Some screenable anti-Dowker spaces

Normal spaces that are not countably paracompact, and countably paracompact, regular spaces that are not normal, are called Dowker and anti-Dowker spaces, respectively. The twin questions of whether there is a screenable Dowker space or a screenable anti-Dowker space are of special interest due to the 1955 theorem of Nagami:

**Theorem A.** A space is paracompact if, and only if, it is screenable, normal, and countably paracompact.

Most of the research surrounding this theorem has had to do with whether countable paracompactness could be dispensed with. This was a very demanding problem, and was first solved by Mary Ellen Rudin using the extra set-theoretic axiom ♦++ [1], and fifteen years later in ZFC by Zoltán Balogh [2].

I have not seen much attention paid to the complementary problem of whether there can be a screenable anti-Dowker space — equivalently, by Nagami’s theorem, a screenable, regular, countably paracompact space that is not paracompact. The purpose of this note is to describe two examples: a simple one obtained under the set-theoretic hypothesis $q_1 > \omega_1$ and a more complicated ZFC example which is also paralindelöf (that is, every open cover has a locally countable open refinement). Both examples are metacompact and subparacompact, so just about every “slight” weakening of paracompactness is realized.

**Definition.** A *countably paracompact* space is a space such that every countable open cover has a locally finite open refinement. A *screenable [metacompact] [subparacompact]* space is one such that every open cover has a σ-disjoint open [point-finite open] [σ-locally finite closed] refinement.

Our examples apply a simplification of the Wage machine [see [3] for a description] to two well-known examples of normal spaces that are not collectionwise normal. Each one has a discrete family $\mathcal{H}$ of closed sets which cannot be put into disjoint open sets. In both spaces, each $H \in \mathcal{H}$ is equipped with a pair of “wings,” and our machine replaces $\bigcup \mathcal{H}$ with two disjoint copies of $\bigcup \mathcal{H}$ and gives each of the two copies of each $H$ one of the “wings” associated with $H$.

The set-theoretic cardinal $q_0 [q_1]$ is the least cardinal $\lambda$ such that some [every] subset of the real line of cardinality $\lambda$ fails to be a Q-set (i.e., a subset $Q$ of $\mathbb{R}$ such that every subset of $A$ is a $G_\delta$ — equivalently, an $F_\sigma$ in the relative topology of $Q$). A well-known consequence of Martin’s Axiom is that $q_0 = c$.

**Example 1.** $[q_1 \geq \omega_2]$ Let $X$ be Heath’s tangent V space. The underlying set is the closed upper half plane. Points not on the x-axis are isolated, points on the x-axis have “tangent V” basic neighborhoods. Given $p = (x, 0)$, these V’s are formed by line segments of length $1/n$ ($n \in \omega \setminus \{0\} = \mathbb{N}$) at 45 degree angle to the
As is well known, this space is a metacompact Moore (hence subparacompact) space that is neither normal nor collectionwise Hausdorff (cwH).

Let $Y$ be a subspace of $X$ consisting of the upper half plane and a Q-set of cardinality $\omega_1$ on the $x$-axis. This is a standard example of a metacompact Moore space that is normal but not cwH, obtained under extra set-theoretic hypotheses. In normal spaces, countable paracompactness is equivalent to countable metacompactness, so $Y$ is countably paracompact. $Y$ is not cwH because the points on the $x$-axis cannot be expanded to a disjoint collection of open sets. This is a simple cardinality argument using the usual topology on the $x$-axis, which is hereditarily Lindelöf.

Let $Y^\dagger$ be the space obtained by replacing the Q-set $Q$ on the $x$-axis by two copies, $Q_0 = \{\langle p, 0 \rangle : p \in Q \}$ and $Q_1 = \{\langle p, 1 \rangle : p \in Q \}$, and giving the points of $Q_0$ the “left wing” of each of the basic tangent V’s, with the “right wings” going to the corresponding points of $Q_1$. That is, the neighborhoods of $\langle (r, 0), 0 \rangle$ are the sets that contain some $V_n(r, 0) = \{\langle (r, 0), 0 \rangle \} \cup \{(x, y) : 0 < y < 1/n, x = r - y\}$, while the neighborhoods of $\langle (r, 0), 1 \rangle$ are the sets that contain some $V_n(r, 1) = \{\langle (r, 0), 1 \rangle \} \cup \{(x, y) : 0 < y < 1/n, x = r + y\}$.

Given an open cover $U$ of $Y^\dagger$, we can refine it to $W_0 \cup W_1 \cup W_2$, where, for $i = 0, 1$, is a family of “wings” one apiece for the points of $Q_i$, left for $Q_0$ and right for $Q_1$; and $W_2$ is the set of all singletons of $\mathbb{R} \times (0, \infty)$ that are not covered by $W_0 \times W_1$. This is a cover of order 2.

The projection map $\pi : Y^\dagger \to Y$ is clearly closed and at most 2-to-1, so countable paracompactness is easily seen to be inversely preserved. Normality fails because if $Q_0$ and $Q_1$ could be put into disjoint open sets $U_i$, then the whole of $Q_0 \cup Q_1$ could be expanded to a disjoint collection of open sets in $Y^\dagger$, and hence $Q$ would also have such an expansion in $Y$, a contradiction.

**Example 2.** Caryn Navy’s space $N$ is described in [4], which includes the proofs that $N$ is paralindelöf and normal but not collectionwise normal. Navy’s space has Baire’s zero-dimensional space $F = D^{\omega_1}$ of weight $\kappa_1$ (where $D$ is the discrete space with underlying set $\omega_1$) playing the role $Q$ played in Example 1. However, rather than a single $V$ at each point, the basic neighborhoods are the sets $[\sigma] = \{f \in F : \sigma \subseteq f\}$ together with a pair of wings attached to each one. Each $\sigma$ is a finite sequence of elements of $\omega_1$. The wings reach into a subspace of isolated points in a counterpart of the open upper half plane in Example 1.

This counterpart is a swarm of copies of $G = 2^\mathcal{T}$ where $\mathcal{T}$ is the family of open subsets of $F$. These are indexed by entwined pairs $\langle \rho, \tau \rangle$, and the right wing of $[\sigma]$ reaches into the copies of $G$ indexed by the $\rho$’s extending $\sigma$, while the left wing into the ones indexed by the $\tau$’s that extend $\sigma$.

One detail missing from [4] is that $N$ is a $\sigma$-space, i.e., it has a $\sigma$-discrete network. This follows from the easy facts that the isolated points form an $F_\sigma$ and that the $[\sigma]$
form a $\sigma$-discrete base of clopen sets for the relative topology on the closed subspace $F$. Thus $N$ is perfectly normal and subparacompact.

With $N^\dagger$ defined analogously to $Y^\dagger$, one proves analogously that $N^\dagger$ is screenable, metacompact, subparacompact, and countably paracompact, but not normal.


