player 2 has a winning strategy.

Another result in [B3] relates proximal spaces to spaces defined by an even more similar game, invented by G. Gruenhage [G1].

Theorem 2.3. [B3, Lemma 5.1] *Every proximal space is a W-space.*

A **W-space** is a space X on which Player 1 has a winning strategy in the following game no matter how $p \in X$ is chosen. On their nth moves, Player 1 plays a neighborhood V_n of p, and Player 2 plays $x_n \in V_n$. Player 1 wins iff $x_n \to p$.

As may be expected, this game is harder for the point-picker to win than the proximal game, because in that game there is more freedom of movement as to where to pick the points. Even the extra way the entourage picker can win the proximal game (getting the intersection of the $D_n[x_n]$ to be empty) is not enough to offset this advantage, as Theorem 2.3 shows.

The converse of Theorem 2.3 does not hold. For example, every first countable space is clearly a W-space; so the Sorgenfrey plane is a W-space, but it is not proximal because it is not normal. Gruenhage showed [G1] that a W-space is Fréchet, and has the property that every countable subspace has a countable base. He also showed that a Σ-product of W-spaces is a W-space [G1].

Problem 6 [7] [8]. *Is the denumerable uniform box power of every compact proximal space proximal? [a W-space?] [Fréchet]?*

Without "compact" the answer to Problem 8 and hence to 6 and 7 is negative, even in the case of metrizable spaces with finer uniformities than the metric, as the following example shows.

Example 2.4. Take the simple metrizable space $X = \omega \times (\omega + 1)$ with the uniformity $\mathfrak{E}$ consisting of all partitions into clopen sets. By the observation at the end of Section 1 and the fact that metric spaces are proximal in the metric uniformity, X is proximal. On the other hand, $(X^\omega, \overline{\mathfrak{E}})$ is not countably tight: there is a subset A of X^ω and a point $\overrightarrow{x}$ in its closure but not in the closure of a countable subset of A.

Let $\overrightarrow{x}$ be the point in the product $(X^\omega, \overline{\mathfrak{E}})$ that satisfies $x(n) = (n, \omega)$ for all n. For each function $f : \omega \to \omega$, take the partition $\mathcal{P}_E$ of X into the parts of each column of all points above the graph of f together with the singletons that are on or below the graph of f. This partition canonically defines a basic open neighborhood $\overline{E}[\overrightarrow{x}]$ of $\overline{x}$ in $(X^\omega, \overline{\mathfrak{E}})$.

Thus $\overrightarrow{x}$ has a base of neighborhoods that is identical with the ones in the usual box product, and if we take the point x_f which is identical with $f : \omega \to \omega$ in the usual ordered pair definition of a function, then the set A of all these x_f has $\overrightarrow{x}$ in its closure, but no set of fewer than $\mathfrak{d}$ of them does. So the countable box power of X is not Fréchet-Urysohn. It is also unknown whether it is normal: it is under Martin's Axiom [R1] [Wm] but we run into a famous box products problem: is $\Box(\omega+1)^\omega$ normal?

This example also suggests that the answer to Problem 3 is negative: the denumerable box power of the irrationals is not normal [R1] [Wm].

An encouraging sign about compact proximal spaces is that they do not admit of subspaces like X in the last example. This is shown by Corollary 2.7 below, which follows from a strong structure theorem.

Theorem 2.5. *Every separable proximal space is metrizable.*

Proof. We first prove Theorem 2.5 in the case where the uniformity has a base of equivalence relations so as to make the central ideas of the proof clearer. The adaptation for general uniformities illustrates some of the tools used in such adaptations.

Let Q be a countable dense subspace of X, and let σ be a strategy for Player A. We will develop a winning counter-strategy for Player B with the help of a countable elementary submodel M of a large enough fragment of the universe such that Q, σ, X, the uniformity $\mathfrak{D}$ on X, and the function $\varphi : \mathfrak{D} \times X \to X$ for which $\varphi(D) = D[x]$ are all members of M. Since Q is countable, every point of Q is also a member of M, by elementarity; also, the topology induced by the uniformity is a member of M.

Let $\mathfrak{B}$ be a base for the uniformity on X such that each member B of $\mathfrak{B}$ induces a partition of X into clopen sets. The partition is countable because each member meets Q, so the individual members are all in M if $B \in M$: elementarity of M implies $|A| \leq \aleph_0 \wedge A \in M \implies A \subset M$.

Let $\{B_n : n \in \omega\}$ be a listing of $\mathfrak{B} \cap M$. The listing will not be in M but that will not affect the proof.

Case I. There exists $p \in X$ such that $\bigcap_{n=1}^{\infty} B_n[p] \neq \emptyset$.

Case II. Otherwise.

In Case I, there exists $r \neq p$ in the intersection. All the points in the intersection are "invisible to M," but they have disjoint closed neighborhoods, say N_p and N_r, and by density of $Q \subset M$, Player B can choose x_n alternately from $D_n[p] \cap N_p \cap Q$ and $D_n[r] \cap N_r \cap Q$. These moves are in M and the strategy σ is in M, and so Player A is constrained by σ to choose D_{n+1} in M, so that $\langle x_n \rangle$ cannot converge to p, while

$$\bigcap_{n=1}^{\infty} D_n[x_n] \supset \bigcap_{n=1}^{\infty} B_n[x_n] = \bigcap_{n=1}^{\infty} B_n[p] \neq \emptyset.$$

In Case II, let τ be the topology on X, and let ν be the metrizable topology whose base is $\bigcup_{n=1}^{\infty} \bigcup \{B_n(q) : q \in Q\}$. If $\tau = \nu$ we are done, otherwise Fréchet property of the two topologies implies that that there is a sequence $\langle p_n : n \in \omega \rangle$ that converges with respect to ν but not with respect to τ. Let its limit be p. There is a closed

τ-neighborhood W of p and a subsequence that ν-converges to p outside W, which we may assume to be the original sequence.

There is a sequence $\langle q_n^i : i \in \omega \rangle$ from Q converging to p_n for each n, and with none of the q_n^i in W. The metrizability of ν implies that there is a sequence from $Q \subset M$ converging to p wrt ν. The sequence itself will not be in M, but Player B can freely choose distinct points from it on each turn, similarly to the method above; only this time, there is no need to distinguish even turns from odd turns. Player A is constrained to choose an entourage in M as long as $x_n \in Q$ for all n, and so Player B wins since $\langle x_n : n \in \omega \rangle$ is not τ-convergent.

For general proximal uniformities, the argument in Case II can go through unchanged once the proof of Case I is suitably modified. It is convenient to do this via a lemma that has a weaker conclusion.

Lemma 2.6. *Every separable proximal space has a coarser metrizable topology.*

Once this is shown, we let ν be such a topology and complete the proof of Theorem 2.5 as in Case II.

Proof of lemma. Contrapositively, let X be a separable space without a coarser metrizable topology. Let Q, σ, M etc. be as before, and let $\mathfrak{B} = \mathfrak{D} \cap M$ where $\mathfrak{D}$ is the uniformity on X. Then $\mathfrak{B}$ is a countable base $\{B_n : n \in \omega\}$ for a uniformity on X, one which can be produced by a pseudometric $\rho \in M$. By the hypothesis on X, there will be points p and r at zero distance. Let U and V be disjoint open subsets of X containing p and r respectively.

We next show that the interiors of the $B_n[m] (m \in X \cap M)$ cover X. These interiors are in M because $B_n[m] \in M$ and because $\bigcup\{G : G \text{ is open and } G \subset B_n[m]\}$ is a member of M even if it is not a subset of M. And the interiors cover X because, as noted above, $\{int(B_n[x]) : x \in X\}$ is an open cover; and it is a member of M. The pseudometric ρ gives a locally finite refinement in M, and this is countable. By elementarity, this refinement is in M and gives a countable subcover of $int(B_n[x]) : x \in X\}$, which is also in M and can be listed there as $\{int(B_n[x_i]) : i \in \omega\}$.

For some x_i, and some $\delta > 0$, the open 2δ-ball $S_{2\delta}(p) = S_{2\delta}(r)$ in the pseudometric ρ is in the interior of $B_n[x_i]$. By density of Q, there will be some $q \in Q$ such that p and r are in $S_\delta(q)$.

Now, if Player A chooses B_n on the jth turn, Player B can pick $q_j \in Q \cap B_n[x_i]$ as above, to be in U if n is even and in V if n is odd. Then whatever B_k is chosen by Player A on turn $j+1$, both p and q are comfortably inside $4B_n[q_j]$ (because they are both in $B_n[x_i]$), and Player B can repeat legally the same procedure that gave q_j, *mutatis mutandis.* $\square$

Spaces with coarser metrizable topologies are called **submetrizable**. Lemma 2.6 suggests the following question.

Problem 9. *Is every submetrizable proximal space metrizable?*

Corollary 2.7. *If $(X, \mathfrak{D})$ is a countably compact proximal space, then the trace of $\mathfrak{D}$ on every countable subset of X has a countable base.*

Proof. Let Q be a countable subspace of X and let S be the closure of Q. Then S is a separable proximal subspace of X, and hence is metrizable, and compact. So the unique uniformity on S has a countable base, as does every uniformity that comes from a metric. Hence its trace on Q also has a countable base. $\square$

The following result shows how important Corollary 2.7 is if Problem 8 is to have an affirmative answer:

Theorem 2.8. [Ny3] *Let $(X, \mathfrak{D})$ be a uniform space. If $(X^\omega, \overline{\mathfrak{D}})$ is Fréchet-Urysohn, then the trace of $\mathfrak{D}$ on every countable subset of X has a countable base.*

The converse is known not to hold, not even for compact spaces, but the counterexample in [Ny3] (the one-point compactification of an Aronszajn tree) is not proximal, nor even a W-space, and so Problem 8 remains open. In fact, Player B has a winning strategy on this space [Ny3].

It is clear from Theorem 2.1 that every Corson compact space (compact subspace of a Σ-product of real lines) is proximal. The converse is open:

Problem 10. *Is every compact proximal space Corson compact?*

A Yes answer would obviously greatly shorten the proof of Corollary 2.7! It would also show that compact proximal spaces have strong hereditary covering properties: Corson compacta are precisely the compact spaces with hereditarily metalindelöf squares [G3]. So if Problem 10 has a negative answer we could still ask what hereditary covering properties proximal compacta have.

On the other hand, we cannot expect any "nice" covering properties to hold for arbitrary proximal spaces unless they put restrictions on the cardinality of the cover to be refined. In [Ny2] it is shown that the countable uniform box power of an uncountable Fort space is not even weakly $\delta\theta$-refinable; and in [B3] it is shown that this uniform box power is proximal.

Nevertheless, there are some possibilities for strengthening normality and countable paracompactness. Normality + countable paracompactness is equivalent to every countable open cover, or every ascending countable open cover $\{U_\alpha : \alpha < \omega\}$ having an **shrinking**, *i.e.,* an open refinement $\{V_\alpha : \alpha < \omega\}$ such that $\overline{V_\alpha} \subset U_\alpha$ for all $\alpha < \omega$ [W, proof of Theorem 2.13]. If the cardinality restriction on the cover is removed, these two characterizations become distinct.

Definition 2.9. A space is **shrinking** [resp. **weak shrinking**] if every open cover [resp. every ascending open cover] has a shrinking.

In [R2], Mary Ellen Rudin showed that every Σ-product of metrizable spaces is shrinking. This suggests the following problems.

Problem 11. Is every proximal space shrinking?

Problem 12. Is every proximal space weak shrinking?

Normal, countably paracompact spaces which are not weak shrinking are very rare; in [R3] Mary Ellen Rudin gives a class of "κ-Dowker" spaces which generalize her famous ZFC Dowker space and she remarks that these are the only examples she knows of normal, countably paracompact spaces that are not weak shrinking.

3. Semi-proximal spaces

Let us call a uniform space **semi-proximal** if Player B does not have a winning strategy in the proximal game. This is not enough to imply proximality, as the game is indeterminate on some spaces. An example will be given after the following theorem.

Theorem 3.1. *Every semi-proximal space is a w-space; that is, the point picking player does not have a winning strategy in the Gruenhage game.*

Proof. This is an easy consequence of Sharma's characterization of w-spaces [Sh]: they are the α_2-Fréchet spaces. [A topological space is α_2 if, whenever σ_n is a sequence converging to a point x for each $n \in \omega$, there is a "diagonal" sequence σ converging to x whose range meets the range of every σ_n.] So if a space X is not a w-space, it is either not Fréchet or it is not α_2.

If X is not Fréchet, Player B in the proximal game can find a point $p \in X$ and a set A such that $p \in \overline{A}$ such that no sequence in A converges to p. Player B can adopt the strategy of playing p itself on odd-numbered moves, and some point of A in even-numbered moves, and then the sequence cannot converge to any point. On move $2n$, this point x_{2n} should be in $D_{2n}[p] \subset 2D_{2n}[x_{n-1}]$ and not just in $4D_{2n-1}[x_{n-1}]$, so that Player B can hop back to p on the next turn. So X is not semi-proximal.

If X is not α_2, the strategy that wins for Player B is similar. This time, Player B singles out a point x that witnesses the failure of α_2, and picks it on odd-numbered moves, while picking a carefully chosen point in the range of σ_n on move $2n$. The choice is guided in the same way as the choice of x_{2n} when X is not Fréchet. So again X is not semi-proximal. $\square$

While proximal spaces and W-spaces are very well behaved under products, w-spaces are not.

Example 3.2. In [Ny1] it is shown how an (ω_1, ω_1)-gap in the power set of ω can be used to construct a pair of w-spaces, each of which has only one nonisolated point, so that the "corner point" of the product cannot be reached with a convergent sequence from the dense set of isolated points [Ny1]. The factor spaces are not first countable, but they are countable, so they are not W-spaces by a theorem in [G1]; so they are not proximal. On the other hand, they are semi-proximal, because of the following partial converse of Theorem 3.1.

Lemma 3.3. *In its fine uniformity, every w-space* [resp. *W-space*] *with a single non-isolated point is semi-proximal* [resp. *proximal*]

Proof. Let X have a single non-isolated point p. The "best" moves of Players A and B essentially turn the proximal game on X into a Gruenhage game on X centered at p. For Player A, $D_n[p]$ is, of necessity, a clopen neighborhood of p, and it is clearly to the advantage of Player A to make D_n induce a partition of X into $D_n[p]$ and the singletons of $X \setminus D_n[p]$. As for Player B, it would be "suicidal" to let $\{x_n\}$ be one of these singletons, because then all the later x_m have to equal x_n, and so converge to it.

In this way, Player A and Player B induce a bijection between these special ways of playing the proximal game and all legal ways of playing the Gruenhage game: the legal moves of the neighborhood picker in the latter are in bijection with the "advantageous" ones of Player A in the former, and the legal moves of the point-picker in the latter then coincide with the "non-suicidal" ones of Player B in the former. With the above constraints, winning strategies of either player in either game become winning strategies of the corresponding player in the other game. □

Corollary 3.4. *The proximal game is indeterminate on the factor spaces of Example 3.2.* □

Lemma 3.3 can easily be extended to scattered paracompact spaces, but we will not need this result here.

Problem 13. *Is every semi-proximal space normal?*

The answer is affirmative if the space is countably compact: it is easy to show that Player B can alternate between two closed sets that cannot be put into disjoint open sets, and any cluster point of $\langle x_n \rangle$ is in the intersection of the sets $D_n[x_n]$, and there are at least two of them. Nevertheless, I conjecture a negative answer to Problem 13.

Problem 14. *If a product of two semi-proximal spaces is Fréchet, must it be semi-proximal?*

Problem 15 [16]. *If every finite subproduct in a countable family of semi-proximal spaces is semi-proximal, must the whole product be semi-proximal? [Fréchet?]*

In problems 14 and 16, "Fréchet" is equivalent to "w-space" everywhere, inasmuch as the product of countably many α_2-spaces is α_2 [No] and we can apply Sharma's characterization of w-spaces as α_2 Fréchet spaces. This also applies to:

Problem 17. *If every finite subproduct in a countable family of semi-proximal spaces is Fréchet, must the whole product be Fréchet?*

Remarkably enough, the following variant of Problems 15 and 16 is still unsolved:

Problem 18. *Is there a ZFC example of a countable family of Fréchet spaces such that every finite subproduct is Fréchet, but the whole product is not Fréchet?*

Examples are known under CH [G2] and MA [T] with the latter even compact, but finding a ZFC counterexample seems to be a formidable problem. As far as Problem 15 is concerned, however, the really relevant general problem is the following:

Problem 19. *Is there a countable family of w-spaces such that every finite subproduct is a w-space, but the whole product is not a w-space?*

For this, there may be no consistency results at all. For example, the factor spaces in [T] are not all w-spaces, because the countable product of countably compact regular w-spaces is a w-space [No].

Finally, here is a problem inspired by Corollary 3.4 and the fact, mentioned earlier, that every proximal space is a Morita P-space.

Problem 20. *Is there a space on which the game used to define Morita P-spaces is indeterminate?*

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