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d-14 Countable Paracompactness, Countable Metacompactness, and Related Concepts

A space is **countably paracompact** (respectively **countably metacompact**) if every countable open cover has a locally finite (respectively, point-finite) open refinement. Despite the superficial similarity in both their names and in some of their respective equivalents, these classes of spaces are very different as far as current-day interests of topologists are concerned. Countably paracompactness generally goes hand in hand with normality – so much so that spaces that are ***normal*** but not countably paracompact are singled out by the term "Dowker spaces" while spaces that are countably paracompact but not normal are widely termed **anti-Dowker** spaces. On the other hand, a quarter of a century ago, so few ***regular*** spaces were known *not* to be countably metacompact that Brian M. Scott [10] referred to the few then-known examples as "almost Dowker spaces". Since then, a considerable variety of regular spaces have been found not to be countably metacompact, including some manifolds [7], even smooth ones such as tangent bundles obtained from smoothings of the long line [9]. Nevertheless, while the term "almost Dowker" would be an overstatement, these spaces are still encountered rather infrequently.

There are many concepts with definitions similar to that of these two, such as that of countable subparacompactness, and that of cb-spaces and weak cb-spaces, but these have attracted relatively little attention and their properties are not so well understood. Since ***subparacompactness*** is treated at some length elsewhere in this encyclopedia, it may be worth pointing out that the various alternative definitions of that concept carry over to **countable subparacompactness**. Thus it makes no difference whether one says "every countable open cover has a $\{\sigma$-locally finite, σ-discrete, σ-closure-preserving, σ-cushioned$\}$ closed refinement", and the only novelty is that it is also equivalent to having a countable closed refinement. Proofs of these equivalences can be found in [3], which is also the seminal paper on the subject of countable metacompactness. It also shows that what were called "countably θ-refinable spaces" are actually the same as countably metacompact spaces, and hence that every countably subparacompact space is countably metacompact. An awkward feature of countable subparacompactness is that it is not implied by countable compactness: the product space $(\omega_1 + 1) \times \omega_1$ is not countably subparacompact.

One can require, in the definitions of countable paracompactness and countable metacompactness, that the refinements be countable as well, and obtain an equivalent condition in either case. The really useful equivalent conditions, however, are the ones that begin with a countable descending sequence of closed sets F_n whose intersection is empty, and expand each F_n to an open set U_n, requiring $\bigcap_{n=0}^{\infty} U_n = \emptyset$ in the case of countable metacompactness and $\bigcap_{n=0}^{\infty} \overline{U_n} = \emptyset$ in the case of countable paracompactness. This makes it obvious that the two properties coincide for normal spaces. Thus one can define a **Dowker space** either as a normal space which is not countably paracompact, or as one that is not countably metacompact. Of course, the property of Dowker spaces that makes them so popular is that they are the normal spaces whose product with $[0,1]$ fails to be normal. Brian Scott [10] found a similar product theorem for ***orthocompactness***: an orthocompact space has orthocompact product with $[0,1]$ iff it is countably metacompact. Countable metacompactness figures in another interesting equivalence due to Norman Howes [4]: a regular linearly Lindelöf space is Lindelöf iff it is countably metacompact. [Recall that a space is called **linearly Lindelöf** if every ascending open cover has a countable subcover.]

Countable paracompactness has many affinities with normality, including a curious set of parallels involving the axioms $\mathrm{V} = \mathrm{L}$ and PMEA. If $\mathrm{V} = \mathrm{L}$, then every normal space and also every countably paracompact space of character $\leqslant \aleph_1$ is ***collectionwise Hausdorff*** [2, 11]. If PMEA, then every normal space of character $< 2^{\aleph_0}$ is ***collectionwise normal***, while every countably paracompact space of ***character*** $< 2^{\aleph_0}$ is collectionwise Hausdorff and expandable [6, 1]. Both of these axioms imply that a subspace of ω_1^2 is normal iff it is countably paracompact [5]. On the other hand, although it is a theorem of ZFC that every normal subspace of ω_1^2 is countably paracompact, the reverse implication is an open problem with interesting set-theoretic equivalents. There are also $\mathrm{V} = \mathrm{L}$ and PMEA theorems for countable metacompactness: $\mathrm{V} = \mathrm{L}$ implies closed discrete subsets of locally countable T_1 countably metacompact spaces are G_δ [8], while PMEA implies closed discrete subsets of T_1 countably metacompact spaces of character $< 2^{\aleph_0}$ are G_δ if every one of their points is a G_δ [1].

Morita P-spaces are an important special class of countably metacompact spaces. These spaces are often classed as "generalized metric spaces" because the normal ones are precisely those normal spaces whose product with every metric space is normal. But they can also be looked at as an interesting example in the theory of topological games. They can be defined as those topological spaces in which the second player has a winning strategy in what might be called the countable metacompactness game. This is a topological game with infinitely many moves indexed by the natural numbers, in which two players take turns playing closed

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sets and open sets in a topological space, and each one has knowledge of past plays but not of future ones. On the nth turn, Player 1 plays a closed set F_n which is a subset of the previously chosen sets F_k, and then Player 2 plays an open set G_n containing F_n. At the end of the game, Player 1 wins iff the set of all F_n have empty intersection but the set of all G_n has nonempty intersection. If either condition fails to obtain, then Player 2 wins.

Morita's original definition was more technical but also lends itself more readily to modification. Given any cardinal number κ and any choice of open sets $G(\alpha_1, \ldots, \alpha_n) \subset X$ for each finite sequence of $\alpha_i \in \kappa$, there are closed sets $F(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ such that for each infinite sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of κ, either

(a) $\bigcup_{n=1}^{\infty} G(\alpha_1, \ldots, \alpha_n) \neq X$, or

(b) $\bigcup_{n=1}^{\infty} F(\alpha_1, \ldots, \alpha_n) = X$.

One variation is to fix $\kappa = \omega$; this gives the class of **$P_{\aleph_0}$-spaces**. If, in addition, one requires that $\bigcup_{n=1}^{\infty} G(\alpha_1, \ldots, \alpha_n) = X$ for each infinite sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of ω, then one defines the class of **weak $P_{\aleph_0}$-spaces**. For $X \times Y$ to be normal for all separable ***metrizable*** Y (respectively, for all separable ***completely metrizable*** Y), it is necessary and sufficient that X be a normal (weak) $P_{\aleph_0}$-space. There are also characterizations along similar lines for all spaces X whose product with a single given metrizable space Y is normal (see [KV, Chapter 18]).

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