Fréchet uniform box products

The uniform box product was introduced by Scott Williams in 2001, but very little was done with it until the recent (2010 and 2012) [B1], [H1] dissertations of Jocelyn Bell and Jeffrey Hankins. Their results had to do with two questions that Williams posed a decade earlier: whether the uniform box product of compact spaces is normal, and whether it is paracompact. Hankins answered the latter question in the negative and there are several fragmentary results on the first question due mostly to Bell [B1] [B2].

The purpose of this paper is to explore the general theme of when a countable uniform box product of Fréchet spaces is Fréchet. On finite products, the uniformity is the same as the usual product uniformity, so it is only denumerable products that are of special interest here. The topology on these depends not only on the topology of X but also the uniformity that is used, as Example 1.3 shows.

Definition 0.1. *A* **Fréchet space** *(or: a* **Fréchet-Urysohn space)** *is a space X such that if a point x is in the closure of a subset A, then there is a sequence from A converging to x. A* **countably tight** *space is one such that if a point x is in the closure of a subset A, then there is a countable $B \subset A$ such that x is in the closure of B.*

Clearly, a space is Fréchet iff it is countably tight and every separable subspace is Fréchet.

The conditions for X^ω to be Fréchet in the uniform box topology are quite restrictive, and much is still not known about them. We let X^ω_u stand for X^ω with the uniform box topology. Uniform spaces are taken to be separated [Definition 1.1 below].

1. Basic definitions and results

Definition 1.1. *A* **diagonal uniformity** *on a set X is a filter $\mathfrak{E}$ of relations on X, called* **entourages** *or* **surroundings** *or* **vicinities** *satisfying the following conditions:*

(1) *$\Delta \subset E$ for all $E \in \mathfrak{E}$, where Δ is the diagonal $\{(x,x) : x \in X\}$*
(2) *If $E \in \mathfrak{E}$, then $E^{-1} \in \mathfrak{E}$, where E^{-1} is the inverse of E, that is, $E^{-1} = \{(y,x) : (x,y) \in E\}$;*
(3) *If $E \in \mathfrak{E}$, then there exists $D \in \mathfrak{E}$ such that $D \circ D \subset E$, where*
$$D \circ F = \{(x,z) : \exists y \in X \text{ such that } (x,y) \in F, (y,z) \in D\}$$

A uniform space $(X, \mathfrak{E})$ is **separated** *if $\bigcap \mathfrak{E} = \Delta$.*

As usual, $D(x)$ means $\{y : \langle x,y \rangle \in D\}$. Given any uniform space $\langle X, \mathfrak{E} \rangle$, the associated topological space has the sets $\{E(x) : E \in \mathfrak{E}\}$ as a base for the neighborhoods of x.

The following concept was introduced by Scott Williams at the 2001 Prague Toposym:

Definition 1.2. *Let $\mathfrak{D}$ be a diagonal uniformity on the space X, and let κ be a cardinal number. For each $D \in \mathfrak{D}$ let*

$$\overline{D} = \{\langle x, y\rangle \in X^\kappa \times X^\kappa : \langle x(\alpha), y(\alpha)\rangle \in D \text{ for all } \alpha \in \kappa\}.$$

The uniformity on X^κ whose base is the collection of all $\overline{D}$ is called **the uniform box product**.

In particular, $\{\overline{D}(x) : D \in \mathcal{D}\}$ is a base for the neighborhoods of $x \in X^\kappa$, and $\overline{D}(x) = \{y : y(\alpha) \in D(x(\alpha))$ for all $\alpha < \kappa\}$.

Applying Definition 1.2 to the usual uniformity $\mathfrak{U}$ on $\mathbb{R}$, we have that $\overline{\mathfrak{U}}$ is an extension of $\ell_\infty(\kappa)$ to all of $\mathbb{R}^\kappa$; and $\ell_\infty(\kappa)$ itself is the component of $\overrightarrow{0}$ in $\mathbb{R}^\kappa$ with the uniform box product.

More generally, X_u^κ is metrizable if the uniformity on X has a countable base (and thus X has a metrizable topology). The following example shows how much this depends on the uniformity used.

Example 1.3. Take the simple metrizable space $X = \omega \times (\omega + 1)$ with the uniformity consisting of all partitions into clopen sets. Take the point $\overline{x}$ in the product that satisfies $x(n) = (n, \omega)$ for all n. For each function $f : \omega \to \omega$, take the partition into the parts of each column of all points above the graph of f together with the singletons that are on or below the graph of f. This partition canonically defines a basic open neighborhood of x in X_u^ω.

Thus x has a base of neighborhoods in X_u^ω that is identical with the ones in the usual box product, and if we take the point x_f which is identical with $f : \omega \to \omega$ in the usual ordered pair definition of a function, then the set of all these x_f has x in its closure, but no set of fewer than $\mathfrak{d}$ of them does.

So X_u^ω fails to be countably tight. The following simple result gives another way X_u^ω can fail to be countably tight.

Theorem 1.4. *If X_u^ω is countably tight, then every countable subset of X is first countable.*

Proof. Let N be a countable subspace of X. If N is not first countable, let $x \in N$ have an uncountable neighborhood base $\mathfrak{B}$ in N of minimum cardinality κ. For each $B \in \mathfrak{B}$ let $x_B \in N^\omega$ have range exactly B. Then no set of fewer than κ points x_B has the point $\overrightarrow{x}$ that is constantly x in N^ω in its uniform box closure, but the whole set $\{x_B : B \in \mathfrak{B}\}$ does. $\square$

We now give a more definitive result for which the foregoing proof was a warm-up. It uses the following notation:

Definition 1.5. *Let $(X, \mathcal{D}$ be a uniform space. Then*

$$u(X, \mathcal{D}) = min\ \{|\mathfrak{B}| : \mathfrak{B} \text{ is a base for } \mathcal{D}\}.$$

We will use simply $u(X)$ if the uniformity is clear from the context.

Theorem 1.6. *Let $(X, \mathcal{D}$ be a uniform space and let $t(X)$ stand for the tightness of X as a topolgical space. Then*

$t(X^\omega_u) \geq u(Q, \mathcal{D} \restriction Q)$ for all countable $Q \subset X$.

Proof. Let Q be a countable subset of X. Make $x : \omega \to Q$ have infinite preimage for all $q \in Q$. Let $\{D_\alpha : \alpha < \kappa\}$ be a base for the relative uniformity on Q, of minimal size.

For each $\alpha < \kappa$ and each $i \in \omega$, define $x_\alpha(i)$ in such a way that, for each $q \in Q$, we have:

$$D_\alpha(q) = \{x_\alpha(i) : i \in \omega, \text{ and } x(i) = q\}.$$

In other words, we sprinkle the $x_\alpha(i)$ for each α in such a way that, whenever $x(i) = q$, we put $x_\alpha(i)$ into $D_\alpha(q)$, and we vary the choice from one of the infinitely many such i such that $x(i) = q$ to the next, so as to fill out all of $D_\alpha(q)$.

Claim 1. $x \in c\ell\{x_\alpha : \alpha < \kappa\}$.

Claim 2. If $\Gamma \subset \kappa$, $|\Gamma| < \kappa$, then $x \notin c\ell\{x_\alpha : \alpha \in \Gamma\}$.

Corollary 1.7. *If X^ω_u is Fréchet, then $u(Q) = \omega$ for all countable $Q \subset X$ and hence for all countable subsets of X^ω_u. In particular, every countable subset of X^ω_u is second countable.*

This is a very strong restriction on when X^ω_u is Fréchet. Its second sentence is reminiscent of Corollary 3.4 in Gruenhage's paper [G] where he introduced a topological game between two players, subsequently dubbed "the hero" and "the villain" in [Ny1].

The game utilizes a point j ("the jail") in a space X, and consists of moves indexed by the natural numbers. On each move, the hero picks a neighborhood of p. The villain plays a point within the neighborhood the hero has just chosen. When all ω moves have been played, the hero wins if the points picked by the villain converge to j, otherwise the villain wins.

Gruenhage [G] referred to spaces in which the first player has a winning strategy no matter how j is chosen as "W-spaces," and showed:

Theorem. [G, Corollary 3.4]: *Every W-space is a Fréchet space space in which every countable subset is second countable.*

This necessary condition is not sufficient for a space X to be a W-space, nor for X^ω_u to be Fréchet. The following example shows this.

Example 1.8. If X is the one point compactification of an Aronszajn tree with the interval topology, then every countable subset of X^ω_u is first countable, and X is Fréchet, but X is not a W-space, and X^ω_u is not countably tight.

Note that there is no need to specify the uniformity on X since there is only one uniformity on any compact space. Fred Galvin was the first to show (unpublished) that Example 1.8 is not a W-space. Theorem 1.12 below generalizes this.

The following two examples show that neither one of "X is a W-space" and "X^ω_u is Fréchet" implies the other, not even for compact spaces (where the uniformity is unique).

Example 1.9. *The Alexandroff double arrow space* A is $[0,1] \times \{0,1\}$ with the lexicographical order inducing the interval topology. This makes A into a compact, perfectly normal space whose (unique) uniformity $\mathfrak{U}_A$ has a base $\mathcal{B}$ consisting of partitions of A into clopen intervals.

It is easy to see that $\mathcal{B}$ has cardinality $\mathfrak{c}$ and that, more generally, $u(\mathfrak{U}_A) = \mathfrak{c}$. This is also true of any dense subspace in the relative uniformity, because the completion is (A, U_A) itself. In particular, this is true of the countable subspace $\mathbb{Q} \times \{0,1\}$. It follows from Theorem 1.6 that the tightness of A^ω_u is $\mathfrak{c}$; in particular, A^ω_u is not Fréchet.

On the other hand, A is a W-space, as is any first countable space.

The next example involves the following concepts.

Definition 1.10. Given a limit ordinal α of countable cofinality, a **ladder at** α is a strictly ascending sequence of ordinals less than α whose supremum is α. A **ladder system on** ω_1 is a family $\mathfrak{L} = \{L_\alpha : \alpha \in \gamma \cap \Lambda\}$, where Λ stands for the set of countable limit ordinals, and each L_α is a ladder at α.

The following example was shown by Arhange'skii (who attributed it to Michael Wage) to be a compact space whose space of continuous real-valued functions is Lindelöf in the topology of pointwise convergence [A1]:

Example 1.11. Given a ladder system $\mathfrak{L}$ on ω_1, the space $X_\mathfrak{L}$ has $(\omega_1 \times \{0\}) \cup (\Lambda \times \{1\})$ as underlying set. Points of $\omega_1 \times \{0\}$ are isolated, while a set containing $\langle \alpha, 1 \rangle$ is a neighborhood of $\langle \alpha, 1 \rangle$ iff it contains a cofinite subset of $L_\alpha \times \{0\}$.

Let $X_\mathfrak{L} + 1$ denote the one point compactification of $X_\mathfrak{L}$. Then $(X_\mathfrak{L} + 1)^\omega_u$ is Fréchet, but $X_\mathfrak{L} + 1$ is not a W-space.

The proof that the one-point compactification of $X_{\mathfrak{L}}$ is not a W-space is very similar to the usual proof for that of an Aronszajn tree. In fact, both fall under a rather general theorem.

Theorem 1.12. *Let X be a locally compact space which is the union of a strictly ascending ω_1-sequence of open subspaces $\{X_\alpha : \alpha < \omega_1\}$ for which the following is a stationary subset of ω_1: the set E of all $\alpha \in \omega_1$ such that X_α has nonempty boundary and $\bigcup\{X_\xi : \xi < \alpha\} = X_\alpha$.*

Then if each X_α has a countable dense subset, the one-point compactification of X is not a W-space.

Proof. We may assume without loss of generality that X is Fréchet. Let σ be a strategy for the hero, and let M be a countable elementary submodel of a large enough H_θ, such that $\langle X_\alpha : \alpha < \omega_1\rangle$, σ, and the topology on X are all members of M, and such that $\omega_1 \cap M = \delta \in E$.

The extra point ∞ is the jail that the villain can avoid if the hero's strategy σ is known. Before the game begins, the villain picks a point $p \in \overline{X_\delta} \setminus X_\delta$, and a sequence $\langle x_n : n \in \omega\rangle$ converging to p, such that $x_n \in M$ for all n. This can be done as follows. Let $\alpha_n \nearrow \delta$. Since $X_{\alpha_n} \in M$, there is a countable dense subset D_n of X_{α_n} such that $D_n \in M$. Let $D = \bigcup_{n=0}^{\infty} D_n$. Then p is in the closure of D, and $D \subset M$, so we can use the Fr'echet property to get $\langle x_n : n \in \omega\rangle$ with range a subset of D.

The hero's first move picks out an open neighborhood $\sigma(\emptyset)$ of ∞ whose complement $K(\emptyset)$ is a compact subset of X. Since $\sigma \in M$, $K(\emptyset) \in M$, and so there exists $\beta_1 < \delta$ such that $K(\emptyset) \subset X_{\beta_1}$. Indeed, M "thinks" δ is ω_1, so no compact subset of $M \cap X$ in M can be "cofinal" in X_δ. So the villain some x_{n_1} in this sequence that is in $X_\delta \setminus X_{\beta_1}$.

On his kth move, after the hero has played a neighborhood V which his strategy calls for, the villain picks $n_k \geq k$ such that $x_{n_k} \in X_\delta \setminus X_{\beta_k}$ for some β_k such that $X \setminus V \subset X_{\beta_k}$.

In the end, the villain's chosen points converge to p and so they cannot converge to ∞. $\square$

There are many kinds of spaces satisfying the hypothesis in the first sentence of Theorem 1.12. One kind consists of locally compact spaces in which the set of nonisolated points is uncountable and closed discrete: just let X_0 be the union of the isolated points with all but ω_1 of the nonisolated ones, then add the remaining nonisolated points one at a time. Another example is that of scattered locally compact spaces of Cantor-Bendixson height ω_1, with X_α the complement of the αth Cantor-Bendixson derivative of X. Clearly, if X is separable then all the hypotheses are satisfied in either case. Of course, neither Example 1.8 nor Example 1.11 is separable.

In the case where each X_α is compact metrizable, as in 1.8 and 1.11, all the hypotheses of Theorem 1.12 are satisfied, and the spaces described are what Arhangel'skii called "bambou" spaces in [A2].

2. More examples and some questions about the Fréchet property

Example 2.1. Let $2C$ denote the Alexandroff duplicate of the Cantor set. This is the Cartesian product of the Cantor set C with $\{0,1\}$, with the following topology. Points of $C \times \{1\}$ are isolated, and a base for the neighborhoods of $\langle r, 0\rangle$ is given by sets of the form $V \times \{0,1\} \setminus \{\langle r, 1\rangle\}$ where V is an open interval of C containing $\langle r, 0\rangle$.

As is well known, $2C$ is compact and first countable (and therefore a W-space), but it is not perfectly normal, and so is generally thought of as being “not as nice” as the double arrow space A. However, it does have the property that every countable subspace is metrizable, which A lacks; more strongly, $2C^\omega_u$ is Fréchet; in fact, it is a W-space.

Problem. If X is a W-space such that X^ω_u is Fréchet, must X^ω_u be a W-space?

Gruenhage [G] showed that the denumerable Tychonoff product of W-spaces is a W-space, but we are dealing with a different topology on the product.

The following topology was introduced in [Ny2] and some of its basic properties are shown there.

Definition 2.2. *Let T be a tree. The* **coarse wedge topology** *on T is the one whose subbase consists of all sets of the form $V_t = \{s \in T : s \geq t\}$ and their complements.*

Definition 2.3. *A tree T is* **chain-complete** *if every chain has a supremum. The* **chain-completion** *of a tree T is the tree $\hat{T}$ that adds to T a supremum t_C for each downwards-closed chain C in T that lacks a supremum in T, so that $c < t_C$ in $\hat{T}$ iff $c \in C$, while t_C is below every upper bound (if any exist) of C in T.*

Example 2.4. *If X is a chain-completion of an Aronszajn tree, then X is not metrizable in the coarse wedge topology, but X^ω_u is first countable.*

Trees in general are well behaved in the coarse wedge topology. Our next theorem is about them and involves a game invented by Jocelyn Bell [B3]. It is called the proximal game, and is a game of ω moves played on a uniform space $(X, \mathcal{D})$ between two players, A and B. The game is easiest to describe if the uniformity has a base of equivalence relations; this will always be the case when X is compact and totally disconnected — equivalently, the Stone space of a Boolean algebra — as are all compact examples in this paper. An equivalence relation partitions a uniform space into clopen sets.

In the proximal game, Players A and B alternate, with Player A picking a partition in $\mathcal{D}$ and Player B picking a point of X on each move according to the following rules. On the first move, there are no restrictions. On move $n+1$, Player A must pick a partition that is a refinement of the one he picked on the nth move; Player B must pick x_{n+1} from the member of the nth partition from which x_n was picked. And so, by induction, all later x_m have to be taken from the member of the nth partition where x_n is situated, for all $n \geq 1$.

On the uniform spaces with bases of equivalence relations, Player A wins iff either
(I) there exists $z \in X$ such that $x_1, x_2 \dots$ converges to z or
(II) $\bigcap_{n=1}^{\infty} D_n[x_n] = \emptyset$. Otherwise Player B wins.

The game for general uniform spaces is a bit more complicated and is described in [B3]. We do not need it in this paper.

Theorem. *A tree is proximal in the coarse wedge uniformity if, and only if, it is of height* $\leq \omega_1$.

Proof. One implication is easy: if T is of height $> \omega_1$ then it has a copy of $\omega_1 + 1$ and the following is a winning strategy for Player B that only requires a memory of the last moves played by the two players.

Let W be a branch of T containing a point on level ω_1 and let w be this point. If n is even, Player B lets $x_n = w$; if $n > 1$ is odd, Player B lets x_n be a point strictly below w inside $D_n[x_{n-1}]$ and not just inside $D_{n-1}[x_{n-1}]$. Of course, $D_n[x_{n-1}] = D_n[w]$. Then on the next move, Player B is free to pick w again because $w \in D_n[x_n]$. Then $\langle x_n : n \in \omega \rangle$ has at least two cluster points, w itself and at least one inside the countably compact subspace (in fact, a copy of ω_1) of all points strictly below w.

The proof of the converse is complicated enough to be interesting, yet simple enough to be grasped by anyone with a basic understanding of trees.

3. A Corson compact L-space

In this section we use some of the machinery of the coarse wedge topology to give an example that may be the first consistent example of a Corson compact L-space. They cannot be constructed in ZFC because MA_{ω_1} implies there are no compact L-spaces at all.

Definition 3.1. *A tree is* **complete** *if it is rooted and chain-complete.*

The term is partly motivated by the fact that a tree is a complete semilattice wrt infima iff it is rooted and Dedekind complete.

A corollary of the following theorem is that every complete tree is compact Hausdorff in the coarse wedge topology.

Theorem 3.2. [Ny2, Corollary 3.5] *A tree is compact Hausdorff in the coarse wedge topology iff it is chain-complete and has only finitely many minimal elements.*

Theorem 3.3. *A complete tree is Corson compact in the coarse wedge topology iff every chain is countable.*

Proof. A necessary and sufficient condition for a compact space being Corson compact is that it have a point-countable T_0 separating cover by cozero sets—equivalently, open F_σ-sets [MR]. If the complete tree has an uncountable chain, then it has a copy of $\omega_1 + 1$, does not have such a cover, thanks in part to the Pressing-Down Lemma (Fodor's Lemma).

Conversely, if every chain is countable, then the clopen sets of the form V_t clearly form a T_0-separating, point-countable cover. □

Let us call a tree **uniformly ω-ary** if every nonmaximal point has denumerably many immediate successors. Then we have:

Theorem 3.4. *If there is a Souslin tree, there is a Corson compact L-space.*

Proof. As is well known, every Souslin tree has a subtree T in which every point has more than one successor at every level above it. Thus every point of T has denumerably many successors on the next limit level above it. And so, a uniformly ω-ary Souslin tree results when we take the subtree S of all points on limit levels of T.

Claim. The chain completion $\hat{S}$ of S is an L-space in the coarse wedge topology.

Once the claim is proved, Corson compactness follows from Theorem 3.3.

Proof of Claim. Since initial segments of $\hat{S}$ are closed, $\hat{S}$ is not separable. In the proof that $\hat{S}$ is hereditarily Lindelöf, uniform ω-aity plays a key role: if the tree were finitary, every point on a successor level would be isolated.

We make use of the elementary fact that a space is hereditarily Lindelöf if (and only if) every open subspace is Lindelöf. Let W be an open subspace of $\hat{S}$, and let W_0 be the set of points $t \in W$ such that $V_t \subset W$. Let $A = \{a \in W_0 : a \text{ is minimal in } W_0\}$. Then W_0 is the disjoint union of the clopen wedges V_α $(a \in A)$, and A is countable by the Souslin property.

If $x \in W \setminus W_0$, then there is a basic clopen subset of W based on x, of the form $V_x \setminus (V_{x_1}, \dots, V_{x_n})$ where $n \geq 1$. There are no more than n immediate successors of x below one of the x_i, and if s is one of the other immediate successors of x, then $V_s \subset V_x \setminus (V_{x_1}, \dots, V_{x_n})$, so $s \in W_0$. But then $s \in A$ also, since any V_t containing V_s properly must also contain x, contradicting $x \in W \setminus W_0$. So $W \setminus W_0$ is countable, and we have countably many basic clopen sets whose union is W. □

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