COMPACTIFICATIONS AND REMAINDERS OF MONOTONICALLY NORMAL SPACES

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Abstract. Monotonically normal spaces have many strong properties, but poor preservation properties. For example, there are locally compact, monotonically normal spaces whose one-point compactifications are not monotonically normal, and hence have no monotonically normal compactifications. We give two classes of such spaces, and give a pair of necessary conditions for spaces of pointwise countable type to have, respectively, compactifications or remainders that are monotonically normal. We show that a monotonically normal, locally compact space has a monotonically normal compactification if it is either locally connected or countably compact, and show that this latter condition cannot be weakened to "σ-countably compact."

1. Introduction

In [3] we began a study of when a monotonically normal space can have a monotonically normal compactification. In particular, we gave a necessary and sufficient criterion for a locally compact space to have one [see below]. As Mary Ellen Rudin already noted in [12], this is equivalent to the one-point compactification being monotonically normal, inasmuch as identifying the remainder in a compactification of a locally compact space to a single point preserves monotone normality.

Theorem 1.1. [3] *The one point compactification of a monotonically normal, locally compact space X is monotonically normal if, and only if, X is weakly orthocompact.*

Recall that a family of open sets is called *interior-preserving* if, and only if, every intersection of sets belonging to the family is open, and

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that a space is *weakly orthocompact* if every directed open cover of the space has an interior-preserving open refinement.

Weak orthocompactness was introduced by B.M. Scott in [14]. Scott showed in [14] that weak orthocompactness is preserved in various topological operations. We can use his results and Theorem 1.1 to establish invariance properties for the class $\mathcal{LM}$ of locally compact spaces with a monotonically normal compactification. This way we see, for instance, that the class is invariant under perfect mappings (both ways). Trivially, weak orthocompactness, and hence also the class $\mathcal{LM}$, is invariant under topological direct sums. As a consequence of the mentioned results, we obtain the following Locally Finite Sum Theorem: a locally compact space X belongs to $\mathcal{LM}$ provided that X has locally finite cover by closed subsets belonging to $\mathcal{LM}$.

Since it is not always easy to ascertain whether a space is weakly orthocompact, we also give a number of other conditions under which a space has a monotonically normal compactification. This includes spaces that are not locally compact, for which it is also interesting to know when their remainders are monotonically normal.

We also continue the study, begun in [3], of spaces that we call "utterly normal" and "UNO". These are successive strengthenings of monotone normality which may, for all we know, be equivalent to it; but, as shown in [3], if UNO is equivalent to monotone normality, then stratifiable spaces are M_1, solving a problem that is now 55 years old!

Definition 1.2. A *magnetic base system* for a space X is a collection $\{\mathcal{B}_x : x \in X\}$ where each $\mathcal{B}_x$ is a base for the neighborhoods of x, with the following property. If $B_x \in \mathcal{B}_x$ and $B_y \in \mathcal{B}_y$ and $B_x \cap B_y \neq \emptyset$ then either $x \in \overline{B_y}$ or $y \in \overline{B_x}$. A magnetic base system is *open [closed] [clopen]* if each member of each $\mathcal{B}_x$ is open [closed] [clopen].

A space is *utterly normal [UNO] [utterly ultranormal]* if it is a regular space that has a(n) [open] [clopen] magnetic base system.

The following characterizations of monotone normality, due to Borges [2], make it easy [3] to show that every utterly normal space is monotonically normal.

Theorem 1.3. [2] *The following are equivalent for a space X.*

(1) *There is an assignment of an open neighborhood $h(x,U) =: U_x$ of x to each pair (x,U) such that U is an open neighborhood of x, and that if $U_x \cap V_y \neq \emptyset$, then either $x \in V$ or $y \in U$.*

(2) *There is an assignment as in* (1) *with the additional property that if $x \in U \subset V$, then $U_x \subset V_x$.*

(3) X *is monotonically normal.*

The key to (1) $\implies$ (2) is to let $H(x,U) = \bigcup\{V_x : x \in V \subset U\}$. The key to (2) $\implies$ (3) is to let $G(F_1, F_2) = \bigcup\{H(x, X \setminus F_2) : x \in F_1\}$. Conversely, we let $U_x = G(\{x\}, X \setminus U)$ and use the fact that $U_x \subset G(\{x\},\{y\})$ for all $y \notin U$.

In [2], statements (1) and (2) have an extra condition that $H(x,U) \subset U$, but this is automatic from these statements as they stand if the space is T_1; then, in fact, $\overline{H(x,U)} \subset U$. This may have been first observed only recently, in [16].

The usual definition of monotone normality, motivating the terminology, is as follows:

Definition 1.4. A space X is *monotonically normal* provided that there is an operator $G(\_,\_)$ assigning to each ordered pair $\langle F_0, F_1\rangle$ of disjoint closed subsets an open set $G(F_0, F_1)$ such that
(a) $F_0 \subset G(F_0, F_1)$
(b) If $F_0 \subset F_0'$ and $F_1' \subset F_1$ then $G(F_0, F_1) \subset G(F_0', F_1')$
(c) $G(F_0, F_1) \cap G(F_1, F_0) = \emptyset$

In Section 2, we give a pair of necessary conditions for spaces of pointwise countable type to have, respectively, monotonically normal remainders and monotonically normal compactifications. In Section 3, we give a general class of locally compact, UNO spaces which do not have monotonically normal compactifications. In Section 4, we show that a locally compact, monotonically normal space has a monotonically normal compactification if it is either countably compact or locally connected. Examples from Section 3 show that "countably compact" cannot be weakened to "every closed discrete subspace is countable."

Henceforth in this paper, "space" will mean "Tychonoff space," since the center of interest has to do with compactifications, by which is meant compact Hausdorff spaces in which a given space embeds as a dense subspace.

2. Properties at infinity give necessary conditions

Recall that a Tychonoff space X is said to have *property P at infinity* if its Stone-Čech remainder $\beta X \setminus X$ has property P. Also recall:

Definition 2.1. A space is *of pointwise countable type* if every point is contained in a compact set C with a countable outer base of neighborhoods. A space is *strongly paracompact* if every open cover has a star-finite open refinement, by which is meant an open refinement $\mathcal{U}$ such that every $U \in \mathcal{U}$ meets at most finitely many other members of $\mathcal{U}$. A space X is *[strongly] paracompact at infinity* if $\beta X \setminus X$ is [strongly] paracompact.

First countable spaces and locally compact spaces are of pointwise countable type. In the case of locally compact spaces, this follows from an application of Urysohn's lemma to a compact neighborhood N of a point and an open neighborhood with compact closure that contains N. Taking N to $\{0\}$, we let $C = f^{-1}\{0\}$.

Henriksen and Isbell showed in [5] that every first countable linearly orderable space is paracompact at infinity. By applying a deep theorem of M.E. Rudin [13] and by modifying the proof given in [5], we obtain a strengthening of this result.

Theorem 2.2. *Let X be a space of pointwise countable type. If X has a monotonically normal compactification, then X is strongly paracompact at infinity.*

Proof. Assume that C is a monotonically normal compactification of X. By the Basic Theorem in [13], there exists a compact linearly orderable space L and a continuous onto mapping $g : L \to C$. There exists a closed subset K of L such that the mapping $f = g\restriction_K$ from K to C is irreducible and onto. Note that the subspace $f^{-1}(X)$ of K is dense and of pointwise countable type. The compact subspace K of L is linearly orderable. Let $\prec$ be a linear order on K such that K has the order topology determined by $\prec$.

We show that $C \setminus X$ is strongly paracompact. Then since strong paracompactness is inversely preserved by perfect mappings, it follows that X is strongly paracompact at infinity.

Let $\mathcal{G}$ be an open cover of $C \setminus X$, and let $\mathcal{H}$ be a family of open subsets of C such that $\mathcal{G} = \{H \setminus X : H \in \mathcal{H}\}$. Set $J = \bigcup \mathcal{H}$ and $T = f^{-1}(J)$. Denote by $\mathcal{I}$ the collection of all maximal $\prec$–intervals contained in the open subset T of K. Then $\mathcal{I}$ is a disjoint open cover of T. As a consequence, $\mathcal{I}$ is locally finite in T.

We show that, for every $I \in \mathcal{I}$, there exists a σ-compact set $I' \subset I$ such that $I \setminus f^{-1}(X) \subset I'$. Let $I \in \mathcal{I}$, and let a and b be the left and right end–points of I, respectively. If the set $M = \{a, b\} \setminus I$ is empty, then I is compact and we can set $I' = I$. Assume that $M \neq \emptyset$. For each $p \in M$, we define a set K_p as follows. By maximality of I, we have $p \in K \setminus T \subset f^{-1}(X)$. It follows, since $f^{-1}(X)$ is of pointwise countable type, that there exists a compact subset K_p of $f^{-1}(X)$ such that $p \in K_p$ and K_p has a countable outer base in $f^{-1}(X)$. Note that K_p is a G_δ-set in K. Now the set $I' = I \setminus \bigcup\{K_z : z \in M\}$ contains the set $I \setminus f^{-1}(X)$. Moreover, I' is an F_σ-subset of the compact set $\overline{I} = I \cup M$ and hence I' is σ-compact.

For each $I \in \mathcal{I}$, the subset $I^* = f(I')$ of C is σ-compact. The mapping $f \restriction_T : T \to J$ is perfect and it follows, since the family $\mathcal{I}$ is locally finite in T, that the family $\mathcal{I}^* = \{I^* : I \in \mathcal{I}\}$ is locally finite in J. The subspace $J^* = \bigcup \mathcal{I}^*$ of J is strongly paracompact, since the subspace has a locally finite cover by σ-compact sets. We have $J^* \subset J = \bigcup \mathcal{H}$ and it follows, since $\mathcal{H}$ is an open family, that $\mathcal{H}$ has a star-finite open refinement in the subspace J^*.

Since $\mathcal{I}$ covers T and $I \setminus f^{-1}(X) \subset I'$ for each $I \in \mathcal{I}$, we have $T \setminus f^{-1}(X) \subset \bigcup\{I' : I \in \mathcal{I}\}$ and hence $C \setminus X = J \setminus X \subset J^*$. It follows from the foregoing that the family $\mathcal{G} = \{H \setminus X : H \in \mathcal{H}\}$ has a star-finite open refinement in the subspace $C \setminus X$. □

Every suborderable space has a linearly orderable compactification, and such a compactification is monotonically normal. Hence we see that every suborderable space of pointwise countable type is strongly paracompact at infinity.

With the help of an important theorem of Balogh and Rudin, we can extend the result in 2.2 with "strongly" omitted, to monotonically normal remainders.

Theorem 2.3. *Let X be a space of pointwise countable type. Then every monotonically normal closed subspace of a remainder of X is paracompact. In particular, if X has a monotonically normal remainder, then X is paracompact at infinity.*

Proof. Let K be a compactification of X and let F be a monotonically normal closed subspace of the remainder $K \setminus X$. To prove that F is paracompact it suffices, by a theorem of Balogh and Rudin [1], to show that no closed subspace of F is homeomorphic with a stationary subset of a regular uncountable cardinal.

Assume on the contrary that there exists a closed subset S of F and a regular uncountable cardinal κ such that S is homeomorphic with a stationary subset of the ordinal space κ. By compactness, there exists a point $z \in K$ such that each neighborhood of z contains κ-many points of S. We have $z \notin S$, since every point of S has a neighborhood containing less than κ-many points of S. It follows, since S is closed in

$K \setminus X$, that $z \in X$. Let C be a compact subset of X such that $z \in C$ and C has a countable outer base in X. Then C is a G_δ-set in K and it follows, since $C \cap S = \emptyset$, that there exists an open subset G of K containing S such that $|S \setminus G| = \kappa$. Let V be a neighborhood of z in K such that $\overline{V} \subset G$. Now $C \setminus G$ and $\overline{V} \cap S$ are disjoint closed subsets of S of cardinality κ, but this is a contradiction, since S is homeomorphic with a stationary subset of κ. □

We know of no general classes of spaces that admit a monotonically normal remainder besides the easy examples of locally compact spaces, suborderable spaces, and separable metrizable spaces. In particular, the following problem is open.

Problem 1. Does every metrizable space have a monotonically normal remainder?

In connection with this problem, note that there are metrizable spaces which do not have a monotonically normal compactification. An example is any non-separable hedgehog [8]. However, these spaces have separable metrizable remainders in a compactification constructed in [8].

In an upcoming paper [7], one of us shows that if X is Nagata's bow-tie space, then the subspace $Y = \{(z, u) \in X : u \text{ is rational } \}$ is a first countable, stratifiable space that fails to be paracompact at infinity and hence has no monotonically normal remainder.

3. Classes of complementary counterexamples

In this section, we define a pair of general classes of locally compact, utterly ultranormal spaces without monotonically normal compactifications.

We use the notation $A^{(\alpha)}$ for the αth Cantor-Bendixson derivative of A, so that $A^{(1)}$ is derived set of A. We write Λ for the set $\omega_1^{(1)}$ of countably infinite limit ordinals and Λ_2 for $\Lambda^{(1)} = \omega_1^{(2)}$. We use $\nu_n \nearrow \alpha$ as shorthand for "$\langle \nu_n : n \in \omega \rangle$ is strictly increasing with supremum α."

Example 3.1. *We let $\mathfrak{T}$ designate the set of topologies τ on ω_1 in which, to each point α there are associated $B(\alpha) \subset [0, \alpha]$ and $B(\alpha, \xi) = B(\alpha) \cap (\xi, \alpha]$ for each $\xi < \alpha$, such that:*

(1) *$\{B(\alpha, \xi) : \xi < \alpha\}$ is a base for the neighborhoods of α [we allow $\xi = -1$ in case $\alpha = 0$].*

(2) *If* $\alpha \in \Lambda$ *then* $\alpha = sup(B(\alpha) \setminus \Lambda) = sup([0,\alpha] \setminus [B(\alpha) \cup \Lambda])$.
(3) *If* $\alpha \in \Lambda$ *and* $\alpha < \beta$, *then there exists* $\xi < \alpha$ *such that* $B(\alpha, \xi) = B(\beta) \cap (\xi, \alpha]$.
(4) *If* $\gamma_n + k_n \nearrow \alpha$ *where* $\alpha \in \Lambda$ *and* $k_n \in \omega \setminus \{0\}$, *and* $\gamma_n \in \Lambda \cup \{0\}$, *and* $\gamma_n + k_n \in B(\alpha)$ *for all* n, *then* $k_n \to \infty$.

Note that (4) is automatic for $\alpha \in \Lambda \setminus \Lambda_2$ but otherwise there is tension between (4) and (3) which, together with (2), accounts for the lack of a monotonically normal compactification when $\tau \in \mathcal{T}$. Of course, in the presence of (3), we need only verify (2) for $\alpha \in \Lambda \setminus \Lambda_2$; moreover, (2) and (3) have the corollary that (ω_1, τ) has a dense set of isolated points, which are the same as they are in the usual topology (*i.e.,* 0 and successor ordinals).

Before going on to some other basic properties shared by all $\tau \in \mathfrak{T}$, we define a class $\mathfrak{T}^*$ of "complementary" spaces sharing many properties with those of $\mathfrak{T}$.

Example 3.2. *We let* $\mathfrak{T}^*$ *designate the set of topologies* τ *on* ω_1 *in which, to each point* α *there are associated* $B^*(\alpha) \subset [0, \alpha]$ *and* $B^*(\alpha, \xi) = B^*(\alpha) \cap (\xi, \alpha]$ *for each* $\xi < \alpha$, *satisfying* (1), (2), *and* (3) *of Example 3.1 with* B^* *in place of* B *everywhere, and:*
(4*) *If* $\gamma_n + k_n \nearrow \alpha$ *where* $\alpha \in \Lambda$ *and* $\gamma_n \in \Lambda \cup \{0\}$ *and* $k_n \in \omega$, *and there exists* $N \in \omega$ *such that* $k_n \leq N$ *for all* $n \in \omega$, *then* $\langle \gamma_n + k_n \rangle$ *is eventually in each* $B^*(\alpha, \xi)$ $(\xi < \alpha)$.

Note that the conclusion of Example 3.2 is equivalent to τ-convergence of $\langle \gamma_n + k_n \rangle$ to α. The comments following Example 3.1 also apply to each $\tau \in \mathfrak{T}^*$. It is also easy to see that we obtain a space in $\mathfrak{T}^*$ if we define $B^*(\alpha) = [0, \alpha] \setminus B(\alpha)$ for a space in $\mathfrak{T}$; and that if we define $B(\alpha) = [0, \alpha] \setminus B^*(\alpha)$ for a space in $\mathfrak{T}^*$, then we get a space in $\mathfrak{T}$. Despite the difference between (4) and (4*), the two kinds of spaces have much in common.

Lemma 3.3. *If* $\tau \in \mathfrak{T} \cup \mathfrak{T}^*$, *then* τ *is finer than the usual topology on* ω_1, *each* $B(\alpha, \xi)$ *[resp. each* $B^*(\alpha, \xi)$*] is* τ*-open,* (ω_1, τ) *is first countable, and the relative topology on* Λ *is its usual topology.*

Proof. The first three properties are clear from (1) and from the fact that $B(\alpha) \subset [0, \alpha]$, and it is clear from (3) that $B(\beta, \xi) \cap \Lambda = [0, \beta] \cap \Lambda$ for all $\beta \in \omega_1$. This also applies to B^* in place of B everywhere. □

Lemma 3.4. *If* $\tau \in \mathfrak{T} \cup \mathfrak{T}^*$, *then* (ω_1, τ) *is locally compact and has a base of clopen sets of the form* $B(\alpha, \xi)$ *[resp.* $B^*(\alpha, \xi)$*].*

Proof. Obviously, $B(0) = \{0\}$ and $B(\xi + 1, \xi)$ are compact for all successor ordinals $\xi + 1$. If $\beta \in \Lambda$ then, since $B(\beta)$ is countable, it is enough to show that $B(\beta)$ is countably compact. These comments also hold for B^* in place of B.

Let A be an infinite subset of $B(\beta)$. Then A contains a strictly ascending sequence σ of ordinals. Let $sup(ran(\sigma)) = \alpha$. If $\alpha = \beta$ then $\sigma \to \alpha$ by (1), while if $\alpha < \beta$, then (1) and (3) have the same effect, implying that α is a limit point of A in $B(\beta)$. The proof for B^* in place of B is the same. $\square$

In the following lemma, $A \subseteq^* B$ means $A \setminus B$ is finite, while $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$ both hold.

Lemma 3.5. *If $B(\alpha)$ and $B^*(\alpha)$ are as in Examples 3.1 and 3.2, then $B(\alpha) =^* B(\beta) \cap [0, \alpha]$ (and therefore $B^*(\alpha) =^* B^*(\beta) \cap [0, \alpha]$) whenever $\alpha < \beta$, $\alpha, \beta \in \Lambda$.*

Proof. There is a quick topological proof using the compactness of these sets, but it is instructive to see that (3) in 3.1 is equivalent to the first conclusion [and then its counterpart for 3.2 is just a matter of notation].

It is trivial to see that $B^*(\alpha) =^* B^*(\beta) \cap [0, \alpha]$ implies that there is $\xi < \alpha$ as in (3): one need only get past the finitely many points in the symmetric difference $S = B^*(\alpha)\Delta(B^*(\beta) \cap [0, \alpha])$ and use the fact that $\alpha \in \Lambda$.

Inversely, we show that the assumption that S is infinite leads to a contradiction with (3). Let α be the least (limit) ordinal for which there exists $\beta > \alpha$ such that S is infinite. Choose $\xi < \alpha$ such that $B(\alpha) \cap (\xi, a] = B(\beta) \cap (\xi, a]$. Clearly $S \subset [0, \xi]$, so ξ is infinite, and if λ is the greatest limit ordinal $\leq \xi$, then all but finitely many points of S are in $[0, \lambda]$. But this contradicts minimality of α: if $S \cap B(\alpha) \cap B(\lambda)$ is infinite, then $B(\lambda) \setminus B(\beta)$ is infinite; while if it is finite, then either $B(\lambda) \setminus B(\alpha)$ is infinite, violating $B^*(\lambda) =^* B^*(\alpha) \cap [0, \lambda]$, or $B(\beta) \cap [0, \lambda] \setminus B(\lambda)$ is infinite. $\square$

The proof of the following theorem is made easy by the fact that the topology on Λ is its usual topology, and by the corollary that $[0, \alpha] \cap \Lambda$ is compact for all α.

Theorem 3.6. *Let $\tau \in \mathfrak{T} \cup \mathfrak{T}^*$. Then (ω_1, τ) is utterly ultranormal, but its one-point compactification is not monotonically normal.*

Proof. Let $\mathcal{B}_\xi = \{\{\xi\}\}$ if $\xi \notin \Lambda$ and let $\mathcal{B}_\gamma = \{B(\gamma, \xi) : \xi < \gamma\}$ if $\gamma \in \Lambda$. The system $\{\mathcal{B}_\alpha : \alpha \in \omega_1\}$ is magnetic. To show this, it is enough verify that if α and β are both limit ordinals such that $\alpha < \beta$, then $\alpha \in B(\beta,\eta)$ whenever $B(\alpha,\xi) \cap B(\beta,\eta) \neq \emptyset$. And this follows easily from (3) and the fact that $B(\beta,\eta) \cap \Lambda = (\eta,\beta] \cap \Lambda$; similarly for R^* in place of B. Since each $B(\alpha,\xi)$ and $B^*(\alpha,\xi)$ is clopen, (ω_1,τ) is utterly ultranormal.

Now let X denote the one-point compactification of (ω_1,τ) where $\tau \in \mathfrak{T} \cup \mathfrak{T}^*$, and let ∞ denote the extra point in X. Suppose there were a Borges operator $(\cdot)_x$ on X; then its restriction to (ω_1,τ) would also be a Borges operator. For each $\lambda \in \Lambda_2$ let $\lambda' = \lambda + \omega^2$ be next ordinal in Λ_2.

If $\tau \in \mathfrak{T}$, let $\gamma(\lambda) + n(\lambda)$, where $\gamma(\lambda) \in \Lambda$, be the least element of $B(\lambda',\lambda)_{\lambda'}$. By (3), $n(\lambda) > 0$ for all λ.

By a simple cardinality argument, there exists $n \in \omega$ for which there is a strictly ascending sequence $\langle \lambda_k \rangle$ in Λ_2 such that $n(k) = n$ for all k. Then by (4), $\lambda_k + n$ converges to ∞ in X. Let $\gamma = sup_k \lambda_k$ and let $V = X \setminus ([0,\gamma] \cap \Lambda)$. Now V_∞ contains all but finitely many of the $\lambda_k + n$. But $\infty \notin B(\lambda'_k, \lambda_k)$ and $\lambda'_k \notin V$, a contradiction to $(\cdot)_x$ being a Borges operator.

If $\tau \in \mathfrak{T}^*$, let $V(\lambda) = X \setminus (\Lambda \cap [0,\lambda']) = ([0,\lambda'] \setminus \Lambda) \cup (\lambda',\omega_1) \cup \{\infty\}$. Let $\gamma(\lambda) + n(\lambda)$, where $\gamma(\lambda) \in \Lambda$, be the least element of $V(\lambda)_\infty$. Clearly, $n(\lambda) \neq 0$ for all $\lambda \in \Lambda_2$. Then there exists $n \in \omega$ and a strictly ascending sequence $\langle \lambda_k \rangle$ in Λ_2 such that $n(k) = n$ for all k. Let $\gamma = sup_k \lambda_k$. Then $\lambda_k + n$ converges to γ. But $\infty \notin [0,\gamma]_\gamma$ and $\gamma \notin V(\gamma)$, a contradiction to $(\cdot)_x$ being a Borges operator. □

Corollary 3.7. No topology in $\mathfrak{T} \cup \mathfrak{T}^*$ is weakly orthocompact.

Proof. This is an immediate consequence of Theorem 3.6, Theorem 1.1, and the easy fact that a space with a clopen partition into weakly orthocompact spaces is itself weakly orthocompact. □

It is remarkable how little of (ω_1,τ) was needed in the proof of Theorem 3.6 to show that X is not monotonically normal. In particular, $W = \bigcup\{B(\lambda',\lambda) : \lambda \in \Lambda_2\}$ is nonstationary: it is disjoint from the club set $\Lambda_3 = \Lambda_2^{(1)} = \Lambda^{(2)}$. However, this does not mean that the one-point compactification of W is not monotonically normal: it was necessary in the proof of 3.6 to find a neighborhood of ∞ which omits infinitely many of the λ_k.

We also had no need of the Axiom of Choice (AC) in the proof. The only place where this requires explanation is the observation that $N_n = \{\lambda \in \Lambda_2 : n(\lambda) = n\}$ is infinite for some n. Now ZF is enough to show that $\omega \times \omega$ is infinite, and if each N_n were finite (or even countable), we could define a surjective function $f : \omega \times \omega \to \Lambda_2 \setminus \Lambda_3$ by letting $f(n,k)$ be the $k+1$st element of N_n unless $|N_n|$ is finite, with k_n elements, in which case we let $f(n,k)$ be the k_nth element for all $k \geq k_n$. But it is a theorem of ZF that the image of a countable set is countable, and that $\Lambda_2 \setminus \Lambda_3$ is uncountable. So we can take the least n such that N_n is infinite, and define $\langle \lambda_k \rangle_{n=1}^{\infty}$ by recursion, by letting λ_k be the kth member of N_n.

However, AC cannot be avoided altogether if $\mathfrak{T}$ is to be nonempty. In fact, $\mathfrak{T} \neq \emptyset$ is equivalent to a well-known and much used concept in set-theoretic topology: the existence of a ladder system on ω_1.

Definition 3.8. Given a limit ordinal α, a *ladder at* α is a strictly ascending sequence of ordinals less than α whose supremum is α. Given an ordinal θ, a *ladder system on* θ is a family

$$\{L_\alpha : \alpha \in \theta,\ \alpha \text{ is a limit ordinal of countable cofinality}\}$$

where each L_α is a ladder at α.

Lemma 3.9. *If $\mathfrak{T} \neq \emptyset$, then there is a ladder system on ω_1*

Proof. The existence of any collection $\{B(\alpha) : \alpha \in \Lambda\}$ of subsets of ω_1 satisfying (2) and (4) of Example 3.1 is already enough to imply that there is a ladder system on ω_1. In fact, it is easy to see that if $\alpha \in \Lambda$ and we let

$$\alpha_n = min\{\xi : \forall \gamma + k \geq \xi\ (g \in \Lambda \cup \{0\}) \implies k \geq n\}$$

and we let $L_\alpha = \{\alpha_n : n \in \omega\}$ then $\{L_\alpha : \alpha \in \Lambda\}$ is a ladder system on ω_1. ☐

To show the converse of 3.9 we now construct a specific example of $\tau \in \mathfrak{T}$, given a ladder system $\{L_\alpha : \alpha \in \Lambda\}$.

Example 3.10. We build $\{B(\nu) : \nu \in \omega_1\}$ by recursion. Let $B(0) = \{0\}$ and, if $\nu = \xi + 1$, let $B(\nu) = B(\xi) \cup \{\nu\}$. If $\alpha \in \Lambda \cup \{0\}$ and $\nu = \alpha + \omega$ let $B(\nu) = \{\nu\} \cup B(\alpha) \cup \{\alpha + 2n : n \in \omega\}$.

It remains to define $B(\alpha)$ when $\alpha \in \Lambda_2$. Given $\alpha_n \in L_\alpha$, let $S(\alpha, 0) = B(\alpha_0)$ and, for $n > 0$, let

$$S(\alpha, n) = B(\alpha_n, \alpha_{n-1}) \cap (\Lambda \cup \{\gamma + k : \gamma \in \Lambda \cup \{0\} \text{ and } k \geq n\}$$

and let $B(\alpha) = \{\alpha\} \cup \bigcup_{n=0}^{\infty} S(\alpha, n)$.

Theorem 3.11. *The following are equivalent:*

(i) $\mathfrak{T} \neq \emptyset$

(ii) There is a collection $\{B(\alpha) : \alpha \in \Lambda\}$ *of subsets of* ω_1 *satisfying (2) and (4) of Example 3.1*

(iii) There is a ladder system on ω_1.

Proof. It is obvious that (i) implies (ii), and it was shown in the proof of Lemma 3.9 that (ii) implies (iii). To show that (iii) implies (i), it is enough to show that Example 3.10 witnesses (1) through (4) in the definition of $\tau \in \mathfrak{T}$ if we define $B(\alpha, \xi)$ to equal $B(\alpha) \cap (\xi, \alpha]$.

Since $\alpha_n \nearrow \alpha$, (4) in Definition 3.1 is obviously satisfied, and (3) follows by induction, using limit ordinals below α. Indeed, if $\gamma \in B(\alpha_n, \alpha_{n-1}) \cap \Lambda$ then $B(\alpha_n, \xi) \cap [0, \gamma] = B(\gamma, \xi)$ for some $\xi < \gamma$, and by (4) applied to γ there exists $\eta \geq \xi$ such that $\eta < \gamma$ and all successor ordinals between η and γ are in $S(\alpha, n)$.

Condition (2) is satisfied because, as remarked after the description of Example 3.1, we need only show it for $\Lambda \setminus \Lambda_2$, and it is obvious for those ordinals.

Finally, to show (1), we need to see that $\mathcal{B} = \{B(0)\} \cup \{B(\alpha, \xi) : \xi < \alpha \in \omega_1\}$ is a base for a topology on ω_1. Obviously, $\bigcup B = \omega_1$. If $\nu \in B(\alpha, \xi) \cap B(\beta, \eta)$ and $\nu \notin \Lambda$ then $\{\nu\} \in \mathcal{B}$, while if $\nu \in \Lambda$ then since $\nu \leq \min\{\alpha, \beta\}$, then (3) applied to ν and α and β gives $\mu < \nu$ such that $B(\nu, \mu) = B(\alpha) \cap (\mu, \nu] = B(\beta) \cap (\mu, \nu]$. $\square$

Now we come to two contrasting theorems about $\mathfrak{T}$ and $\mathfrak{T}^*$:

Theorem 3.12. *Let* $\tau \in \mathfrak{T}$. *Then* Λ *is a* G_δ *in* (ω_1, τ). *In fact,* $D_n = \{\lambda + n : \lambda \in \Lambda \cup \{0\}\}$ *is a closed discrete subspace for each* $n \in \omega \setminus \{0\}$.

Proof. That D_n is closed is an easy consequence of (1) and (4) in 3.1, and discreteness is immediate from Lemma 3.3. So $\bigcup_{n=1}^\infty D_n$ is an F_σ, and its complement is Λ. $\square$

Theorem 3.13. *Let* $\tau \in \mathfrak{T}^*$. *Then* (ω_1, τ) *is is* σ*-countably compact. In particular, it is* ω_1*-compact, i.e., every closed discrete subspace is countable.*

Proof. Let D_n be as in the preceding proof. It is clear from (4^*) that $D_n \cup \Lambda$ is countably compact for each $n \in \omega \setminus \{0\}$, and closed, and ω_1 is obviously the union of these countably many subspaces. Countable extent (*i.e.*, ω_1-compactness) is an easy consequence. $\square$

The following notes will not be used after this section.

Historical notes.

Note 1. The axiom of there being a ladder system on ω_1 was first used by G.H. Hardy in 1903 to construct an uncountable well-orderable subset of $\mathbb{R}$ [4]. Hardy thought that he had avoided all use of AC, but it was soon noted that, while the existence of individual ladders at $\alpha \in \omega_1$ follows from ZF, there are so many possible ladders at each α that some form of AC is required to produce an entire ladder system on ω_1.

A rigorous proof of this had to await Cohen's 1963 proof that AC is ZF-independent and subsequent work showing that the existence of an uncountable well-orderable subset of $\mathbb{R}$ is also ZF-independent. A good reference for this last fact is [6], where many other weakenings of AC are compared and catalogued. The existence of a ladder system on ω_1 is not one of them, however. One of us (Nyikos) has been collecting statements equivalent to this axiom and has an online file [9] showing their equivalence and their relation to some other weakenings of AC.

Note 2. Example 3.10 needs only minor modifications to produce a space homeomorphic to Mary Ellen Rudin's example [12] of a monotonically normal, locally compact space without a monotonically normal compactification. The following example gives these modifications.

Example 3.14. On ω_1, follow the construction in Example 3.10, with the difference that $B(\nu) = B(\alpha) \cup (\alpha, \nu]$ when $\nu = \alpha + \omega, \alpha \in \Lambda \cup \{0\}$. This does not exactly give a member of $\mathfrak{T}$ but we still have the following modifications of (2) in 3.1, enough for obtaining the main features of (ω_1, τ):
(2′) If $\alpha \in \Lambda$ then $\alpha = sup(B(\alpha) \setminus \Lambda)$ and if α is a limit of limit ordinals, then $\alpha = sup([0, \alpha] \setminus [B(\alpha) \cup \Lambda])$.

Our copy of Rudin's space has ω_1 with the resulting topology as a subspace. To produce X, we add points converging to each limit ordinal α from above, beginning with $\alpha + 1$, so that

$$X = \omega_1 \cup \{\alpha + \frac{1}{2^n} : \alpha \in \Lambda, n \in \omega\}$$

with the obvious order:

$$\alpha < \alpha + \frac{1}{2^{n+1}} < \alpha + \frac{1}{2^n} \leq \alpha + 1 \text{ for all } \alpha, n.$$

The sets of the following form give a base for X:

$$B(\alpha; \xi, k) = B(\alpha, \xi) \cup \{\beta + \frac{1}{2^n} : \beta \in \Lambda, \beta < \alpha, n \in \omega\} \cup \{\alpha + \frac{1}{2^n} : n \geq k\}$$

Rudin's space has underlying set $\omega_1 \times (\omega + 1)$ and the topology induced by the bijection $\omega_1 \times (\omega+1) \to X$ that takes $\langle \xi, \omega \rangle$ to the ξth limit ordinal, and takes $\langle \alpha, n \rangle$ to $\alpha + \frac{1}{2^{n+1}}$ and takes $\langle \nu, n \rangle$ to $\alpha + n$ when $\nu = \alpha + \omega, \alpha \in \Lambda$.

4. Positive results on countably compact spaces and locally connected spaces

We begin with an auxiliary result. Recall that a family $\mathcal{L}$ of sets is *well-monotone* provided that inclusion is a well-order on $\mathcal{L}$.

Lemma 4.1. *Let X be monotonically normal and locally compact. Then X has an open cover $\mathcal{U} \cup \bigcup_{s \in S} \mathcal{V}_s$, where $\mathcal{U}$ is a point-finite family of sets with compact closure, every $\mathcal{V}_s$ is well-monotone and consists of sets with paracompact closure, and the family $\{\bigcup \mathcal{V}_s : s \in S\}$ is discrete.*

Proof. We use the well-known fact that every monotonically normal space X is hereditarily collectionwise normal, and has the following powerful properties if it is locally compact:

The Balogh-Rudin Covering Property [1]*:* If $\mathcal{U}$ is an open cover of X, then $X = V \cup \bigcup \mathcal{W}$, where $\mathcal{W}$ is a discrete family of copies of stationary subsets of ordinals of uncountable cofinality, and V is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.

The Balogh-Rudin Paracompactness Criterion (locally compact case) [1]*:* X is paracompact if, and only if, it contains no closed copy of a regular uncountable cardinal.

By the Balogh-Rudin Covering Property, X has a σ-disjoint open family $\mathcal{G}$ such that $\overline{G}$ is compact for each $G \in \mathcal{G}$ and a discrete closed family $\{F_s : s \in S\}$ such that $\bigcup_{s \in S} F_s = X \setminus \bigcup \mathcal{G}$ and each F_s is homeomorphic with an ordinal. By a result of Rudin [11], the (monotonically normal) subspace $\bigcup \mathcal{G}$ of X is countably paracompact. It follows that $\mathcal{G}$ has a point-finite open refinement $\mathcal{U}$.

By collectionwise normality, the family $\{F_s : s \in S\}$ has a discrete open expansion $\{W_s : s \in S\}$. Let $s \in S$. There exists an ordinal λ such that we can write $F_s = \{x_\alpha : \alpha < \lambda\}$ so that the mapping $\alpha \mapsto x_\alpha$ is a homeomorphism $\lambda \to F_s$. Let the sets $G(F_0, F_1)$ verify monotone normality of X as in Definition 1.4, and for each $\alpha < \lambda$, let $V_\alpha = G\big(\{x_\beta : \beta \leq \alpha\}, \{x_\beta : \alpha < \beta < \lambda\} \cup (X \setminus W_s)\big)$. Then $\mathcal{V}_s = \{V_\alpha : \alpha < \lambda\}$ is a well-monotone open family and, for every $\alpha < \lambda$, we have $\{x_\beta : \beta \le \alpha\} \subset V_\alpha$ and $\overline{V_\alpha} \cap \big(\{x_\beta : \alpha < \beta < \lambda\} \cup (X \setminus W_s)\big) = \emptyset$. Note that $F_s \subset \bigcup \mathcal{V}_s$. We show that every set in $\mathcal{V}_s$ has paracompact closure.

Assume on the contrary that there exists $\gamma < \lambda$ such that $\overline{V_\gamma}$ is not paracompact. The Balogh-Rudin Paracompactness Criterion, stated more generally for Čech complete spaces in Corollary 2.3 of [1], implies that there exists a closed set $H \subset \overline{V_\gamma}$ which is homeomorphic with a regular uncountable cardinal. We have $H \cap F_s \subset \overline{V_\gamma} \cap F_s = \{x_\alpha : \alpha \le \gamma\}$ and the set $\{x_\alpha : \alpha \le \gamma\}$ is compact. It follows that $H \setminus F_s$ contains a set homeomorphic with a regular uncountable cardinal. This, however, is impossible, since $H \setminus F_s \subset W_s \setminus F \subset \bigcup \mathcal{G} = \bigcup \mathcal{V}$ and a regular uncountable cardinal cannot have a point-finite cover by open sets with compact closures.

Since $\bigcup \mathcal{V}_s \subset W_s$ for each s, the family $\{\bigcup \mathcal{V}_s : s \in S\}$ is discrete. □

Theorem 3.13 shows that countable compactness cannot be weakened to σ-countable compactness in our next theorem:

Theorem 4.2. *Every locally compact, countably compact, monotonically normal space X has a monotonically normal one-point compactification and is therefore UNO.*

Proof. We use Lemma 4.1, and we denote by $\mathcal{H}$ the cover $\mathcal{U} \cup \bigcup_{s \in S} \mathcal{V}_s$ in the lemma. Note that every well-monotone family is interior-preserving. As a consequence, the cover $\mathcal{H}$ is interior-preserving. Since X is countably compact, every set in $\mathcal{H}$ has compact closure. Hence X has an interior-preserving cover by relatively compact sets and this implies that X is weakly orthocompact. By Theorem 1.1, $X + 1$ is monotonically normal. □

Now we recall a theorem that might still be unknown had not the second author been inspired by a theorem in which Alan Dow played an essential role: the theorem that PFA(S)[S] implies every hereditarily normal manifold of dimension > 1 is metrizable. In addition, Alan helped Frank Tall repair a hole in [15] where it was claimed that normal locally compact spaces are ω_1-collectionwise Hausdorff under PFA(S)[S]; he did this by replacing PFA(S)[S] with MM(S)[S] and modifying the proof. This is crucial in part (2) of the following theorem.

Only conditions (2) and (3) in the following theorem appear in the Main Theorem of [10], but (1) appears there also as a separate theorem.

Theorem 4.3. *Let X be a locally compact, locally connected space. If either*

(1) *X is monotonically normal or*

(2) *MM(S)[S] holds and X is hereditarily normal or*

(3) *PFA or PFA(S)[S] holds and X is normal and hereditarily strongly collectionwise Hausdorff,*

then every component of X is the disjoint union of an open Lindelöf space L and at most countably many closed countably compact spaces S_n. Moreover, the boundary of L is discrete.

Theorem 4.4. *Let X be a locally compact, locally connected, and monotonically normal space. Then the one-point compactification of X is monotonically normal, and therefore X is UNO.*

Proof. Let C be a connected component of X. Write $C = L \cup \bigcup\{S_n : n \in \alpha\}$ as in Theorem 4.3, where $\alpha \in \omega \cup \{\omega\}$. We show that the family $\mathcal{S} = \{S_n : n \in \alpha\}$ is discrete. Let $p \in C$. Then p has a connected compact neighborhood K which meets at most one point of the boundary of L. Let $k \in \alpha$ be such that $K \cap \partial L \subset S_k$. Now K is the union of the mutually disjoint closed sets $K \cap (\overline{L} \cup S_k)$ and $K \cap S_n$, $n \in \alpha \setminus \{k\}$, and it follows from Sierpinski's theorem on continua that only one of those closed sets is non-empty. As a consequence, the neighborhood K of p meets at most one of the sets S_n, $n \in \alpha$.

By Theorem 4.2, every S_n has a monotonically normal compactification. The subspace $\overline{L}$ is the union of the Lindelöf space L and the countable set ∂L, and hence $\overline{L}$ is Lindelöf. As a consequence, also $\overline{L}$ has a monotonically normal compactification. It follows that C has a locally finite closed cover by sets with monotonically normal compactifications. By the Locally Finite Sum Theorem for this property, also C has a monotonically normal compactification.

Finally, the space X has a monotonically normal compactification, since a locally connected space is the topological direct sum of its connected components. □

We close with a corollary of Theorem 4.3 which does not extend to the spaces in $\mathfrak{T}$, and with a problem that this corollary suggests.

Corollary 4.5. *Every monotonically normal, locally compact, locally connected space is the topological direct sum of ω_1-compact subspaces.*

Proof. It suffices to show that each component of the space is ω_1-compact.

The sets S_n in 4.3 are countably compact, so every closed discrete subspace of their union is countable. Also, every closed discrete subspace of a Lindelöf space is easily shown to be countable. $\square$

Example 3.1 shows that "locally connected" cannot be eliminated from this corollary, even if we replace it with "locally countable." Let $X = (\omega_1, \tau) \in \mathfrak{T}$. A partition of X into clopen sets has to have exactly one member of the partition [call it G] contain a co-countable subset of Λ. Obviously, G contains uncountably many successor ordinals. So there exists a positive integer k such that $\{\gamma+k : \gamma \in \Lambda \text{ and } \gamma+k \in G\}$ is uncountable. But this is a closed discrete subspace of X.

Problem 2. If a locally compact space has a monotonically normal compactification, is it the topological direct sum of ω_1-compact subspaces?

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