An elementary topological property of Hilbert space, with applications to Erdős space, and some generalizations

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One of the central results of this paper is an elementary yet striking interplay between the norm topology and the natural product topology on Hilbert space, ℓ_2. The natural product topology is associated with the classical definition of Hilbert space as the space of square-summable sequences with the norm $\|\cdot\|_2$ which we will simply write as $\|\cdot\|$. This gives an embedding (in the algebraic, not the topological sense) of ℓ_2 into the product space $\mathbb{R}^\omega$. [For notational convenience, we make the set ω of non-negative integers the domain for all our sequences.] The natural product topology is strictly coarser than the weak topology, but is well known to be equivalent to it on norm-bounded sets.

Since ℓ_2 is paracompact, every locally finite collection of subsets expands to a locally finite collection of open sets [E]. The other topologies are also paracompact, since they are coarser than the norm topology and hence Lindelöf. There are locally finite collections in the norm and weak topologies that are not locally finite in the coarser topologies; however, within each topology one can expand any locally finite collection to a locally finite collection of open sets. Our first theorem extends this fact by expanding any collection $\mathcal{S}$ of sets that is locally finite in the norm topology to a family of sets that is open in the product topology and hence in the weak topology, yet is locally finite in the norm topology. This result is apparently new even where the weak topology is concerned. Yet it is a corollary of an even more general result [Lemma 1] which states that any countable family of subsets of ℓ_2 can be expanded to a family of product-open sets that is norm-locally finite at every point where the original family was norm-locally finite.

The proofs easily extend to any separable Banach space with a Schauder basis, provided that the product topology is defined with respect to the Schauder basis in the natural way. In Sections 3 and 4 we also extend the first theorem to the Hilbert spaces $\ell_2(\Gamma)$ and some other classes of non-separable Banach spaces, via a more complicated proof.

Our applications of the theorem for ℓ_2 involve Erdős space, the subgroup $\mathbb{E}$ of ℓ_2 consisting of those points whose coordinates are all rational. Already back in 1940 [ő], Erdős showed that $\mathbb{E}$ does not have a base of clopen sets in the norm topology, even though any two points can be separated by a clopen set—that is, if $x \neq y$ then there is a clopen subset C of $\mathbb{E}$ with the norm topology such that $x \in C$ and $y \notin C$. Beyond this, nothing seems to have been done until now to improve our understanding of the clopen subsets of $\mathbb{E}$. The results in Section 5 represent a big jump in our understanding, but our knowledge of the clopen subsets of $\mathbb{E}$ is still fragmentary. Section 6 uses that knowledge to produce new TVS topologies on Hilbert space that are between the norm topology and the product topology. With few exceptions, the topologies are not locally convex.

1. The basic theorem and some generalizations.

We will employ the following notation. Given $x = \langle x_n : n \in \omega \rangle \in \ell_2$, we define $spt(x)$, the support of x, to be its set of nonzero coordinates. Given a subset A of ω, we define $x \restriction A$ to be the point in ℓ_2 which agrees with x on A and is 0 elsewhere. We use the von Neumann convention that every natural number equals the set of its predecessors in ω, so that $x \restriction n$ means $x \restriction [0, n) \cap \omega$.

Given a set A, we denote the set of all functions from A to $\mathbb{R}$ as $\mathbb{R}^X$. A base for the product topology on $\mathbb{R}^X$, also known as the topology of pointwise convergence, is the collection of all sets of the form

$$V(f; x_1, \ldots x_n; \delta) = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \delta \text{ for all } i\}$$

with $\delta > 0$, as f ranges over $\mathbb{R}^X$ and $\{x_1, \ldots x_n\}$ over all finite subsets of X. We extend the expression "product topology" any subspace of $\mathbb{R}^X$.

1.1. Definition. Let X be a topological space, let $x \in X$ and let A be an indexing set. A family of sets $\{S_a : a \in A\}$ is *locally finite at* x if x has a neighborhood meeting S_a for only finitely many $a \in A$. A family is *locally finite* if it is locally finite at every point of X. An *expansion* of $\{S_a : a \in A\}$ is a family $\{R_a : a \in A\}$ such that $S_a \subset R_a$ for all $a \in A$.

One feature of Definition 1.1 is that a set cannot be repeated infinitely many times in the indexing of a locally finite family. Some papers require that distinct S_a expand to distinct R_a in the definition of an expansion. In the interests of simplicity, we do not require this here; however, if $\{R_a : a \in A\}$ is locally finite, then each R_a extends at most finitely many members of $\{S_a : a \in A\}$.

1.2. Lemma. *Let $\{S_n : n \in \omega\}$ be a family of subsets of ℓ_2. There is a choice of product-open sets $U_n \supset S_n$ such that if x is a point of ℓ_2 at which $\{S_n : n \in \omega\}$ is locally finite in the norm, then $\{U_n : n \in \omega\}$ is also norm-locally-finite at x.*

Proof. Let $V_n = \{x \in \ell_2 : \exists s \in S_n(\|s - x\| < 1/2^n)\}$. It is easy to see that V_n is norm-open and that $\{V_n : n \in \omega\}$ is norm-locally finite at every point at which $\{S_n : n \in \omega\}$ is locally finite in the norm. Let $P_n = \{p \in V_n : spt(p) \text{ is finite }\}$. For each $p \in P_n$ let $k_n(p) = max\{n, max(spt(p)) + 1\}$. Let U_n be the set of all points in ℓ_2 which agree with some point p of P_n in the first $k_n(p)$ coordinates. Clearly, $V_n \subset U_n$.

U_n is open in the product topology. If $x \in U_n$, let $p \in P_n$ satisfy $p = x \restriction k_n(p)$. Let $m = k_n(p)$. There exists $\varepsilon > 0$ such that the ε-ball centered on p is a subset of V_n. Then $y \in P_n$ whenever $spt(y) \subset m$ and $|p(i) - y(i)| < \frac{\varepsilon}{m}$ for all $i \in m$. Therefore, the basic product-open set $V(p; 0, \dots m-1; \frac{\varepsilon}{m}) \cap \ell_2$ is a subset of U_n containing x.

Finally, suppose that $\{U_n : n \in \omega\}$ is not locally finite at x. Let $z_j \to x$ in the norm, where $z_j \in U_{n(j)}$ for some $n(j) \geq j$. Let $m(j) \geq n(j)$ satisfy $z_j \restriction m(j) \in V_{n(j)}$. Then

$$\|x \restriction m(j) \ - z_j \restriction m(j)\| \ = \ \|(x - z_j) \restriction m(j)\| \ \leq \ \|x - z_j\| \to 0$$

and $x \restriction m(j) \to x$ and so $z_j \restriction m(j) \to x$ all in the norm. But $z_j \restriction m(j) \in V_{n(j)}$, hence $\{V_n : n \in \omega\}$ is not norm-locally finite at x. $\square$

The following is immediate from Lemma 1.2.

1.3. Corollary. *Let $\mathcal{V}$ be a denumerable family of norm-open subsets of ℓ_2. There is an expansion $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$ of $\mathcal{V}$ to a family of product-open sets, such that if x is in the closure of $\bigcup \mathcal{U}$ without being in the closure of any U_V, then x is already in the closure of $\bigcup \mathcal{V}$.* $\square$

1.4. Theorem. *Each locally finite collection of subsets of $\langle \ell_2, \| \cdot \| \rangle$ expands to a family of product-open sets that is locally finite in $\langle \ell_2, \| \cdot \| \rangle$.*

Proof. Since ℓ_2 is separable, every locally finite collection of subsets is countable. Now apply Lemma 1. $\square$

With one tiny change in the proof, we can extend Lemma 1 to all separable Banach spaces with Schauder bases, and then both Corollary 1 and Theorem 1 then extend immediately. To define the product topology in this setting, one uses the natural isomorphism between a Banach space E with Schauder basis $\{e_n : n \in \omega\}$ and the set X of all sequences s from ω to $\mathbb{R}$ for which there is $y_s \in E$ such that $s(n)$ is the coordinate of y_s with respect to e_n. The norm is simply transferred from E to X and the product topology is even coarser than the weak topology on X: the coordinate map taking each s to $s(n)$ is obviously both linear and continuous.

Of course, X and the isomorphism depend very much on the basis chosen. But whatever the basis, we can extend the proof of Lemma 1 by multiplying $\|x - z_j\|$ by a constant factor L in the displayed formula. This is because every Schauder basis can be associated with a constant L such that for every vector x and every positive integer n we have $\|x \restriction n\| \leq L\|x\|$ [S,] [Of course, we can take $L = 1$ in the case of ℓ_p spaces.] And so we have:

1.5 Corollary. *Let E be a Banach space with a denumerable Schauder basis B and let $x \in E$. Each countable family of subsets of E which is locally finite at x with respect to the norm has an expansion to a family of sets which are open in the relative product topology associated with B and norm-locally-finite at x.* □

Without a Schauder basis there is usually no natural definition for a product topology, but we can use the fact that every separable Banach space isometrically and isomorphically embeds into $C[0,1]$ with the supremum norm [M] to show that every locally finite collection of sets in a separable Banach space (or even just a separable normed vector space) expands to a family of weakly open sets that is locally finite in the norm. Lemma 1.2 also extends:

1.6. Theorem. *Let X be a separable normed vector space and let $\{S_n : n \in \omega\}$ be a (countable) family of subsets of X. There is a choice of weakly open sets $U_n \supset S_n$ such that if x is a point of X at which $\{S_n : n \in \omega\}$ is locally finite in the norm, then $\{U_n : n \in \omega\}$ is also norm-locally-finite at x.*

Proof. Let X be identified with a subspace of $C[0,1]$, via the embedding of X in its completion and [M,]. Let $\mathcal{T}$ be the product topology that is defined on $C[0,1]$ with respect to some Schauder basis and let $\mathcal{T} \restriction X$ be the relative topology on X. Extend Lemma 1 to produce $\mathcal{T}$-open sets U'_n that are locally finite (in the norm) at any point of $C[0,1]$ at which the family of S_n's is locally finite. Each U'_n is weakly open in $C[0,1]$, so $U_n = X \cap U'_n$ is weakly open in X. □

2. Other product topologies

Of course, the product topology used in the foregoing proof has very little to do with the product topology most naturally associated with $C[0,1]$, which is not even metrizable: the topology of pointwise convergence. It is therefore perhaps surprising that there is a similar proof of a similar theorem with respect to this product topology as well. The proof uses much the same ideas of squeezing the points of the norm-open sets progressively more strongly as n increases.

2.1. Theorem. *Let $C_p[0,1]$ denote $C[0,1]$ with the topology of pointwise convergence. Let $\{S_n : n \in \omega\}$ be a (countable) family of subsets of $C_p[0,1]$. There is a choice of C_p-open sets $U_n \supset S_n$ such that if $\{S_n : n \in \omega\}$ is locally finite in the norm at $g \in C[0,1]$, then $\{U_n : n \in \omega\}$ is also locally finite at g in the norm.*

Proof. Let $\mathbb{Q} \cap [0,1] = \{q_n : n \in \omega\}$, with $q_0 = 0$, $q_1 = 1$. Each $f \in C[0,1]$ is uniformly continuous, so that if $f \in V_n$, there exists $m = m(f,n) \geq n$ such that

$|f(r) - f(q_i) < 1/2^n$ for all r between q_i and the adjacent member(s) of $\{q_0, \ldots, q_m\}$. Let

$$U_n = \{h \in C[0,1] : \exists f \in S_n \text{ such that } |f(q_i) - h(q_i)| < \frac{1}{2^n} \text{ for all } i \leq m(f,n)\}$$

$$= \bigcup \{V(f; q_1, \ldots q_{m(f,n)}; \frac{1}{2^n}) : f \in S_n\}$$

Now let g have points from infinitely many U_n in every norm neighborhood. For each $j \in \omega$ let $k(j) \geq j$ be so large that $|g(r) - g(q_i)| < 1/2^j$ for all r between q_i and the adjacent member(s) of $\{q_0, \ldots, q_{k(j)}\}$. Choose $n(j) \geq k(j)$ so that there is $h_j \in U_{n(j)}$ such that $\|g - h\|_\infty < 1/2^j$. Let $f_j \in V_{n(j)}$ satisfy

$$|f_j(q_i) - h_j(q_i)| < \frac{1}{2^{n_j)}} \text{ for all } i \leq m\}.$$

where $m = m(f_j, n(j))$. Since $j \leq n(j) \leq m$ we have $|g(q_i) - f_j(q_i)| < 1/(2^{j-1})$ for all $i \leq n(j)$. Now if $r \in [0,1]$ let $r \in [q_i, q_j)$ where q_i and q_j are adjacent members of $\{q_0, \ldots q_m$; then

$$|g(r) - f_j(r)| \;\leq\; |g(r) - g(q_i)| + |g(q_i) - f_j(q_i) + |f_j(q_i) - f_j(r)| \leq \frac{1}{2^{j-2}}.$$

Hence $\|g - f_j\| \leq 1/(2^{j-2})$ and $f_j \to g$ as $j \to \infty$. $\square$

The proof of Theorem 1.6 established something a little stronger than its statement: it showed that there is a single metrizable linear topology on X that is coarser than the weak topology, from which the sets U_n can always be taken. While the product topology in Theorem 2.1 is not metrizable, we really only used the basic product neighborhoods associated with rational numbers. Indeed, there is a natural algebraic embedding of $C[0,1]$ into the metrizable space $\mathbb{R}^{\mathbb{Q} \cap [0,1]}$ due to the fact that every continuous real-valued function on $[0,1]$ is determined by its values on $\mathbb{Q}$. Hence our proof of Theorem 2.1 gives an alternative way of proving the strengthened version of Theorem 1.6.

In some spaces, there is a natural product topology to which Theorem 1.4 does not extend. For example, $C[0,1]$ has a natural product topology in which every nonempty open set is dense in the L_p norm $[1 \leq p < \infty]$. Thus it is impossible to even have an infinite collection of product-open sets that is locally finite in the L_p norm. Of course, $\langle L_p[0,1], \|\cdot\|_p\rangle$ does have a Schauder basis and it is possible to define a product topology with respect to that.

Theorem 1.4 also does not extend to many non-separable spaces, not even if they are complete. The space ℓ_∞ is an example. Even if one takes advantage of the fact that ℓ_∞ is essentially $\mathrm{C}(\beta\omega)$ with the uniform metric, thereby gaining $2^{\mathfrak{c}}$ new basic product open sets, it still cannot be done. In fact, as will be shown in Section 4, ℓ_∞ even has a countable closed discrete subspace D which cannot be expanded to a point-finite collection of weakly open sets, even though the weak topology on ℓ_∞ is strictly finer than the product topology. Moreover, D is a subset of c_0, where it can be expanded to a countable locally finite family of product-open sets.

3. Extensions to some non-separable spaces

In contrast to the case of ℓ_∞, there are some nonseparable Banach spaces to which Theorem 1.4 does extend. This includes $\ell_p(\Gamma)$ for every set Γ and every p such that $1 \leq p < \infty$.

3.1. Theorem. *Let $\{V_a : a \in A\}$ be a family of subsets of $\ell_p(\Gamma)$ that is locally finite with respect to the norm. There is an expansion to a family $\{U_a : a \in A\}$ of product-open sets that is locally finite with respect to the norm.*

Proof. Since $\ell_p(\Gamma)$ is metrizable, it is paracompact, and so every locally finite collection of subsets expands to a locally finite collection of open sets. So we may assume without loss of generality that each V_a is open. For each $a \in A$ let $P_a = \{p \in V_a : spt(p) \text{ is finite}\}$. Let U_a be the set of all points which agree with some point of P_a on its support. Since each point x of $\ell_p(\Gamma)$ can be approximated arbitrarily closely by $x \restriction F$ for some finite subset F of A, V_a is a subset of U_a, and the proof that U_a is open is like the proof that U_n is open in Lemma 1.2, with $spt(p)$ replacing $k_n(p)$.

To see that $\{U_a : a \in A\}$ is locally finite, suppose on the contrary that every neighborhood of x meets U_a for infinitely many a. Pick distinct $a_n \in A$ and $z_n \in U_{a_n}$ such that $z_n \to x$. Pick $z_n^* \in F_{a_n}$ agreeing with z_n on $spt(z_n^*)$. Let $S_n = spt(z_n^*)$ and let $\{\gamma_n : n \in \omega\} = spt(x) \cup \bigcup_{n\in\omega} S_n$. Our goal is to define a Cauchy subsequence of $\langle z_n^* : n \in \omega\rangle$, contradicting local finiteness of $\{V_a : a \in A\}$.

For each positive integer m, define $A_m \subset \{\gamma_i : i < m\}$ by induction so that $S_n \cap \{\gamma_i : i < m\} = A_m$ for infinitely many n, and so that $A_n \cap \{\gamma_i : i < m\} = A_m$ whenever $n \geq m$. Fix $\delta > 0$. Pick m so that $\|x - x \restriction \{\gamma_i : i < m\}\| < \delta$ and $\|x - z_n\| < \delta$ for all $n \geq m$.

Claim. If $M, N \geq m$ and $S_M \cap \{\gamma_i : i < m\} = S_N \cap \{\gamma_i : i < m\} = A_m$, then $\|z_M^* - z_N^*\| < 6\delta$.

Assuming the claim, define $k_m \geq m$ for all $m \geq 1$ so that $S_{k_m} \cap \{\gamma_i : i < m\} = A_m$. Then $\langle z^*_{k_m} : m \geq 1 \rangle$ is the desired Cauchy subsequence.

Proof of Claim. We have

$$(*) \qquad \|x \restriction A_m - z_N \restriction A_m\| < \delta \qquad \text{and}$$

$$(**) \qquad \|x \restriction (S_N \setminus A_m) - z_N \restriction (S_N \setminus A_m)\| < \delta$$

because $N \geq m$ implies $\|x - z_N\| < \delta$.

Moreover, $\|x \restriction \{\gamma_i : i \geq m\}\| < \delta$ and so

$$\|x \restriction (S_N \setminus A_m)\| = \|x \restriction (S_N \cap \{\gamma_i : i \geq m\})\| < \delta$$

also. It follows from $(**)$ that $\|z_N \restriction (S_N \setminus A_m)\| < 2\delta$.

We have the same facts with M in place of N. So we have

$$\|z_N \restriction A_m - z_M \restriction A_m\| < 2\delta$$

and

$$\|z_N \restriction (S_N \setminus A_m) - z_M \restriction (S_M \setminus A_m)\| < 4\delta.$$

But $z^*_N = (z_N \restriction A_m) + (z_N \restriction [S_N \setminus A_{k+1}])$, and similarly for z^*_M. So these last two displayed formulas give $\|z^*_M - z^*_N\| < 6\delta$, as desired. □

The foregoing proof works verbatim for $c_0(\Gamma)$ with the supremum norm. There is also a natural generalization of an unconditional basis to arbitrary index sets Γ which gives a further generalization of Theorem 3.1. The generalization is to a linearly independent set $\{e_\gamma : \gamma \in \Gamma\}$ such that for each vector x there is a family of vectors $\{x_\gamma = r_\gamma e_g : \gamma \in G\}$ is *summable to* x.

This is easily seen to be equivalent to having $r_\gamma = 0$ for all but countably many γ and having $\sum_n x_{\gamma_n} = x$ no matter how one lists the nonzero coordinates as $\gamma_0, \gamma_1, \ldots$. Calling such a Γ an unconditional basis even if it is uncountable, and defining the product topology with respect to it in the natural way, we arrive at:

3.2. Theorem. *Let X be a Banach space with an unconditional basis. Every locally finite collection of subsets of X can be expanded to a locally finite collection of product-open sets.*

Problem 1. *Can Theorem 5 be extended to all Banach spaces in which the norm locally depends on finitely many coordinates?*

4. Examples and counterexamples of the form $C(K)$.

We will now prove the result about ℓ_∞ promised at the end of Section 2. Let $Z = \{\chi_a : a \text{ is a finite subset of } \omega\}$. Then Z is a countable closed discrete subspace of $(c_0, \|\cdot\|_\infty)$ and hence of ℓ_∞. Then by Theorem 1.6 it can be expanded to a locally finite collection of sets open in the weak topology of c_0. [Indeed, the sets can be open in the natural product topology of c_0 as indicated after Theorem 1.4.] However, this is no longer possible in the larger space ℓ_∞. This will follow from some general theorems applied to the following example.

Example. Let T be the full ω-ary tree of height $\omega+1$. That is, elements of T are finite sequences (with domain some $n \in \omega$) and ω-sequences of natural numbers, ordered by end extension. Let S stand for the set of finite sequences. Let T be given the interval topology. This has as a base all $\{s\} \subset S$ and all sets of the form

$$(s,t] = \{x \in T : s < x \leq t\}.$$

With this topology, T is locally compact and Hausdorff, and S is its set of isolated points. Let K be the one-point compactification of T. In $C(K)$, let

$$D = \{\chi_{\hat{s}} : s \in S\}, \text{ where } \hat{s} = \{x \in S : x \leq s\}.$$

Claim. If $\mathcal{V} = \{V_s : s \in S\}$ is a family of product-open sets in $C(K)$ such that $\chi_{\hat{s}} \in V_s$, then there is a clopen subset A of K such that $\chi_A \in V_s$ for infinitely many $s \in S$.

Proof of claim: It is enough to show the claim in the case where V_s is of the form $V(\chi_{\hat{s}}; b_1, \ldots b_n; \delta)$. Let $b_s \subset K$ be the set of coordinates on which V_s is restricted. The set A is built by induction. Let $s_0 \in S$. If s_n has been defined, let s_{n+1} be an immediate successor of s_n in T which does not have any points of b_{s_n} above it. When the induction is finished, let t be the supremum of the s_n in T and let

$$A \;=\; \hat{t} \;=\; \hat{s_0} \cup \{s_n : n \in \omega\} \cup \{t\}.$$

Then A is clearly clopen in K, and $\chi_A \in V_{s_n}$ for all n. Indeed, χ_A agrees with each $\chi_{\hat{s_n}}$ on the domain of the latter, and no point of b_{s_n} outside $\hat{s_n}$ is in A, so $\chi_A \in V_{s_n}$. $\square$

A similar claim holds for $C(\beta\omega)$, which is naturally isomorphic to ℓ_∞, and the set Z defined before this example can be regarded as being a subset of either space. Let $\phi : \omega \to K$ be a 1-1 function with range S. Using the universal property of the Stone-Čech compactification, ϕ extends uniquely to a continuous function $\phi^\beta : \beta\omega \to K$. Since ϕ^β is surjective, the induced map $\psi : C(K) \to C(\beta\omega)$ (defined by $\psi(f) = f \circ \phi^\beta$) is an isometry and an isomorphism with its range. Thus if W_s is open in the weak topology of $C(\beta\omega)$ and contains $\psi(\chi_{\hat{s}})$, the preimage of W_s under ψ is open in the weak topology of $C(K)$ and contains $\chi_{\hat{s}}$. But since K is scattered, the weak topology of $C(K)$ coincides with the product topology [R]. *Letting $V_s = \psi^{-1}(W_s)$, we let A be as in the claim above, and then $\psi(\chi_A)$ is in W_s for infinitely many $s \in S$.*

5. The clopen subsets of Erdős space.

A corollary of Erdős's results mentioned in the introduction is that the topology τ on $\mathbb{E}$ whose base is the set of all clopen subsets of $\langle \mathbb{E}, \|\cdot\| \rangle$ is a strictly coarser Tychonoff topology. We will now prove that the two topologies have the same countable closed subsets, and derive some interesting consequences.

5.1. Lemma. *Let D be a countable norm-closed subset of $\mathbb{E}$, not containing $\overrightarrow{0}$. There is a product-open subset of ℓ^2 containing D whose trace on $\mathbb{E}$ is a norm-clopen set missing $\overrightarrow{0}$.*

Proof. Let $D = \{d_n : n \in \omega\}$. In Lemma 1.2, let $S_n = \{d_n\}$ and follow the proof, choosing V_n so that its closure misses $\overrightarrow{0}$. For each n let $W_n \subset U_n$ be a basic product-open set containing d_n whose trace on $\mathbb{E}$ is clopen. This can simply be arranged by having the coordinates in which W_n is restricted use intervals with irrational endpoints for doing the restricting. Now by Lemma 1.2, the only points of ℓ_2 which have each neighborhood meeting infinitely many W_n are in the norm-closure of D, hence are either in D or else have at least one irrational coordinate. So $\bigcup\{W_n : n \in \omega\}$ traces a clopen set on $\mathbb{E}$ containing D and missing $\overrightarrow{0}$. $\square$

5.2. Theorem. *A countable subset of $\mathbb{E}$ is τ-closed iff it is norm-closed.*

Proof. By translation-invariance, Lemma 5.1 implies every norm-closed countable subset of $\mathbb{E}$ is an intersection of norm-clopen sets, and is therefore τ-closed. $\square$

If one follows the proof of Lemma 1.2, one may wind up shrinking the sets W_n more than necessary. For example, if D is closed discrete in ℓ_2, then one can let each V_n be of diameter r times the distace from d_n to its nearest neighbor(s) where r is any positive number $< 1/2$. Then any basic open set of the following form can be chosen for W_n. Let A be a finite subset of ω such that there are irrational numbers $p_n(i)$ and $q_n(i)$ satisfying $p_n(i) < d_n(i) < q_n(i)$ for each $i \in A$, and such that the following set is a subset of V_n:

$$Y(p_n, q_n) = \{y \in L : spt(y) \subset A \text{ and } p_n(i) < y(i) < q_n(i) \text{ for all } i \in A.\}$$

Let $W_n = \{z \in \ell_2 : z \restriction A \in Y(p_n, q_n)\}$.

Alan Dow observed the following corollary of Theorem 5.2.

Theorem 5.3.. *Every τ-convergent sequence in $\mathbb{E}$ is norm-convergent. Thus $\langle \mathbb{E}, \tau \rangle$ is not a sequential space.*

Proof. Suppose there were a τ-convergent sequence σ that is not norm-convergent. Since τ is coarser than the norm topology, this would imply that the range of some one-to-one subsequence of σ is closed discrete in the norm topology. But then by Theorem 5.2, this range is also τ-closed-discrete, contradicting τ-convergence to $\overrightarrow{0}$. $\square$

Theorem 5.2 leads even more directly to the negation of a generalization of sequentiality, introduced by Moore and Mrówka in [MM].

5.4. Definition. A topology is *determined by countable closed sets* [resp. *countably tight*] if a set A is closed if (and only if) $c\ell(B)$ is a subset of A whenever $B \subset A$ and $c\ell(B)$ is countable [*resp.* and B is countable].

In what came to be called "the Moore-Mrówka problem," they asked whether every compact Hausdorff countably tight space is determined by countable closed sets. This has been shown to be independent of the usual (ZFC) axioms of set theory [BDFN] [B]. They also remarked that the problem was open for arbitrary Hausdorff spaces. A ZFC counterexample was provided by I. Juhász and Weiss [JW] [N]. Just from the name, one might infer that $\langle \mathbb{E}, \tau \rangle$ is another counterexample, inasmuch as it has the same countable closed sets as the norm topology does. This is indeed the case: one can let A be any set which is norm-closed but not τ-closed, and any subset of A with countable τ-closure has the same norm-closure which is thus a subset of A.

It is interesting to compare and contrast the counterexample in [JW] with this one. The one in [JW] is constructed by transfinite induction and is neither hereditarily separable nor hereditarily Lindelöf. $\langle \mathbb{E}, \tau \rangle$ has both properties and is defined in an elementary way using only ZF; only the countable axiom of choice is needed to verify that τ is not determined by countable closed sets. On the other hand, the Juhász-Weiss space is pseudo-radial [JW] while $\langle \mathbb{E}, \tau \rangle$ is not.

5.5. Definition. A space is *pseudo-radial* if its topology is determined by well-ordered nets. In other words, if a set A is not closed, there is a point x outside A and a well-ordered net in A converging to x.

5.6. Theorem. *The space $\langle \mathbb{E}, \tau \rangle$ is hereditarily separable (hence countably tight) and hereditarily Lindelöf, but not pseudo-radial.*

Proof. Erdős space is separable metrizable and hence both hereditarily separable and hereditarily Lindelöf. Since τ is a coarser topology, it has both of the latter properties.

To show that $\langle \mathbb{E}, \tau \rangle$ is not pseudo-radial, we use the fact that every point is a G_δ. So, if ξ is a net of uncountable cofinality that is not eventually constant and p is any point of $\mathbb{E}$, then p has a neighborhood missing a cofinal subnet of ξ. Thus the only convergent well-ordered subnets that are not eventually constant have cofinal convergent subsequences. But Theorem 5.3 shows that these are not enough to determine the topology. For instance, the complement of the open unit ball in $\mathbb{E}$ is norm-closed and hence sequentially closed (but is also dense!) in $\langle \mathbb{E}, \tau \rangle$. $\square$

The following generalization of pseudo-radiality has been studied in connection with a famous unsolved problem of general topology, the M_1-M_3 problem [MSK]:

5.7. Definition. A space X is said to be WAP if for every non-closed subset A there is a point $x \in c\ell(A) \setminus A$ and a subset B of A such that $x \in c\ell(B)$ and x is the only point of $c\ell(B)$ that is not also in A.

Problem 3. *Is $\langle \mathbb{E}, \tau \rangle$ a WAP space?*

But the most basic unsolved problem about this space is the following.

Problem 4. *Is $\langle \mathbb{E}, \tau \rangle$ a topological group (equivalently, a topological vector space over $\mathbb{Q}$)?*

Because translation and scalar multiplication are separately continuous in ℓ^2, $\langle \mathbb{E}, + \rangle$ is a semitopological group with continuous inverse in the topology τ. So Problem 4 boils down to asking whether addition is jointly τ-continuous. This is a problem even where the clopen sets of Theorem 6 are concerned:

Problem 5. *Let $\mathbb{E}$ be Erdős space and let C be a clopen nbhd of $\overrightarrow{0}$ in $\mathbb{E}$ defined by $C = \mathbb{E} \setminus \bigcup\{W_n : n \in \omega\}$, where W_n is as in the proof of Lemma 2. Is there a clopen nbhd of $\overrightarrow{0}$ whose sum with itself is a subset of C?*

If D is finite there is a simple solution: follow the remark preceding Theorem 5.3 and let K be the set of all points of $\mathbb{E}$ that are between $\frac{1}{2}p_n$ and $\frac{1}{2}q_n$ for some n, in every coordinate where $\frac{1}{2}p_n$ (equivalently, $\frac{1}{2}q_n$) is nonzero. But this idea is not feasible for infinite D, not even in the following example where the members of D are all a distance of 1 from their nearest neighbors.

5.8. Example. Let $D = \{d_n : n \in \omega\}$ be the following closed discrete subspace of ℓ^2: d_n is the point which is $1/2^n$ in the first 2^{2n} terms, and 0 in all other terms. The points of D are on the unit sphere of ℓ_2, any pair of successive points is one unit apart, and other pairs are even further from each other. In defining W_n as in the remark following the proof of Theorem 4, we can let p_n be the point all of whose nonzero coordinates are $1/(2^n\pi)$ below the nonzero coordinates of d_n. Given d_n, one might try replacing W_n with the set G_n of all points which are of absolute value greater than $\frac{1}{2}p_n(i)$ in at least one coordinate $i \in spt(p_n)$. This is a set that is open in the product topology and whose trace on $\mathbb{E}$ is clopen in $\mathbb{E}$. Also, $H_n = \mathbb{E} \setminus G_n$ is symmetrical with respect to the origin, and $H_n + H_n \subset \mathbb{E} \setminus G_n$. It is easy to see that the set of G_n's does not have clopen union. The following point x is in the closure of the G_n without being in any G_n: let $x(k) = 1/2^{n+2}$ when $k = 2^{2n}$ and $x(j) = 0$ whenever j is not a number of this form. If we let z_n be the point of G_n which satisfies $z_n(2^{2n}) = 1/2^n$ and agrees with x elsewhere, it is easy to see that the sequence of z_n's converges to x in norm.

However, the following set K is clopen and does solve the problem in the affirmative where this particular choice of W_n is concerned.

5.9. Example. Let $f : \omega \to \mathbb{R}$ be defined as follows. $f(0) = 1/4\pi$, $f(i) = 1/8\pi$ for $i \in \{1, 2, 3\}$, and in general $f(n) = 1/(2^{n+2}\pi)$ for $i \in [2^{n-1}, 2^n - 1] \cap \omega$. Let K be the set of all sequences in $\mathbb{E}$ such that the absolute value of the kth coordinate is greater than $f(k)$ in less than half of the coordinates k in the interval $[2^{n-1}, 2^n - 1]$, no matter what n is.

Now if x and y are in K, then for each n there exists $k \in [2^{n-1}, 2^n - 1]$ such that $x(k)$ and $y(k)$ are both less than $f(k)$, which in turn is less than half of $1/2^n - 1/(2^n\pi)$. Thus if W_n is as in Example 1, then $x + y \notin W_n$ for any n. Therefore $K + K$ is contained in the set $C = \mathbb{E} \setminus \bigcup\{W_n : n \in \omega\}$. Moreover, K is clopen. The proof of this is very similar to that of the following lemma, which extends the notation $x \restriction A$ to any function from ω to $\mathbb{R}$ in the obvious way for all finite A: $f \restriction A$ is the element of ℓ^2 which agrees with $f \restriction A$ on A and is zero elsewhere.

5.10. Lemma. *Let f be any positive real sequence which is not in ℓ^2; in other words, the sequence of ℓ^2 norms $\|f \restriction [0, n]\|$ increases without bound. For $x \in \ell_2$ and any interval $I = [m, n] \subset \omega$, let*

$$A_I(x) = \{k : k \in I \textit{ and } |x(k)| \geq f(k)\}. \tag{1}$$

For positive real numbers $r \le 1$ and $\epsilon \le 1$, let $G(f,r,\epsilon)$ be the set of all sequences σ in ℓ_2 such that, for each I satisfying $\|f \restriction I\| \ge r$, we have $\|f \restriction A_I(\sigma)\| < \epsilon\|f \restriction I\|$. If $f(i)$ is irrational for all $i \in \omega$, then $K = \mathbb{E} \cap G(f,r,\epsilon)$ is clopen in $\mathbb{E}$.

Proof. If $p \notin K$, then for some I satisfying $\|f \restriction I\| \ge r$, we have $\|f \restriction A_I(p)\| \ge \epsilon\|f \restriction I\|$. Then clearly the set of all points x such that $|p(i) - x(i)| < |p(i)| - f(i)$ for all $i \in I$ is a product-open set in ℓ^2 that contains p and misses K. Hence K is closed even in the product topology of $\mathbb{E}$.

To show $G = G(f,r,\epsilon)$ is norm-open, let $y \in G$. Let $j \in \omega$ be so large that $\|y \restriction [j,\infty)\| < r\epsilon$ and $\|f \restriction [0,j-1]\| > r$. Let $k \ge j$ be such that $\|f \restriction [j,k]\| > r$.

Let $\delta = min\{\varepsilon, \nu\}$ where $\varepsilon = min\{f(i) - |y(i)| : i \in [0,k] \setminus A_{[0,k]}(y)\}$ and $\nu = r\epsilon - \|y \restriction [j,\infty)\|$. Then $B(y,\delta) = \{x \in \mathbb{E} : \|y - x\| < \delta\}$ is an open subset of G containing y. Indeed, let $I = [m,n]$ and suppose first that $n \le k$. In this case, $A_I(x) \subset A_I(y)$ and so $\|f \restriction A_I(x)\| < \epsilon\|f \restriction I\|$ for all $x \in B(y,\delta)$. So now suppose $n > k$. Then by the first case, $\|f \restriction A_{[m,j-1]}(x)\| < \epsilon\|f \restriction [m,j-1]\|$, while

$$\|x \restriction [j,\infty)\| \;\le\; \|y \restriction [j,\infty)\| + \|x - y \restriction [j,\infty)\| \;<\; r\epsilon, \text{and}$$

$$\|f \restriction A_{[j,n]}(x)\| \;<\; r\epsilon \;<\; \epsilon\|f \restriction [j,n]\|.$$

Now $\|f \restriction A_{[m,n]}(x)\| = \sqrt{\|f \restriction A_{[m,j-1]}(x)\|^2 + \|f \restriction A_{[j,n]}(x)\|^2}$
$< \epsilon\sqrt{\|f \restriction [m,j-1]\|^2 + \|f \restriction [j,n]\|^2} = \epsilon\|f \restriction [m,n].$ $\square$