Hilbert's First and Second Problems and the foundations of mathematics

Peter Nyikos, University of South Carolina, Columbia, SC 29208 USA

In 1900, David Hilbert gave a seminal lecture in which he spoke about a list of unsolved problems in mathematics that he deemed to be of outstanding importance. The first of these was Cantor's continuum problem, which has to do with infinite numbers with which Cantor revolutionised set theory. The smallest infinite number, $\aleph_0$, 'aleph-nought,' gives the number of positive whole numbers. A set is of this cardinality if it is possible to list its members in an arrangement such that each one is encountered after a finite number (however large) of steps. Cantor's revolutionary discovery was that the points on a line cannot be so listed, and so the number of points on a line is a strictly higher infinite number ($\mathfrak{c}$, 'the cardinality of the continuum') than $\aleph_0$. Hilbert's First Problem asks whether any infinite subset of the real line is of one of these two cardinalities. The axiom that this is indeed the case is known as the Continuum Hypothesis (CH).

This problem had unexpected connections with Hilbert's Second Problem (and even with the Tenth, see the article by M. Davis and the comments on the book edited by F. Browder). The Second Problem asked for a proof of the consistency of the foundations of mathematics. Some of the flavor of the urgency of that problem is provided by the following passage from an article by Simpson in the same volume of JSL as the article by P. Maddy:

'We must remember that in Hilbert's time, all mathematicians were excited about the foundations of mathematics. Intense controversy centered around the problem of the legitimacy of abstract objects. Weierstrass had greatly clarified the role of the infinite in calculus. Cantor's set theory promised to raise mathematics to new heights of generality, clarity and rigor. But Frege's attempt to base mathematics on a general theory of properties led to an embarrassing contradiction. Great mathematicians such as Kronecker, Poincaré, and Brouwer challenged the validity of all infinitistic reasoning. Hilbert vowed to defend the Cantorian paradise. The fires of controversy were fueled by revolutionary developments in mathematical physics. There was a stormy climate of debate and criticism. The contrast with today's climate of intellectual exhaustion and compartmentalization could not be more striking.

' ... Actually, Hilbert saw the issue as having supramathematical significance. Mathematics is not only the most logical and rigorous of the sciences but also the most spectacular example of the power of "unaided" human reason. If mathematics fails, then so does the human spirit. I was deeply moved by the following passage [13, pp.370–371]: "The definitive clarification of the nature of the infinite has become necessary, not merely for the special interests of the individual sciences but for the honor of human understanding itself." '

Hilbert was already aware, at the time of his 1900 lecture, of some connection between the provability of the consistency of a mathematical theory and the decidability of statements by the axioms of the theory. But it was Kurt Gödel who showed the true nature of this connection in the process of showing that Hilbert's Second Problem has a negative solution:

The First Incompleteness Theorem. *Every recursively axiomatizable theory rich enough to include the Peano Axioms contains statements whose truth cannot be decided within the theory. In particular, Peano Arithmetic itself can be used to formulate true statements about the natural numbers that are not provable within Peano Arithmetic.*

[**Query:** Is Fermat's Last Theorem one of these statements? Wiles has shown it follows from the usual ZFC axioms; but does it already follow from the Peano axioms?]

The Second Incompleteness Theorem. *Every recursively axiomatizable theory rich enough to include the Peano Axioms is incapable of demonstrating its own consistency.*

Another fundamental discovery of Gödel was:

The Completeness Theorem. *Every consistent set of axioms has a model.*

Together with the first incompleteness theorem, this has been a source of a wealth of mathematics as well as such paradoxical facts as the following: it is impossible to unambiguously formalize the distinction between "finite" and "infinite". The "featherless biped" definition of an infinite set as one that can be put into one-to-one correspondence with a proper subset of itself does not work; neither does the more natural definition of a finite set as one that can be put into one-to-one correspondence with $\{0, \ldots, n\}$ for some natural number n: the very concept of "natural number" cannot be formalized in a way that makes it clear that our intuitive concept of a natural number is intended.

GOĎEL [1940] also gave a partial solution to Hilbert's First Problem by showing that the Continuum Hypothesis (CH) is consistent if the usual Zermelo-Fraenkel (ZF) axioms for set theory are consistent. He produced a model, known as the Constructible Universe, of the ZF axioms in which both the Axiom of Choice (AC) and the CH hold. Then Cohen showed in 1963 that the negations of these axioms are also consistent with ZF; in particular, CH can fail while AC holds in a model of ZF. Cohen's technique for producing such models was generalized by Scott, Solovay, and Shoenfeld and a huge variety of models of ZFC (ZF plus AC) has been produced in the years since then, affecting many areas of mathematics. The books by DALES AND WOODIN, FREMLIN, KUNEN, KUNEN AND VAUGHAN, MONK, and RUDIN as well as the articles by EKLOF and ROITMAN, and the articles of Blass reviewed by

Nyikos give some idea of how great a variety of topics these independence results have been relevant to.

Topology has been affected perhaps more than any other field, and the following gives a small sample. Recall the Heine-Borel theorem: *Every open cover of a closed bounded subset of the real line has a finite subcover.* The conclusion provides also the definition of *A compact topological space.* The conclusion of another famous topological theorem, the Bolzano-Weierstrass theorem, is the basis for a weaker concept:

Definition. A topological space is *countably compact* if every infinite subset has an accumulation point.

A strengthening of countable compactness, not shared by all compact spaces, is that of *sequential compactness:* every sequence has a convergent subsequence.

These three concepts agree for all metrizable spaces (those spaces whose topology is given by a distance function to the non-negative reals that is symmetric, puts distinct points at a positive distance from each other, and satisfies the triangle inequality). Compact metrizable spaces have lots of other properties not shared by compact topological spaces in general, so it is perhaps surprising that the question of when a compact space is metrizable can be very simply settled:

Theorem. [Sneider, 1945] *A compact space is metrizable if, and only if, it is Hausdorff and has a G_δ-diagonal; that is, the diagonal $\{(x,x) : x \in X\}$ is a countable intersection of open sets.*

This theorem was extended to all regular countably compact spaces by J. Chaber in 1975. One might naturally expect these two theorems to either stand or fall together if "G_δ-diagonal" is weakened to "small diagonal":

Definition. A space has a *small diagonal* if, whenever A is an uncountable subset of $X \times X$ that is disjoint from the diagonal Δ, there is a neighborhood U of Δ such that $U \setminus A$ is uncountable.

But in fact, this is not the case. On the one hand, we know that CH implies every compact Hausdorff space with a small diagonal is metrizable. We do not know whether ZFC implies this as well; but, be that as it may, the corresponding statement about regular countably compact spaces is independent not only of ZFC, but also of CH. On the one hand, Gary Gruenhage has shown that in a model of CH constructed by Todd Eisworth and Peter Nyikos, the statement is true– every countably compact regular space with a small diagonal is metrizable; on the other hand, Oleg Pavlov has constructed a counterexample in Gödel's Constructible Universe, the very model that originally established the consistency of CH!

Annotated Bibliography

J. Barwise, ed. *Handbook of Mathematical Logic,* North-Holland, 1977.

[Part B, on set theory, has many consistency and independence results, including applications to topology. The article by J. P. Burgess in Part B gives a fine introduction to forcing. The article by Smorynski on Gödel's Incompleteness Theorems is one of the very few treatments I have seen that does not leave holes plugged only by hand-waving. After the clearest treatment I have ever seen of the Second Incompleteness Theorem, he even points out, "In Section 2.1, we have been guilty of cheating in two places" and then goes on to make the necessary repairs. There is also a significant article by Harrington about an incompleteness in Peano Arithmetic at the end.]

F. Browder, ed. *Mathematical Developments Arising from Hilbert Problems,* Proceedings of Symposia in Pure Mathematics XXVIII, American Mathematical Society, 1974. Dewey Decimal number 510.82 A51 28.

[Includes a reprint of the English translation of Hilbert's article. The article on Hilbert's first problem, by D. A. Martin, expounds on the significance of consistency and independence proofs, and of large cardinal axioms. There are articles on the second problem by Kreisel and on the tenth by the co-solvers, Martin Davis (see reference to an article by him below), Yuri Matijasevic, and Julia Robinson. A quote from their article: "The consistency of a recursively axiomatizable theory is equivalent to the assertion that some definite Diophantine equation has no solutions."]

H.G. Dales and W.H. Woodin, *An Introduction to Independence for Analysts,* Cambridge University Press, 1987.

[An eloquent preface introduces a self-contained treatment of the set-theoretic independence of a basic problem in functional analysis: If X is compact, Hausdorff and infinite, is every homomorphism from $C(X, \mathbb{C})$ into any Banach algebra continuous? Answer: No if CH for *every* such infinite X, but it is also consistent that the answer is Yes for *every* such X!]

M. Davis, "Hilbert's Tenth Problem is Unsolvable," Amer. Math. Monthly 80 (1973) 233–269.

[A spellbinding exposition with complete proofs, not merely of the tenth problem but about how its solution impacts the foundations of mathematics in completely unexpected ways. Included is a very concrete treatment of Gödel's First Incompleteness Theorem in terms of Diophantine equations. If we ever contact an extraterrestrial intelligence and want to impress it with what human beings are capable of, this would be the article I'd recommend to be transmitted to them.]

P. Eklof, "Whitehead's problem is undecidable," Amer. Math. Monthly 83 (1976) 775–788.

[The set-theoretic independence of the problem of whether every Whitehead group is free.]

D. H. Fremlin, *Consequences of Martin's Axiom,* Cambridge University Press, 1984.

[Includes many applications to topology, measure theory, and algebra of Martin's Axiom and the negation of CH, as well as of some weaker axioms which also deny CH.]

GÖDEL, K.

[1940] "The consistency of the axiom of choice and of the generalized continuum hypothesis," Ann. Math. Studies no. 3, Princeton University Press, 1940.

[1947] What is Cantor's continuum problem? *Amer. Math. Monthly*, **54**: 515–25.

[The first article gives the proof of its main results in full; the second explains, *inter alia*, why Gödel believed the Continuum Hypothesis to be "dubious" in spite of its consistency.]

A. Kanamori and M. Magidor, "The evolution of large cardinal axioms in set theory," pp. 99–275 in: *Higher Set Theory* G. H. Muller and D. S. Scott, eds., Lecture Notes in Mathematics no. 669, Springer-Verlag, 1978.

[A dramatic article on large cardinal axioms with a wealth of information and proofs.]

K. Kunen, *Set Theory: An Introduction to Independence Proofs*, Elsevier Science Publishers, 1980.

[Together with Burgess's article referenced above, this provides a fine understanding of how forcing is done and why its results are consistent with ZFC.]

K. Kunen and J. Vaughan, eds. *Handbook of Set-Theoretic Topology*, (North-Holland, Amsterdam), 733–760 (1984).

[Still the most comprehensive single source of information about the subject.]

P. Maddy, "Believing the axioms, I" and "Believing the axioms, II", J. Symbolic Logic 53 (1988) 481–511 and 736–764.

[A highly readable pair of articles in which a philosopher looks at CH and at large cardinal axioms, and reasons for believing or disbelieving them.]

J. D. Monk, *Cardinal Invariants on Boolean algebras,* Birkhäuser Verlag, 1996.

[Contains many consistency and independence results.]

P. Nyikos, untitled review, J. Symbolic Logic 57 (1992) 763–766.

[A review of 7 papers authored or co-authored by Andreas Blass, giving applications of forcing to algebra, analysis, and topology]

J. Roitman, "The uses of set theory," *The Mathematical Intelligencer* 14 (1) (1992) 63–69.

[An entertaining and informative article which pointedly leaves out all applications to general topology, Boolean algebra, Whitehead groups, and measure theory, in order to better make the point that set-theoretic consistency results and specialized set-theoretic techniques are are useful in unexpected places in mathematics.]

M. E. Rudin, "Lectures on Set Theoretic Topology," American Mathematical Society, 1975.

[This booklet made it clear how profoundly general topology was being remade by set-theoretic consistency and independent results.]