LOCALLY COMPACT, LOCALLY CONNECTED, MONOTONICALLY NORMAL SPACES

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Abstract. A remarkable global structure theorem is shown for locally compact, locally connected, monotonically normal spaces. The conclusion of the theorem is strong enough to imply several properties that follow from monotone normality, including collectionwise normality, countable paracompactness (in fact, that every countable open cover has a star-finite open refinement) and a powerful theorem of Balogh and Rudin on refinement of open covers of a monotonically normal space. It also implies that every locally compact, locally connected, monotonically normal space has a monotonically normal one-point compactification.

1. Introduction

Compact monotonically normal spaces have remarkably strong properties, not hinted at in the definition of monotone normality (Definition 2.4). Foremost of these is the theorem that they are precisely the continuous images of compact ordered spaces. The monumentally difficult proof of this fact, known as Nikiel's Conjecture, was found by Mary Ellen Rudin [13].

Locally compact spaces with monotonically normal one-point compactifications also have a very strong structure theory, exemplified by the following recent theorem by the author [11], where it is also shown to characterize these spaces among the locally compact, monotonically normal spaces:

Theorem 1.1. *Every locally compact space with a monotonically normal one-point compactification is a topological direct sum of σ-ω-bounded spaces, each of which is the union of an open Lindelöf subset and of a discrete family of closed, ω-bounded subsets. If the space is totally disconnected, it is a topological direct sum of ω-bounded subspaces.*

Recall that *an ω-bounded space* is one in which every countable set has compact closure. As is most often the case in general topology, the prefix σ- denotes being the countable union of what follows. To say that a space X is *the topological direct sum* of subspaces satisfying Property P is to say that X has a partition into open (hence also closed) subspaces satisfying P. A wealth of topological properties are satisfied by X itself if they are satisfied by each of the summands.

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The main result of this paper, Theorem 2.1, is that locally compact, locally connected, monotonically normal spaces have the properties in Theorem 1.1 with some added features. This was the assumption used in [4] to show that they have monotonically normal one-point compactifications.

This is significant because monotone normality is very often not preserved in passing to the one-point compactification. In [4], two classes of totally disconnected, locally compact spaces are described where it is not preserved. The spaces in one class are σ-ω-bounded, but not the topological direct sum of ω-bounded subspaces; the ones in the other are not even the topological direct sum of ω_1-compact subspaces. [Recall that a space is called ω_1-*compact* if it is of *countable extent*, meaning that every closed discrete subspace is countable.]

In [11] an intermediate class in models of ♣ is described: ω_1-compact but not the direct sum of σ-ω-bounded subspaces. On the other hand, it is also shown in [11] that the PID axiom implies that every locally compact, ω_1-compact, monotonically normal space is σ-ω-bounded.

It is hoped that [11] and this paper will be the first steps towards a comprehensive structure theory of locally compact, monotonically normal spaces. This theme will be taken up again in the last section.

Theorem 1.1 can be used to analyze compact monotonically normal spaces by a process of successively removing points to give locally compact subspaces. The author has begun such a process in the locally connected case [10], in hopes of substantially simplifying the proof of Nikiel's Conjecture for this case, which is featured in the following beautiful generalization of the Hahn-Mazurkiewicz theorem:

Theorem A. *The continuous images of generalized arcs are precisely the compact, connected, locally connected, monotonically normal spaces.*

With $[0, 1]$ in place of generalized arcs, and metrizability in place of monotone normality, this is the classic Hahn-Mazurkiewicz theorem. [A *generalized arc* is a linearly orderable compact connected space.] A thorough and excellent survey of the history of these theorems, and also of Nikiel's Conjecture, was provided by the late Sibe Mardešić [5].

All through this paper, "space" means "Hausdorff topological space"; but attention is confined almost exclusively to locally compact (hence Tychonoff), hereditarily normal spaces, especially monotonically normal ones.

2. The main theorem: statement and consequences

The main result of this paper is formulated with one of the two standard (and very different) definitions of locally connected spaces in mind: they are spaces in which the components of every open set are open. This makes them the topological direct sum of their components.

In a sequel to this paper, it will be shown how, in $MM(S)[S]$ models, the following theorem (which is the main result of this paper) extends to the class of locally compact, locally connected, hereditarily normal spaces. It is an unsolved problem whether they are all monotonically normal in such models.

Theorem 2.1. *Let X be a locally compact, locally connected, monotonically normal space. Every component of X has a Lindelöf subspace L such that $X \setminus L$ is the union of a (possibly empty) countable discrete collection $\mathcal{S}$ of closed, connected, ω-bounded, noncompact subspaces.*

Moreover, if $\mathcal{S}$ is not empty, then each $S \in \mathcal{S}$ includes uncountably many cut points of the component and has exactly one point in the closure of L.

Moreover, each $S \in \mathcal{S}$ is a "string of beads" in which there is a set C of cut points of the whole component, such that C is homeomorphic to an ordinal of uncountable cofinality, and each "bead" is the connected 2-point compactification of an open subspace whose boundary consists of the two extra points, successive members of C. The union of these "beads" comprises the entire string.

Each member S of $\mathcal{S}$ (if any) maps onto a connected ordered space that is naturally associated with the set C that goes with S. This is one of a class of spaces introduced by Cantor in the same 1883 paper [2] where he introduced infinite ordinals. Cantor order-embedded each ordinal θ into a connected, linearly ordered space (here denoted $\theta^{\#}$) by inserting a copy of $(0, 1)$ between each ordinal and its successor. The $+$ operation extends in the obvious way to give us $\alpha + r$ for all $r \in \mathbb{R}^+$, and $\alpha \in \theta$.

Corollary 2.2. *Let $S \in \mathcal{S}$ and $C \subset S$ be as in Theorem 2.1. Let θ be the unique ordinal with which C is order-isomorphic when C is given the disconnection order, $x < y$ iff y is in the component of $X \setminus \{x\}$ that includes a tail (final segment) of C, and let $f : C \to \theta$ be the unique surjective order-preserving function.*

Then there is a continuous surjective function $g : S \to \theta^{\#}$ extending f.

Proof. Identify C with θ for notational purposes. For each $\alpha \in \theta$ let B_α be the "bead" whose boundary is $\{\alpha, \alpha + 1\}$. Let $g_\alpha : B_\alpha \to [0, 1]$ be a continuous function such that $g_\alpha(\alpha) = 0$ and $g_\alpha(\alpha + 1) = 1$. Let $g : S \to \theta^{\#}$ be the function that takes $\alpha + r$ to $\alpha + g_\alpha(\alpha + r)$ for all $r \in [0, 1]$. This is a well-defined function since $g_\alpha(\alpha + 1) = \alpha + 1 = g_{\alpha+1}(\alpha + 1 + 0)$ for all α. It is surjective since each B_α is connected and so each g_α is surjective. Continuity is easy to show if one uses basic open intervals of the form $[0, \epsilon)$ and $(\beta + 1 - \epsilon, \alpha + \epsilon)$ for successors $\alpha = \beta + 1$, and $(\xi, \gamma + \epsilon)$ when γ is a limit ordinal and $\xi < \gamma$. $\square$

The subspace L of Theorem 2.1 could be taken to be closed. Its boundary is the set $\{p_n : n \in \omega\}$, where $\overline{L} \cap S_n = \{p_n\}$, so adding it preserves the Lindelöf property. The closure of L is connected as well: a disconnection of L would put each p_n into one component or another, and when the S_n are added on, we get a disconnection of the component of X, a contradiction.

This argument can be used to show that if L is just assumed to be paracompact in the main theorem, then L is actually Lindelöf. This is because of a powerful old theorem [3, 5.1.27]:

Theorem 2.3. *Every locally compact, paracompact space is the topological direct sum of (open, hence also closed) Lindelöf subspaces.*

For the same reason, the main theorem could also omit all mention of the countability of $\mathcal{S}$, and yet the actual wording would be a quick corollary.

The subspace L can, alternatively, be made open by starting with a preliminary choice, and taking its union with the first ω beads along each bead string. The union is σ-compact and hence Lindelöf. And, since it includes the points p_n, it is also connected.

The following definition of monotone normality is given for the sake of completeness:

Definition 2.4. A space X is *monotonically normal* if there is an operator $G(\_,\_)$ assigning to each ordered pair $\langle F_0, F_1\rangle$ of disjoint closed subsets an open set $G(F_0, F_1)$ such that
(a) $F_0 \subset G(F_0, F_1)$
(b) [normality] $G(F_0, F_1) \cap G(F_1, F_0) = \emptyset$.
(c) [monotonicity] If $F_0 \subset F_0'$ and $F_1' \subset F_1$ then $G(F_0, F_1) \subset G(F_0', F_1')$

There is a very different characterization, due to Borges [11, Theorem 4.5], that makes it easy to show that *monotone normality is a hereditary property.* But instead of using either definition to show the main theorem, we will be using three strong properties of monotonically normal spaces.

Property 2.5. *Every monotonically normal space is (hereditarily) collectionwise normal.*

Definition 2.6. A space is *collectionwise normal (CWN)* [resp. *strongly collectionwise Hausdorff (scwH)]* if for every discrete family $\mathcal{F}$ of closed sets [resp. singletons] there is a discrete family $\{U_F : F \in \mathcal{F}\}$ of open sets such that $F \subset U_F$ and such that $U_{F_1} \cap U_{F_2} = \emptyset$ whenever $F_1 \neq F_2$.

The usual definition of CWN has "disjoint" in place of "discrete," but it is an elementary, well-known exercise that the two definitions are equivalent [3].

Property 2.7. *In a monotonically normal space, every open Lindelöf subset has (Lindelöf closure and) a hereditarily Lindelöf boundary.*

This second property follows quickly from the first property and from the fact that every monotonically normal space of countable spread (i.e., every discrete subspace is countable) is hereditarily Lindelöf [12]. Property 2.5 implies that the boundary of an open Lindelöf subspace L is of countable spread, in the following way. Let D be a discrete subspace of $\overline{L} \setminus L$. Then D is closed discrete in the relative topology of $L \cup D$; and Property 2.5 implies it can be expanded to a discrete collection of relatively open sets in $L \cup D$, each meeting L; but every discrete collection of sets in a Lindelöf space is countable. [When applied to collections of sets rather than to point sets in a topological space, "discrete" means that every point of the space has a neighborhood meeting at most one member of the collection.]

The third key property is a striking result of Balogh and Rudin [1].

Property 2.8. *Let X be monotonically normal, and let $\mathcal{U}$ an open cover of X. Then $X = V \cup \bigcup \mathcal{E}$, where $\mathcal{E}$ is a discrete family of closed subspaces, each homeomorphic to a stationary subset of a regular uncountable cardinal, and $V = \bigcup \mathcal{V}$ is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.*

In locally compact spaces, the stationary subsets in $\mathcal{E}$ can be taken to be the cardinals themselves (see Theorem 3.4 below). This is what will be directly used here.

Theorem 2.1 is so strong that it implies both properties 2.5 and 2.8. In fact it shows a little more.

Theorem 2.9. *Let X be a locally compact, locally connected space that has the properties in the conclusion of Theorem 2.1. Then X is collectionwise normal (CWN) and countably paracompact.*

Proof. It is enough to show 2.10 for the individual components since they are (cl)open by local connectedness. It is an elementary exercise to show that a topological direct sum is CWN iff each summand is CWN, and the same is true of countable paracompactness, which means that every countable open cover has a locally finite refinement.

By Theorem 2.1, each component is the union of countably many closed countably compact subsets; for example, the subspace L is the union of countably many compact ones. So every discrete collection of closed subsets is countable, and CWN follows from the elementary exercise that every normal space is "$\aleph_0$-collectionwise normal."

As for countable paracompactness, we can actually show that every countable open cover $\mathcal{U}$ has a star-finite open refinement. For each n, let $\{p_n\} = S_n \cap \overline{L}$. Now, countable compactness of S_n implies that $S_n \setminus \{p_n\}$ has a finite cover $\mathcal{W}_n$ by (open) sets of the form $U \cap S_n \setminus \{p_n\}$. Then $\mathcal{W} = \bigcup_{n=0}^{\infty} \mathcal{W}_n$ is star-finite.

Now if $L \setminus \bigcup \mathcal{W}$ is compact, then it has a finite cover $\mathcal{U}_L$ by sets of the form $U \cap L$ where $U \in \mathcal{U}$, and then $\mathcal{W} \cup \mathcal{U}_L$ is a star-finite open refinement of $\mathcal{U}$.

In the general case, we fall back on the theorem [3, 5.3.11] that a regular Lindelöf space is *strongly paracompact*, which means that every open cover has a star-finite open refinement. For each n, let V_n be an open neighborhood of p_n in the relative topology of S_n. Let $\mathcal{G}$ be a (countable) star-finite relatively open refinement of the trace of $\mathcal{U}$ on $\overline{L} = L \cup \{p_n : n \in \omega\}$. For each n, and each of the finitely many G in $st(p_n, \mathcal{G}) = \{G \in \mathcal{G} : p_n \in G\}$, let $G' = G \cup (V_n \cap S_n)$. Since p_n is a cut point of X, G' is an open neighborhood of p_n. Let $\mathcal{G}' = \{G' : G \in st(p_n, \mathcal{G}), n \in \omega\}$. Then $\mathcal{W} \cup \mathcal{G}'$ is a star-finite open refinement of $\mathcal{U}$. □

Theorem 2.10. *Let X be a locally compact, locally connected space that has the properties in the conclusion of Theorem 2.1. If $\mathcal{U}$ is an open cover of X, then $X = \bigcup \mathcal{V} \cup \bigcup \mathcal{W}$, where $\mathcal{W}$ is a discrete family of copies of regular uncountable cardinals, and $\mathcal{V}$ is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.*

Proof. This too can be proven just by using the components: the discrete families and partial refinements can be done on each component separately and the union taken over all components. So we assume X is connected.

If X is Lindelöf, then $\mathcal{V}$ can simply be a countable subcover of $\mathcal{U}$. Otherwise, by Theorem 2.1, $X = \bigcup\{S_n : n \in \omega\} \cup L$ where L is an open Lindelöf subspace and $\mathcal{S} = \{S_n : n \in \omega\}$ is a collection of "bead strings."

Fix $S_n \in \mathcal{S}$ and let C_n be as in Theorem 2.1, identified with an ordinal θ_n of uncountable cofinality. For each $\alpha \in \theta_n$, let B_n^α be the "bead" whose boundary consists of $\{\alpha, \alpha + 1\}$. Let $\mathcal{V}(\alpha, n)$ be the set of intersections of the interior of B_n^α ($= B_n^\alpha \setminus \{\alpha, \alpha+1\}$) with a finite $\mathcal{U}$-subcover of B_n^α. An elementary application of the axiom of choice shows that

$$\bigcup\{\mathcal{V}(\alpha, n) : \alpha \in \theta_n, n \in \omega\}$$

is the union of a countable family of disjoint open sets. Then a countable subcover of L can be trivially added. □

3. The case of Lindelöf degree ω_1

This section deals with a non-Lindelöf space X as in the main theorem, with the additional properties of being connected, and of Lindelöf degree ω_1 (that is, every open cover has a subcover of cardinality $\leq \aleph_1$). This is symbolized by $\ell(X) = \omega_1$. In a locally compact space, the latter property is easily shown to be equivalent to being the union of $\aleph_1$ open sets with compact closures. Such open sets form the most natural base for a locally compact space, but it is often useful to use other kinds of basic open sets. For example:

Lemma 3.1. *Every locally compact space has a base of open Lindelöf subsets. If in addition the space is locally connected, then the open Lindelöf connected subsets form a base.*

Proof. The first sentence has a quick proof using the fact that locally compact spaces are Tychonoff. But here is a unified proof for both sentences.

Let X be locally compact. For each $p \in X$ and open set G such that $p \in G$, let G_0 be an open set such that $\overline{G_0}$ is compact and $p \in G_0$, $\overline{G_0} \subset G$. Suppose G_n has been defined so that $\overline{G_n}$ is compact and $\overline{G_i} \subset G_n$ for all $i < n$. Cover the boundary of G_n with finitely many open subsets whose closures are compact and contained in G. Let G_{n+1} be the union of G_n with these finitely many open sets; then $\overline{G_n} \subset G_{n+1}$ and the induction proceeds through ω. Let $H = \bigcup_{n=0}^{\infty} G_n$; H is an open Lindelöf (because σ-compact) neighborhood of p contained in G.

If X is, in addition, locally connected, let G_0 be connected, and let all the open sets covering the boundary of each G_n be connected and meet G_n. An easy induction shows that each G_n is connected and so H is connected. $\square$

Recall the concept of a canonical sequence [7].

Definition 3.2. A *canonical sequence in a space* X is a well-ordered family $\Sigma = \langle X_\xi : \xi \in \omega_1 \rangle$ of open subspaces of X such that $\overline{X_\xi}$ is Lindelöf and $\overline{X_\xi} \subset X_\eta$ for all $\xi < \eta$ in ω_1, and $X_\alpha = \bigcup \{X_\xi : \xi < \alpha\}$ for all limit ordinals α.

With a slight abuse of language, we let $\bigcup \Sigma$ mean $\bigcup \{X_\xi : \xi \in \omega_1\}$.

The proof of the main theorem 2.1 begins with the construction of a canonical sequence in non-Lindelöf components of X as described. For components of Lindelöf degree $> \omega_1$, the later stages of the construction are given in the following section.

The proof of the following theorem appears in [7], but is included for the convenience of the reader.

Theorem 3.3. *Let X be a locally compact space, such that every Lindelöf subset has Lindelöf closure. If $\ell(X) = \omega_1$, then there is a canonical sequence $\Sigma = \langle X_\alpha : \alpha \in \omega_1 \rangle$ for X such that $\bigcup \Sigma = X$, and such that each X_α is connected, open, Lindelöf, and properly contained in X_β whenever $\beta > \alpha$.*

Proof. Let X_0 be a Lindelöf open subspace. By Property 2.7, $\overline{X_0}$ is Lindelöf.

In general, if X_α has been defined, we cover its boundary with countably many Lindelöf open sets, each of which meets the boundary of X_α, and let $X_{\alpha+1}$ be the union of X_α with the added sets. If α is a limit ordinal and X_ξ has been defined for all $\xi < \alpha$, let $X_\alpha = \bigcup \{X_\xi : \xi < \alpha\}$. Since α is countable, X_α is Lindelöf, etc. $\square$

The following strengthening of Property 2.8 for locally compact spaces was a key step in the proof of Theorem 1.1 in [11].

Theorem 3.4. *Let $\mathcal{U}$ be an open cover of a locally compact monotonically normal space X. Then X has a discrete collection $\mathcal{C}$ of closed copies of regular uncountable cardinals, such that $X \setminus \bigcup \mathcal{C}$ has a σ-disjoint cover $\mathcal{V}$ by open sets refining the trace of $\mathcal{U}$ on $X \setminus \bigcup \mathcal{C}$.*

Let $\mathcal{U}$ be a cover of $X = \bigcup \Sigma$ by open sets with compact closures, so that each is a subset of some X_α. As in Theorem 3.4, let $X = \bigcup \mathcal{V} \cup \bigcup \mathcal{C}$ where $\mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_n$, and each $\mathcal{V}_n$ is a disjoint collection of open sets. By taking components, we may assume each member of each $\mathcal{V}_n$ is connected.

Each member of $\mathcal{C}$ is a copy of ω_1, inasmuch as each C in $\mathcal{C}$ is countably compact and noncompact, and the cofinality of Σ is ω_1. Similarly, there is some $\alpha_C < \omega_1$ such that $C \cap X_\alpha \neq \emptyset$ for all $\alpha > \alpha_C$.

For ease of comparison with the following section, we make a change of notation $Y = X$, $Y_\alpha = X_\alpha$ for all $\alpha \in \omega_1$. Let

$$\Gamma = \{\alpha : \text{ for all } V \in \mathcal{V}, (V \cap Y_\alpha \neq \emptyset \implies V \subset Y_\alpha)\}$$

Lemma 3.5. Γ *is a club.*

Proof. Closedness is trivial, and we use a standard leapfrog argument to show that it is unbounded. Let α_0 be any countable ordinal. The members of each $\mathcal{V}_n$ are connected, and $\overline{Y_{\alpha_0+1}} \setminus Y_{\alpha_0+1}$ is hereditarily Lindelöf and disconnects Y, and $\mathcal{V}_n$ is a disjoint collection for each n. Therefore, at most countably many members of each $\mathcal{V}_n$ (and hence of $\mathcal{V}$) that meet Y_{α_0} will also meet $Y \setminus Y_{\alpha_0+1}$. Since these members of $\mathcal{V}$ have compact closure, they are all contained in some Y_β, $(\beta < \omega_1)$. Let the least such $\beta > \alpha_0$ be α_1. Now by induction we get a strictly increasing sequence $\langle \alpha_n : n \in \omega \rangle$ with α_{n+1} defined from α_n in the same way that α_1 was defined from α_0. Let $\alpha = sup_n \alpha_n$. Then $\alpha \in \Gamma$, because $Y_\alpha = \bigcup \{Y_{\alpha_n} : n \in \omega\}$. $\square$

By definition of Γ, the members of $\mathcal{V}$ miss all the boundaries $F_\alpha = \overline{Y_\alpha} \setminus Y_\alpha$ such that $\alpha \in \Gamma$. The only sets in the cover of Y given by Property 2.8 that meet these boundaries are the members of $\mathcal{C}$. Since they are a discrete family, each point of F_α is isolated in the relative topology of F_α whenever $\alpha \in \Gamma$.

As our first step in pinning down the bead strings in the main theorem 2.1, we show what might be called the "Thorn Lemma":

Lemma 3.6. *Let X be a locally compact, locally connected space with a canonical sequence $\Sigma = \langle X_\alpha : \alpha \in \omega_1 \rangle$. Let α be a limit ordinal and let $p \in \overline{X_\alpha} \setminus X_\alpha$. If p is isolated in the relative topology of $\overline{X_\alpha} \setminus X_\alpha$, and G is an open neighborhood of p with compact closure, such that $G \cap \overline{X_\alpha} \setminus X_\alpha = \{p\}$, then there exists $\xi < \alpha$ such that $\overline{G} \cap X_\alpha \setminus X_\nu$ is (nonempty, and) clopen in the relative topology of $X_\alpha \setminus X_\nu$ whenever $\xi \leq \nu < \alpha$.*

Proof. Since G is a neighborhood of p, and Σ is canonical, $G \cap X_\alpha \setminus X_\mu$ is nonempty for all $\mu < \alpha$. If there is no such ξ, then there is an ascending sequence $\langle \xi_n : n \in \omega \rangle$ whose supremum is α and for which there exists $z_n \in \overline{G} \setminus G \cap (X_\alpha \setminus X_{\xi_n})$. Any accumulation point of $Z = \{z_n : n \in \omega\}$ must be in $\overline{G}$ and can only be p itself, but G is a neighborhood of p missing Z, contradicting compactness of $\overline{G}$. $\square$

For each limit ordinal $\alpha \in \Gamma$ and each $p \in F_\alpha$ there is $\xi < \alpha$ and a compact, connected neighborhood D_p of p such that $D_p \cap \overline{Y_\alpha} \setminus Y_\xi$ is clopen in the relative topology of $\overline{Y_\alpha} \setminus Y_\xi$. In each $C \in \mathcal{C}$, the PDL gives a uniform $\xi_C \in \Gamma$ that works for an unbounded collection of points of C. We may assume, since $\mathcal{C}$ is discrete, that D_p misses all members of $\mathcal{C}$ besides the member in which p is located. And this means that if q is the unique member of $F_\alpha \cap C$, D_q must meet every F_ν, $\xi_C \leq \nu \leq \alpha$, $\nu \in \Gamma$, in the singleton $F_\nu \cap C$. It is now easy to see that each point of C is a cut point of X, and that, more generally,

$$S_C = \bigcup \{D_q \setminus Y_{\xi_C} : q \in C \cap F_\alpha \text{ and } \alpha > \xi_C)\}$$

is a bead string as in the main theorem. Moreover, since S_C is connected, it must meet every F_α such that $\alpha \geq \xi_C$.

Let $\mathcal{S} = \{S_C : C \in \mathcal{C}\}$. Let p_C be the least point of S_C for all $C \in \mathcal{C}$. Then $\{p_C : C \in \mathcal{C}\}$ is the boundary of $\bigcup \mathcal{S}$, and is closed discrete. Let M be the complement of $\bigcup \mathcal{S}$ in X; then $\{p_C : C \in \mathcal{C}\}$ is also the boundary of M.

Since X is connected and $bd(M)$ is closed discrete, $\overline{M}$ is connected by the argument preceding Theorem 2.3. We next use the following well-known fact.

Lemma 3.7. *Let E be a stationary subset of a regular uncountable cardinal. Every σ-disjoint family of bounded open subsets of E has nonstationary union.*

The proof is an easy application of the Pressing-Down Lemma (PDL) applied to the ordinals of E that are limit points of E, and the elementary fact that the union of countably many nonstationary sets is nonstationary.

Now $\overline{M}$ contains no closed copy of a stationary subset of an uncountable regular cardinal, because $\mathcal{V}$ traces a σ-disjoint family of relatively open subsets on it, and each member of $\mathcal{V}$ has compact closure, making the closures of the traces compact subsets of $\overline{M}$. So $\overline{M}$ is paracompact by the following consequence [1, Theorem I] of Property 2.8.

Theorem 3.8. *A monotonically normal space is paracompact if, and only if, it does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

By Theorem 2.3 and the discussion around it, $\overline{M}$ is Lindelöf. It follows that $\bigcup \mathcal{S}$ is countable, completing the proof of the main theorem 2.1 in the case where $\ell(X) = \omega_1$.

4. Transition to the general case

In the general case where $\ell(X)$ is arbitrary, the construction in the preceding section gives a proper open subspace Y inside each X of Lindelöf degree $> \omega_1$. The boundary of Y is composed of points in the closures of the respective closed copies of ω_1, the sets $C \in \mathcal{C}$. The closure of each C is its Stone-Čech compactification, which is well known to be its one-point compactification. But there is no reason why finitely many different C's should not share the same extra point [although infinitely many cannot, because that would put the extra point in the closure of L]. One might even have all the extra points tied together as in the following example:

Example 4.1. *Let $\Pi = \Pi_0 \cup \Pi_1$, where $\Pi_1 = [0, \infty) \times \{\omega_1\}$ and $\Pi_0 = \omega \times L$, where L is the closed long ray $\omega_1^\#$ defined by Cantor. The relative topology on Π_0 is the product* *topology. Points of* Π_1 *have basic open neighborhoods as follows. If* $r \notin \omega$*, let* $B_\epsilon(r, \omega_1) = (r - \epsilon, r + \epsilon) \times \{\omega_1\}$*, and let the base be* $\{B_\epsilon(r, \omega_1) : \epsilon < 1/2\}$*. If* $n \in \omega$*, let*

$$B_{\epsilon,\alpha}(n, \omega_1) = [(n - \epsilon, n + \epsilon) \times \{\omega_1\} \cap \Pi_1] \cup [\{n\} \times (\alpha, \omega_1)]$$

and let the base be $\{B_{\epsilon,\alpha}(n, \omega_1) : \epsilon < 1/2\}$. [Of course, $\cap \Pi_1$ in the displayed formula is redundant except when $n = 0$.

It is easy to see that Π is locally compact, locally connected and connected. Monotone normality is easy to show using the Borges criterion [11, Theorem 4.5]. If we let $\Pi_\alpha = \omega \times [0, \alpha)$ for $\alpha \in \omega_1$, we have a canonical sequence for Π_0 whose union is dense in Π.

Subspaces like Π are inconvenient for proving the main theorem 2.1 in a straightforward manner, because the bead strings we are following in the initial parts of the induction may not be parts of the ultimate bead strings. There are various ways of overcoming this difficulty, and the one we will use in this section was chosen because it lays part of the groundwork for the sequel to this paper.

This method modifies the choice of Y_α so that the bead strings that do not stop within Y will have the extra point at the top as a cut point of the entire component of which Y is a subspace. This process will continue to operate in the following section.

From now on, we will assume X is connected in addition to being locally compact, locally connected, and monotonically normal. We extend the concept of a canonical sequence as follows.

Definition 4.2. Let θ be an ordinal of uncountable cofinality and let Γ be a closed unbounded ("club") subset of θ. A *general* Γ*-sequence in a space* X is a well-ordered family $\Sigma = \langle X_\xi : \xi \in \Gamma \rangle$ of open subspaces of X such that $\overline{X_\xi} \subset X_\eta$ for all $\xi < \eta$ in Γ, and $X_\alpha = \bigcup\{X_\xi : \xi < \alpha\}$ for all limit points α of Γ. If in addition, $\overline{X_\xi}$ is Lindelöf for all $\xi \in \Gamma$, then Σ will be called *a canonical* Γ*-sequence.*

As before, we let $\bigcup \Sigma$ mean $\bigcup\{X_\xi : \xi \in \Gamma\}$ and will usually suppress the Γ- prefix.

It is useful to picture X as a tree, using the components of each open subspace of the form $X \setminus \overline{X_\xi}$ in a general Γ-sequence as the "limbs" that grow from the "fork" at the boundary of X_ξ. Because X is locally connected, each component is open. Note that "limbs" never re-connect: if V is a component of $X \setminus \overline{X_\eta}$ and W is a component of $X \setminus \overline{X_\xi}$ and $\eta < \xi$ then either $W \cap V = \emptyset$ or $W \subset V$; and then, $\overline{W} \subset V$ because X_ξ is an open set containing $\overline{V} \setminus V$. In general, the boundary of each component of $X \setminus \overline{X_\alpha}$ is a nonempty subset of $\overline{X_\alpha} \setminus X_\alpha$.

Our inductive construction will produce a canonical sequence for X for which there are only countably many really large limbs at each fork, as in the following general lemma:

Lemma 4.3. *Let* X *be a locally connected space, with a canonical sequence* Σ *in which each* X_ξ $(\xi \in \Gamma)$ *has hereditarily Lindelöf boundary. Then all but countably many components of* $X \setminus \overline{X_\xi}$ *are subsets of* $X_{\xi+1}$ *whenever* $\xi \in \Gamma$*.*

Proof. Let $\mathcal{V}$ be the (disjoint) collection of all components of $X \setminus \overline{X_\xi}$ that meet $X \setminus X_{\xi+1}$. Connectedness of each $V \in \mathcal{V}$ implies that V meets $\overline{X_{\xi+1}} \setminus X_{\xi+1}$. But this set is hereditarily Lindelöf by Property 2.7 and so cannot contain a family of more than countably many disjoint (relatively) open sets, so $\mathcal{V}$ must be countable. $\square$

Rather than letting Σ be obtained by an arbitrary application of the proof of Theorem 3.3, we will utilize Lemma 4.3 and Theorem 4.4 below to produce a fast-growing "tree" $\{Y_\alpha : \alpha < \omega_1\}$ inside X.

To ensure that each Y_α is connected, we utilize a theorem and a concept in [14, 26.14, 26.15]. If S is a connected space, and $\mathcal{U}$ is any open cover of S, then any two points p, q of S are *connected by a simple chain in* $\mathcal{U}$.

That is, there is a finite sequence $U_0, \dots U_n$ of members of $\mathcal{U}$ such that $p \in U_0 \setminus U_1$, $U_i \cap U_j \neq \emptyset \iff |i - j| \leq 1$, and $q \in U_n \setminus U_{n-1}$.

Theorem 4.4. *Let X be a locally compact, locally connected space in which every open Lindelöf subset has Lindelöf closure, and let V be a connected open subspace of X with Lindelöf boundary. Then there is a connected, Lindelöf open neighborhood $H(V)$ of $bd(V)$ in X.*

Proof. Using Lemma 3.1, cover $bd(V)$ with countably many Lindelöf, connected open sets W_n $(n \in \omega)$. Let $W = \bigcup\{W_n : n \in \omega\}$. The boundary of $W \cap \overline{V}$ is the union of two disjoint subsets, $bd(V)$ and the boundary B in V of $W \cap V$. If $W \cap V$ is not connected, each component meets B, so that if H is a connected, Lindelöf, open subset of V containing B, then $H(V) = W \cup H$ is as desired.

To obtain H, let $\mathcal{D}$ be a cover of V by connected Lindelöf open subsets and let $\{D_n : n \in \omega\} \subset \mathcal{D}$ cover B. For each $i \in \omega$ let $\mathcal{C}_i$ be a simple chain in $\mathcal{D}$ from D_i to D_{i+1}. Let $C_i = \bigcup \mathcal{C}_i$ and let $H = \bigcup_{i=0}^{\infty} D_i \cup C_i$. $\square$

We now build $\{Y_\alpha : \alpha < \omega_1\}$ by induction, using an arbitrary $\{X_\alpha : \alpha < \omega_1\}$ as in Theorem 3.3 as a foundation. Let $Y_0 = X_0$. If $Y_\alpha \supset X_\alpha$ has been defined for $\alpha \in \omega_1$ so that Y_α is connected, open, and Lindelöf, use Lemma 3.1 and Property 2.7 to get a cover $\{G_n : n \in \omega\}$ of the boundary $\overline{Y_\alpha} \setminus Y_\alpha$ of Y_α by connected, open, Lindelöf subsets of X, each of which meets Y_α, and let $S_{\alpha+1} = Y_\alpha \cup X_{\alpha+1} \cup \bigcup_{n=0}^{\infty} G_n$.

Then $S_{\alpha+1}$ is Lindelöf, and as in Lemma 4.3, only countably many components of $X \setminus \overline{Y_\alpha}$ meet $X \setminus \overline{S_{\alpha+1}}$. Let V be such a component. If $\overline{V}$ is Lindelöf, let $H(V) = V$, otherwise let $H(V)$ be as in Theorem 4.4. Let

$$Y_{\alpha+1} = \overline{S_{\alpha+1}} \cup \bigcup\{H(V) : \ V \text{ is a component of } X \setminus \overline{Y_\alpha} \text{ that meets } X \setminus \overline{S_{\alpha+1}}\}.$$

Then $Y_{\alpha+1}$ is clearly connected, open and Lindelöf. Letting $Y_\alpha = \bigcup_{\xi<\alpha} Y_\xi$ whenever α is a countable limit ordinal completes the construction of the canonical sequence $\Sigma(Y) = \langle Y_\alpha : \alpha < \omega_1\rangle$. In particular, $\overline{Y_\alpha}$ is Lindelöf for all α by Property 2.7, and it is connected by the elementary fact that the closure of a connected set is connected. Let $Y = \bigcup\{Y_\alpha : \alpha < \omega_1\}$.

The limbs of Y are either whole limbs of X or initial segments of limbs of X:

Lemma 4.5. *Let $\alpha \in \omega_1$ and let V be a component of $X \setminus \overline{Y_\alpha}$. Then $V \cap Y$ is a component of $Y \setminus \overline{Y_\alpha}$. Moreover, $V \cap Y_\beta$ is connected and $\overline{V \cap Y_\beta} = \overline{V} \cap \overline{Y_\beta}$ for all $\beta > \alpha$. Also, if V has Lindelöf closure in X, then $V \subset Y_{\alpha+1}$.*

Proof. First, suppose V has Lindelöf closure. If $V \cap X \setminus \overline{S_{\alpha+1}} = \emptyset$, then $V \subset S_{\alpha+1} \subset Y_{\alpha+1}$. Otherwise, $H(V) \cap V = V \subset Y_{\alpha+1}$ and so $V \cap Y = V$ is a component of $Y_{\alpha+1} \setminus \overline{Y_\alpha}$ and hence of $Y \setminus \overline{Y_\alpha}$.

If V has non-Lindelöf closure, then $H(V)$ is a connected subset of $Y_{\alpha+1}$ that contains the boundary of $V \setminus S_{\alpha+1}$. Also, $H(V) \cap V$ is an open, connected subset of V and is the only summand in the definition of $Y_{\alpha+1}$ that meets V. Therefore, $H(V) \cap V = H(V) \setminus \overline{Y_\alpha} = V \cap Y_{\alpha+1}$.

The component of $Y \setminus \overline{Y_\alpha}$ that contains $H(V) \cap V$ must be a subset of V; on the other hand, Y is built up by induction in such a way that this component contains all points of $Y \cap V$. Indeed, the boundary of $H(V) \cap V$ in V is the intersection of the boundary of $Y_{\alpha+1}$ with V; and the open summands in the definition of $Y_{\alpha+2}$ are all either subsets of V or disjoint from V, and the union of the former with $Y_{\alpha+1} \cap V$ is connected. The rest of the induction works in the same way, so that $V \cap Y_\beta$ is connected for all $\beta > \alpha$. Obviously,

$$\overline{V \cap Y_\beta} \subset \overline{V} \cap \overline{Y_\beta} = (\overline{V} \cap Y_\beta) \cup [\overline{V} \cap (\overline{Y_\beta} \setminus Y_\beta)].$$

Since Y_β is open, it follows that $(\overline{V} \cap Y_\beta) = V \cap Y_\beta \subset \overline{V \cap Y_\beta}$.

As for $\overline{V} \cap (\overline{Y_\beta} \setminus Y_\beta) = (\overline{V} \cap (\overline{Y_\beta}) \setminus Y_\beta$: points of $\overline{Y_\beta} \setminus Y_\beta$ are in the closures of the individual components of $Y_\beta \setminus \overline{Y_\alpha}$ and there is no overlap where the closures meet $\overline{Y_\beta} \setminus Y_\beta$, and so $\overline{V} \cap \overline{Y_\beta}) \setminus Y_\beta \subset \overline{V \cap Y_\beta} \setminus Y_\beta$, and the reverse containment is trivial. $\square$

Recall the countably many bead strings that have noncompacgt closure in Y. From this lemma it follows that separate bead strings in Y run along different limbs of X, and so the components of $X \setminus \overline{Y}$ are associated with distinct bead strings.

In preparation for the next section, we replace the informal talk of limbs of X with a formal definition of a tree which generalizes the tree $\Upsilon(\Sigma)$ of [7] in the same way that a generalized Γ-sequence generalizes that of a canonical sequence.

Definition 4.6. Let X be a space with a Γ-sequence Σ. *The tree* $\Upsilon(\Sigma)$ (or $\Upsilon(X)$ or simply Υ if the context is clear) has as elements all boundaries $bd(V) = \overline{V} \setminus V$ of components V of some $X \setminus \overline{X_\alpha}$ whose closure $\overline{V}$ is not Lindelöf.

The order on Υ is from "bottom" to "top", *i.e.,* if V_0 is a component of $X \setminus \overline{X_\alpha}$ and V_1 is a component of $X \setminus \overline{X_\beta}$ for some $\beta > \alpha$, then we put $bd(V_0) < bd(V_1)$ iff $V_1 \subset V_0$.

It is easy to see that $bd(V_0) < bd(V_1)$ iff $V_1 \cap V_0 \neq \emptyset$ iff $bd(V_1) \subset V_0$ iff $bd(V_1) \cap V_0 \neq \emptyset$. We use the notation $\Upsilon(\alpha)$ to denote the αth level of Υ (with the 0th level as its first level), i.e., the set of members of Υ whose set of predecessors is of order type α. Note that $\bigcup \Upsilon(\alpha)$ is a (perhaps proper) subset of $\overline{X_\alpha} \setminus X_\alpha$.

The way Υ is defined, some members may form boundaries for more than one component of $X \setminus \overline{X_\alpha}$, including perhaps uncountably many components with compact closures. Also, distinct members of Υ may overlap, but not those from different levels.

A chain (that is, a totally ordered subset) of Υ that is bounded above need not have a (unique) supremum. In particular, if V_ξ is a component of $X \setminus \overline{X_\xi}$ for all $\xi < \eta < \gamma$, and $V_\eta \subset V_\xi$ whenever $\xi < \eta < \gamma$, and $\bigcap\{V_\xi : \xi < \gamma\} \setminus \overline{X_\gamma}$ has components with non-Lindelöf closures, then the boundary of each one of these components has $\{bd(V_\xi) : \xi < \gamma\}$ as its chain of predecessors in Υ.

If p is on the boundary of one of these components (call it V_γ), then (as noted above) $p \in \overline{X_\gamma} \setminus X_\gamma$. Moreover, if G is a connected open neighborhood of p, and $G \cap X \setminus V_\xi \neq \emptyset$, then G meets the boundary of V_ξ (otherwise G would be the union of the disjoint nonempty open sets $G \cap V_\xi$ and $G \setminus \overline{V_\xi}$). Hence, G meets the boundary of every V_η for which $\xi < \eta < \gamma$. By local connectedness, p is in the closure of $\bigcup\{bd(V_\xi) : \xi < \gamma\}$ if γ is a limit ordinal.

There is one important case where a chain that is bounded above in Υ has a unique supremum: this is the case of a transfinite sequence of singletons, as in the ω-sequence case of the Thorn Lemma 3.6, and the easy proof is left to the reader.

5. The proof of the main theorem, completed

If our connected space X is of Lindelöf degree $> \omega_1$, the space Y of the preceding two sections is only a subspace, whose boundary F_{ω_1} is a countable closed discrete subspace of X. Each point of F_{ω_1} is the extra point in the one-point compactification of a bead string of Y. The space may "end" at some of these points, meaning that they have neighborhoods contained in $\overline{Y}$. All others are cut points of X: if A is a component of $X \setminus Y$ whose boundary is $\{p\} \subset F_{\omega_1}$, then p is a cut point of X.

Now A can be subjected to the same analysis that produced Y, with $A \setminus Y$ as our starting point in place of X. It may be that $A \setminus Y$ is is Lindelöf, when the analysis stops after countably many steps with the whole limb accounted for. In this case, A will be part of the Lindelöf subspace L of the main theorem. Otherwise the analysis gives another space like Y at the top of one bead string of Y.

The analysis begins with the point p for which $\{p\} = bd(A)$. Let $B_p(0)$ be a connected relatively open neighborhood of p in $A \setminus Y$ with compact closure. Then $B_p(\alpha)$ is defined exactly like Y_α was, except that now $A \setminus Y$ plays the role that X did in the construction of Y.

For each $\alpha < \omega_1$, let $X_\alpha = Y_\alpha$, let

$$X_{\omega_1+\alpha} = Y \cup \bigcup\{B_p(\alpha) : p \in F_{\omega_1}\}, \text{ and } X_{\omega_1+\omega_1} = \bigcup\{X_{\omega_1+\alpha} : \alpha < \omega_1\}$$

The analysis of X of Lindelöf degree $> \omega_1$ can now be completed in accelerated fashion. It repeats the process that takes us from $Y = X_{\omega_1}$ to $X_{\omega_1+\omega_1}$ in a way that will now be explained.

It is helpful to go along one branch $\mathcal{B}$ of $\Upsilon(X)$ at a time, pausing at each level $\Upsilon(\gamma)$ for which γ is either an ordinal of uncountable cofinality or the supremum of a sequence of such ordinals. On such a level, there is exactly one isolated $\{p\}$ in $\Upsilon(\gamma) \cap \mathcal{B}$. By a slight abuse of language, we call both p and $\{p\}$ "jump points," treating them as prescribed in the preceding paragraph. After each sequence of jumps indexed by a limit ordinal in its natural order, we have a single jump point on the branch $\mathcal{B}$ (as in Lemma 3.6) unless the sequence is cofinal in $\mathcal{B}$.

Call a jump point $\{p\} \in \Upsilon(\gamma)$ "trivial" (a) it is the boundary of just one component of $X \setminus \overline{X_\gamma}$ and (b) the analogue of $\overline{X_{\omega_1}}$ immediately above it and below level $\gamma + \omega_1$ is just a long string of compact beads, whose endpoints have a supremum in $\bigcup \Upsilon(\gamma + \omega_1)$. The following lemma makes it easy to identify these points.

Lemma 5.1. *Let p be a jump point in $\Upsilon(\gamma)$ and let W be the union of all components of $X \setminus \overline{X_\gamma}$ with p in their closure. Then p is trivial if, and only if,* (1) $Z =: \overline{W} \cap \overline{X_{\gamma+\omega_1}}$ *is compact and* (2) $Z \setminus X_{\gamma+\omega_1})$ *is a singleton.*

Proof. Necessity is clear, so we show sufficiency. Compactness of Z implies that all long bead strings in Z meet $\Upsilon(\gamma+\omega_1)$. Condition (2) ensures that exactly one component V of $X \setminus X_\gamma$, [and indeed of $X \setminus X_{\gamma+\alpha}$ for all countable α], has non-Lindelöf closure. Finally, compactness of Z implies that the Lindelöf space L_γ that corresponds to L has compact closure. There is a countable limit α such that $\overline{L_\gamma} \subset X_{\gamma+\alpha}$ and $W \cap X_{\gamma+\alpha}$ has discrete boundary. Exactly one point on this boundary is in the closure of the one component of $X \setminus \overline{X_{\gamma+\alpha}}$ that has non-Lindelöf closure. The closure of this component meets $X_{\gamma+\omega_1}$ in a compact bead string, and $\overline{W} \cap X_{\gamma+\alpha}$ meets all the qualifications of a compact bead [even though it is more natural to think of it as an α-sequence of compact beads]. □

Lemma 5.2. *Any ω-sequence of nontrivial jump points p_n on a branch $\mathcal{B}$ of Υ is cofinal in $\mathcal{B}$.*

Proof. Otherwise, the sequence $\langle p_n : n \in \omega \rangle$ converges on a point p for which $\{p\}$ is on a limit level α of countable cofinality. For each $n \in \omega$ let ν_n satisfy $\{p_n\} \in \Upsilon(\nu_n)$. By the obvious extension of the Thorn Lemma 3.6, all but finitely many p_n are in a compact clopen neighborhood of p, of the form $\overline{V} \cap X_\alpha$ in the relative topology of X_α, where V is a non-Lindelöf component of $X \setminus X_\xi$ for some $\xi < \alpha$, such that Υ has no unbounded branches containing $bd(V)$ that branch off above ξ but before α.

If $p_n \in V$ then $V \cap \overline{X_{\nu_{n+1}}} \setminus X_{\nu_n}$ is easily seen to satisfy the properties of a compact bead with endpoints p_n and p_{n+1}. And now, by Lemma 5.1, the part of $\mathcal{B}$ between p_n and p_{n+1} represents a string of compact beads, no matter how many trivial jumps there are between p_n and p_{n+1}. This contradicts nontriviality of p_n. □

Theorem 5.3. Υ *has at most countably many nontrivial jump points and at most countably many branches that are copies of limit ordinals of uncountable cofinality.*

Proof. Let Ψ be the subtree $\{\{p\} : p \text{ is a nontrivial jump point}\} \cup \Upsilon(0)$ of Υ. Then by Lemma 6.2, Ψ is of height $\leq \omega$; that is, every member of Ψ has at most finitely many predecessors. Also, $\Upsilon(0)$ is countable, and each element $\{p\}$ of Ψ has at most countably many immediate successors in Ψ, and so Ψ is countable.

The branches described in the statement of this theorem go with the sets C in the "Moreover" part of the main theorem. Each one emanates either within X_{ω_1} itself, or within an analogue of X_{ω_1} immediately above a nontrivial jump point, or from a trivial jump point such that all jump points above it (if any) are trivial, and is minimal among the trivial jump points in this category. Let J be the set of these minimal points. If $x \in J$, then $\{x\} \in \Upsilon(\gamma + \omega_1) \cap V$ where V is a component of $X \setminus \overline{X_\gamma}$ with a member $\{p\}$ of Ψ as its boundary. There can only be countably many such x associated with a given $\{p\} \in \Psi$, so J is countable, and thus the set of all branches described in the theorem is countable. □

Finally, for each $x \in J$, we remove $B(x) \setminus \{x\}$ where $B(x)$ is the unbounded string of compact beads that has $\{x\}$ as its boundary, except for x itself. Let L be what remains.

The the proof of the main theorem can now be completed as it was in the case of $\ell(X) = \omega_1$ at the end of Section 3.

6. Future directions

The sequel to this paper, [9], will do for hereditarily normal spaces in $MM(S)[S]$ models what has been done here for monotonically normal spaces just assuming ZFC. The only extra role of $MM(S)[S]$ that is not available in $PFA(S)[S]$ models or from the Proper Forcing Axiom PFA, is the implication that normal spaces are strongly collectionwise Hausdorff (scwH). In the sequel [9], it will be shown how one can substitute "X is normal and hereditarily scwH" for "X is hereditarily normal," and have the global structure of the main theorem 2.1 hold in $PFA(S)[S]$ models and under the PFA.

The main new complication in [9] is that the discrete collection of copies of ω_1 given by monotone normality and Property 2.8 is not *a priori* available to us, but only becomes available after much technical analysis. In fact, the aim of this analysis is almost the opposite: to find a club Γ of ω_1 such that every component of $bd(X_\gamma)$ that is not on an unbounded bead string of X_{ω_1} is infinite. Then some powerful consequences of the PFA, which also hold in $PFA(S)[S]$ models, are used to obtain the conclusion of the main theorem in the $\ell(X) = \omega_1$ case almost exactly as they were used in [8] to show that every normal and hereditarily scwH manifold of dimension $> \omega_1$ is metrizable. Once this is accomplished, the rest of the proof for the hereditarily normal case in $MM(S)[S]$ models is almost the same as in the preceding section; the normal + hereditarily scwH case requires a little extra maneuvering.

After the sequel, the next paper in this series will probe the fine structure of the Lindelöf subspace L and the individual beads. These can be just as complicated as the space itself; a bead could have any number of copies of the whole space hidden away inside it, as could L. But by repeatedly invoking the main theorem and its adaptation in the hereditarily normal $MM(S)[S]$ models, the whole space can be resolved into a subspace of dimension ≤ 1 and a family of disjoint locally compact, locally connected, first countable subspaces such that X is monotonically normal if, and only if, each member of the family is monotonically normal. Whether "first countable" can be strengthened to "perfectly normal" (equivalently, "hereditarily Lindelöf") or monotone normality can be shown outright, is a subject for later research.

Finally, there is the interesting question of how much of the structure that local connectedness gave in the present paper can be obtained without it. There are a number of tricks that can be used to make good use of the collection $\mathcal{C}$ that Theorem 3.4 gives, including an alternative way of getting $\overline{X_\alpha} \setminus X_\alpha$ for countable ordinals α to be countable, closed, and discrete if X is monotonically normal. The more general case of hereditarily normal, locally compact spaces promises to be quite different, however, without local connectedness.

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