THE NORMAL REED SPACE PROBLEM AND A RELATED EXAMPLE

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ABSTRACT. A *Reed* space is a regular space which is the union of countably many open metrizable subspaces. A Reed space is *monotonic* if it is the union of an ascending sequence $\langle M_n : n \in \omega\rangle$ of open metrizable subspaces. An outstanding open problem is whether every normal Reed space is metrizable, and is called the Normal Reed Space Problem. Extensive background information is given about Reed spaces in general, with emphasis on the importance normality plays in this problem. A monotonic Reed space is constructed with many of the properties a counterexample must have. Some suggestions are given for modifying it to produce a nonmetrizable normal Reed space.

1. Introduction

One of the most remarkable unsolved problems in all of topology was posed by G.M. Reed back in the 1990's:

Problem 1. If a normal space is a union of countably many open metrizable subspaces, is it metrizable?

It is very rare to have an unsolved problem that uses only concepts found in most introductory courses in topology. It is even more rare for such problems to be as difficult as Problem 1. Despite intensive efforts to solve it, we do not even have consisency results either way for it, nor for the following, successively more general problems:

Problem 2. Is every normal space with a σ-disjoint base paracompact?

Problem 3. Is every normal, screenable, first countable space paracompact?

This paper introduces the following terminology.

Definition 1.1. A *Reed space* is a regular space which is the union of countably many open metrizable subspaces. A Reed space is *monotonic* if it is the union of an ascending sequence $\langle M_n : n \in \omega\rangle$ of open metrizable subspaces.

Example 2.15 in the next section is a non-monotonic Reed space.

Problem 1 can now be rephrased:

The Normal Reed Space Problem. Is every normal Reed space metrizable?

The wording here alludes to the famous, long-open (1934 – 1983) Normal Moore Space Problem, whose statement has "Moore" instead of "Reed." For the history of this problem, see [17]; and for the mathematics of its solution, see [9].

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The main example of this paper is a ZFC construction of a Reed space which fails rather badly to be normal. But it has many features that it shares with any possible normal nonmetrizable monotonic Reed space. Moreover, the construction is so general that it may give much insight into *why* it failed to be normal, and might pave the way for either a consistent example or a proof that the answer to Problem 1 is affirmative. This seems especially promising in the strongly 0-dimensional case. In normal spaces, this means $Ind\ X = 0$. This is defined the same way as "X is normal," but with "clopen" replacing "open": given disjoint closed sets F_0 and F_1, there are disjoint clopen sets U_0 and U_1 containing F_0 and F_1, respectively. The metrizable open subspaces in the main example are strongly zero-dimensional. This is already enough to give them them a very simple structure, as explained in Section 3. This structure is expounded on in Sections 8 and 9 after the main example is treated in the intervening sections.

The next section gives some useful information about Reed spaces. The sections that follow are almost exclusively exclusively devoted to the class, here designated $\mathcal{M}$, of monotonic Reed spaces. Except for Section 8, its focus is narrowed further to its subclass $\mathcal{ZM}$ of those $X \in \mathcal{M}$ where $Ind(M_n) = 0$ for all M_n in an ascending sequence of open metrizable subspaces whose union is X.

In this paper, "normal" and "regular" are understood to include "Hausdorff." For concepts not defined here, see the standard reference [8] or the standard textbook [28] of general topology.

2. Additional Background

Problem 2 is called "Classic Problem XI" in [18]. It goes back at least to the mid-70's. It is noteworthy in part because both Mary Ellen Rudin [13] and Zoltán Balogh [18], at different times, mistakenly thought they had examples under $\diamondsuit^+$. Yet these two were arguably the best set-theoretic topologists of their time at constructing counterexamples, and by far the best at constructing Dowker spaces.

Definition 2.1. A *Dowker* space is a normal space whose product with $[0, 1]$ is not normal.

Part of the difficulty of these problems lies in the fact that a counterexample for any of them has to be a screenable Dowker space. [Recall that a *screenable* space is one for which every open cover has a σ-disjoint open refinement.]

Remarks 2.2. Obviously, every space wth a σ-disjoint base is screenable and first countable. Bing's Metrization Theorem [8, 4.4.8], [28] states: *a space X is metrizable $\iff$ it is regular and has a σ-discrete base.* From this it is immediate that every union of countably many open metrizable subspaces has a σ-disjoint base.

Every published ZFC example of a Dowker space is patterned after the one of Rudin and the ones of Balogh. A nonmetrizable normal Reed space from ZFC would be very different from any constructed so far. This is even true of any consistent examples: there are only two published examples of screenable Dowker spaces, one by Rudin using $\diamondsuit^{++}$ [23] and one by Balogh in ZFC [3], and both are very far from being first countable. Moreover, one of the most important unsolved problems about Dowker spaces is whether there is a ZFC example of one that is first countable [25] [26].

An obvious contrast between Problem 1 on the one hand, and Problems 2 and 3 on the other, is that the latter two only have "paracompact" as the conclusion, while Problem 1 goes all the way to "metrizable". However, the two conclusions are equivalent, because every paracompact space that is the union of open metrizable subspaces is metrizable. In the case of countably many subspaces, countable paracompactness is enough when combined with normality. [As is well known, paracompact spaces are normal and (trivially) countably paracompact.]

Definition 2.3. A *countably paracompact* [resp. *countably metacompact*] space is one in which every countable open cover has a locally finite [*resp.* point-finite] open refinement.

The following is an old result of G.M. Reed [21], whose unpublished proof is included here with his permission.

Theorem 2.4. *A normal Reed space is metrizable $\iff$ it is countably paracompact.*

Proof. Necessity is clear. To show sufficiency, we use the following characterization [8, 5.2.3] [7]

A space X is normal and countably paracompact $\iff$ each countable open cover $\{U_i\}_{i=0}^{\infty}$ of X has a locally finite open refinement $\{V_i\}_{i=0}^{\infty}$ such that $\overline{V_i} \subset U_i$ for all $i \in \omega$.

Let each U_i be open and metrizable, and for each $n \in \omega$, let $\mathcal{G}_n^i$ be a discrete-in-U_i collection of open sets such that $\bigcup_{n=0}^{\infty} \mathcal{G}_n^i$ is a base for U_i. [Recall Remarks 2.2].

With V_i as above, let $\mathcal{H}_n^i = \{G \in \mathcal{G}_n^i : G \subset V_i\}$. Then $\{\mathcal{H}_n^i\}_{n=0}^{\infty}$ is a base for V_i. In particular, $V_i = \bigcup_{n=0}^{\infty} \mathcal{H}_n^i$, and the closure of each $\bigcup \mathcal{H}_n^i$ in X is a subset of $\overline{V_i}$, so $\mathcal{H}_n^i$ is discrete in X. Consequently, $\{\mathcal{H}_n^i : n \in \omega, i \in \omega\}$ is a σ-discrete base for X. □

It follows from Theorem 2.4 and the next theorem, due to Dowker [7], that a negative answer to Problem 1 gives a Dowker space.

Theorem 2.5. *Let X be a normal space. The following are equivalent:*

(1) *X is countably paracompact.*

(2) *X is countably metacompact.*

(3) *$X \times [0, 1]$ is normal.*

A 1955 theorem of Nagami [16] explains why a counterexample to Problems 2 and 3 is also a Dowker space. Nagami's theorem states that a space is paracompact if, and only if, it is normal, screenable, and countably paracompact — equivalently, by Theorem 2.5, countably metacompact.

In the case of monotonic Reed spaces, normality is not needed in the statement of Theorem 2.4. This follows quickly from the following well-known characterization, shown by Dowker as part of his proof of Theorem 2.5.

Lemma 2.6. *A space X is countably paracompact [resp. countably metacompact] if, and only if, for each sequence $F_n \downarrow \emptyset$ of closed subsets, there is a sequence of open sets G_n such that $F_n \subset G_n$ and such that $\bigcap\{\overline{G_n} : n \in \omega\} = \emptyset$ [resp. $\bigcap\{G_n : n \in \omega\} = \emptyset$].*

Corollary 2.7. *A monotonic Reed space is metrizable $\iff$ it is countably paracompact.*

Proof. Again, necessity is clear. To show sufficiency, make the open cover $\{U_i\}_{i=0}^{\infty}$ in the proof of Theorem 2.4 ascending. Then if $F_i = X \setminus U_i$, we have $F_i \downarrow \emptyset$. Letting V_i be the interior of $X \setminus G_i$, we have $\overline{V_i} \subset U_i$, and we follow the proof of Theorem 2.4, which never used local finiteness of $\{V_i\}_{i=0}^{\infty}$. □

Remark 2.8. *An interesting feature of the preceding proof is that a single open cover — the most natural one — is enough to establish countable paracompactness and hence metrizability in a monotonic Reed space.*

Countable metacompactness is not enough for Corollary 2.7, but G.M. Reed has observed [private communication] that it works if "Moore" is substituted for "metrizable" there.

Definition 2.9. A space X is *developable* if it has a base $\mathcal{B}$ which is the union of countably many open covers $\mathcal{B}_n$ such that, for each choice of $x \in X$ and $B_n \in \mathcal{B}_n$ containing x, $\{B_n : n \in \omega\}$ is a base for the neighborhoods of x. A regular developable space is a *Moore space.*

The following concept makes the substitution precise:

Definition 2.10. *A base $\mathcal{B}$ for a space X is uniform iff, for every $x \in X$, any infinite subfamily of $\mathcal{B}$, each of which contains x, is a base for the neighborhoods of X.*

An old theorem of Bob Heath [14] [8, 5.4.7] [1] states that a space is a metacompact Moore space $\iff$ it has a uniform base. The following is folklore.

Lemma 2.11. *Every space that has a point-finite open cover by metrizable subspaces has a uniform base.*

Proof. Let $X = \bigcup\{(U_\alpha, d_\alpha) : \alpha \in \Gamma\}$, where d_α is a compatible metric on U_α. For each $x \in X$ let

$$A(x) = \{y : y \in U_\alpha \text{ for some } \alpha \text{ such that } x \in U_\alpha\}.$$

For each $n \in \omega$, let

$$B(x,n) = \{y : y \in A(x) \text{ and } d_\alpha(x,y) < 2^{-n} \text{ for all } \alpha \text{ such that } \{x,y\} \subset U_\alpha\}.$$

$B_k(t)$ and $B_k(u)$ are clearly disjoint iff Let $\mathcal{B}_n = \bigcup\{B(x,n) : x \in X\}$. It is routine to verify that $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ is a uniform base for X. □

Theorem 2.12. *Let X be a Reed space. The following are equivalent.*

(1) X *is countably metacompact.*

(2) X *is a metacompact Moore space.*

(3) X *is a Moore space.*

Proof. (1) $\implies$ (2): by Definition 2.6 and Lemma 2.11 and Heath's theorem mentioned just before it.

(3) $\implies$ (1): Every Moore space is countably metacompact. This is well known and is provable in several easy ways. One is that Moore spaces are θ-refinable [11], and a space is countably metacompact $\iff$ it is countably θ-refinable [12]. □

[1] Metacompact spaces are called "weakly paracompact" in [8] and "pointwise paracompact" in [14].

The following corollary was independently shown earlier by G.M. Reed [unpublished] by a somewhat different proof. It shows that "finite" cannot be substituted for "countable" in the Normal Reed Space Problem.

Corollary 2.13. *If a normal space is the union of finitely many open metrizable subspaces, it is metrizable.*

Proof. By Lemma 2.11, Heath's theorem, and Nagami's 1955 theorem stated immediately after Theorem 2.5. □

In the absence of normality, countable metacompactness is much weaker than countable paracompactness. The following well-known old example of a Moore space illustrates this. It is also a Reed space in a very simple way.

Example 2.14. The underlying set for X is the closed upper half plane. Points of the open upper half plane are isolated. Each point x on the x-axis is given neighborhoods along a ray from that point, letting $g(n,x)$ be all points within 2^{-n} of $(x,0)$ along the associated ray. For points of $Q = \mathbb{Q} \times \{0\}$ this ray is the one of slope -1, while for points of $P = (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ it has slope 1. This space is a union of two open metrizable subspaces, $X \setminus P$ and $X \setminus Q$. It is also monotonic: let $Q = \{q_i : i \in \omega\} = Q$ and let $M_n = X \setminus Q \cup \{q_i : i \leq n\}$.

An application of the Baire Category Theorem to the usual topology on the x-axis shows that Example 2.14 is neither normal — P and Q cannot be put into disjoint open sets — nor countably paracompact: the open cover $\{X \setminus Q\} \cup \{g(1,x) : x \in Q\}$ does not have a locally finite open refinement.

On the other hand, Example 2.14 is countably metacompact. There are many different ways to show this using the theory of "generalized metric spaces," but here is a direct proof. Given closed sets $F_n \downarrow \emptyset$, let $U_n = F_n \cup \bigcup\{g(n,x) : x \in F_n \cap (\mathbb{R} \times \{0\})\}$. Clearly, $U_n \downarrow \emptyset$.

Example 2.15. *A Reed space which is not monotonic.* In Example 2.14, replace P and Q with complementary Bernstein subsets B_0 and B_1 of the x-axis. The resulting space X is the union of the open metrizable subspaces $X \setminus B_0$ and $X \setminus B_1$, but a routine application of the Baire Category Theorem to the usual topology on the x-axis shows that any ascending sequence of open subsets whose union is X must have a member meeting both B_0 and B_1 in uncountable dense subsets of some open interval of the x-axis, and this member cannot be normal.

Apart from the failure of X to be monotonic, every other property given for Example 2.14 is satisfied by Example 2.15. In particular, it is countably metacompact, hence a Moore space by Theorem 2.12.

Example 2.16. [Reed, unpublished] *A countably paracompact, nonmetrizable Reed space from a Q-set.* Here "Q-set" designates an uncountable subset Q of the x-axis of which each subset is a G_δ (equivalently, an F_σ) in the relative Euclidean topology of Q. The existence of a Q-set in this sense is ZFC-independent.

This Reed space has a non-Reed prototype known as "Heath's tangent V space," whose underlying set is the upper half plane H together with Q as above. Points of H are isolated, while the neighborhoods of points on the x-axis are initial nontrivial segments of rays that emanate from these poins at slopes of 1 and -1.

This prototype was the first (consistent) example of a nonmetrizable, metacompact Moore space. A standard technique called “the Wage machine” modifies it to produce one that is not normal, but is countably paracompact. The machine is described in [22] in general terms. Its application here is to replace each point on the x-axis by a pair of points, with one using the ray of slope 1 for its base of neighborhoods, and the other of slope of -1. This also produces a Reed space similar to that in Example 2.15. Details are left to the reader.

Example 2.15, Theorem 2.4 and Corollary 2.7 suggest the following weakening of the Normal Reed Space Problem:

Problem 4. Is every normal Reed space monotonic?

Corollary 2.13 is not enough for this. It suggests listing an infinite open cover in a sequence and showing that the union of each initial segment is metrizable. However, the union after finitely many (perhaps two) steps might not even be normal, let alone metrizable.

An affirmative answer to Problem 4 would reduce Problem 1 to what might naturally be called called The Normal Monotonic Reed Space Problem:

Problem 5. Is every normal monotonic Reed space metrizable?

3. Special properties of the main example

The main example of this paper, denoted V, is a (monotonic) Reed space with an ascending sequence of open metrizable subspaces $V_n \uparrow V$. It is non-normal, and so $Ind\ V \neq 0$. However $Ind\ V_n = 0$ for all n. In other words, V is in the class $\mathcal{ZM}$.

For a metrizable space M, $Ind(M) = 0$ is equivalent to M being a copy of a subspace of a countable product of discrete spaces [E, 7.3.15], which might as well be of the same cardinality ≥ 2. This countable power D^ω is known as *Baire's* 0*-dimensional space of weight* $\mathfrak{m}$ where $\mathfrak{m} = |D| \cdot \omega$, and is here denoted $\mathfrak{B}(\mathfrak{m})$.

The natural base for D^ω is the one that is determined by the partitions $\mathcal{P}(n)$ into equivalence classes:

$$[x]_n = \{y \in D^\omega : y_i = x_i \text{ for all } i \leq n\}.$$

Each partition refines the preceding ones, and in $\mathfrak{B}(\mathfrak{m}) = D^\omega$ this means chopping up each $[x]_{n-1}$ into $\mathfrak{m}$-many pieces to produce $\mathcal{P}(n)$. This even applies if we define $[x]_{-1}$ as D^ω.

Our main example V not countably metacompact; in other words, it satisfies (1) and (2) of the following definition. And it also satisfies (3).

Definition 3.1. A space is *strongly Dowker* [resp. *strongly almost Dowker*] if it is a normal [resp. regular] space with a countable collection of closed sets F_k such that

(1) $F_k \downarrow \emptyset$,

(2) If G_k is an open set containing F_k, then $\bigcap_{k=0}^{\infty} G_k \neq \emptyset$ and

(3) F_{k+1} is nowhere dense in the relative topology of F_k for all $k \in \omega$. Equivalently:

(3′) $F_k \setminus F_{k+1}$ is dense in F_k for all k.

The motivation for these terms will be given in Section 8, along with a proof that every nonmetrizable monotonic Reed space is strongly Dowker [Theorem 8.1]. Moreover, this proof shows that the ascending sequence $\langle M_n : n \in \omega \rangle$ of open metrizable subspaces can be chosen so that $\langle F_n = X \setminus M_n : n \in \omega \rangle$ witnesses (3). The space V is a strongly *almost* Dowker space witnessed by $F_k = V \setminus V_k$ for all $k \in \omega$.

There are several features shared by all monotonic Reed spaces X in class $\mathcal{ZM}$ in which $M_{k+1} \setminus M_k (= F_{k+1} \setminus F_{k+2})$ is nowhere dense in M_{k+1} for all $k \in \omega$. The first feature does not require $X \in \mathcal{ZM}$, but the other two do require it.

- For all $k > 0$, $X_k =: M_k \setminus M_{k-1}$ is relatively closed in M_k and, by $(3')$, each neighborhood of each point of X_k meets all X_i for which $i \leq k$. Here $X_0 =: M_0$.

- Each open subset U of M_k is a union of disjoint basic clopen subsets: for each $x \in U$, take the least n for which $[x]_n \subset U$. Here $[x]_n$ is determined by a given embedding of M_k in $\mathfrak{B}(\mathfrak{m})$. In the main example, M_k is an actual copy $\mathfrak{B}(\mathfrak{c})$, and is designated V_k.

- If M_k is identified with a subspace of $\mathfrak{B}(\mathfrak{m})$, each partition $\mathcal{P}_k(n) = \{M_k \cap [x]_n : x \in M_k\}$ of M_k traces a basic partition $\mathcal{P}'_k(n)$ on X_k which identifies it also as a subspace of $\mathfrak{B}(\mathfrak{m})$. The traces on M_{k-1} of each $P \in \mathcal{P}_k(n)$ are clopen in the relative topology of M_{k-1}, but need not be members of $\mathcal{P}_{k-1}(\ell)$ for any ℓ. In the main example, they are unions of denumerably many members of $\mathcal{P}_{k-1}(n+1)$.

4. The main example defined

For X and M_k as above, we substitute V and V_k, respectively, in the main example, but continue to use X_k as above. Topologically, each X_k is $\{k\} \times [0,1]_D^\omega$, where $[0,1]_D$ refers to $[0,1]$ with the discrete topology. In particular, $X_0 = V_0$ is thus canonically homeomorphic to $\mathfrak{B}(\mathfrak{c})$, as is X_k for all k. For $k > 0$ it takes some work to define a homeomorphism from V_k to $\mathfrak{B}(\mathfrak{c})$, as might be expected from the non-normality, etc. of V.

We adopt the following notation:

$$X_k = \{p : p = (k; r_0, \dots r_i, \dots), r_i \in [0,1]\},\ V_k = \bigcup_{i=0}^{k} X_i, V = \bigcup_{i=0}^{\infty} X_i, \text{ and } F_k = V_k^c = X \setminus V_k.$$

The usual base for the relative topology on X_k is here denoted:

$$\mathcal{B}_k = \bigcup_{n=0}^{\infty} \mathcal{B}_k^n, \text{ where } \mathcal{B}_k^n = \{B_k(\overline{s}) : \overline{s} = (r_0, \dots r_n)\}$$

where $B_k(\overline{s}) = \{p \in \{k\} \times [0,1]^\omega : p \restriction_{dom(\overline{s})} = \overline{s}\}$. By the bracket notation, $B_k(\overline{s}) = [p]_n$ if $\{k\} \times [0,1]_D^\omega$ is identified with $\mathfrak{B}(\mathfrak{c})$.

There is a high degree of self-similarity built into V, as is already apparent from the definitions above. This makes a description of the topology below comparatively concise.

We begin the definition of the topology τ by letting $\mathcal{W}_0 = \mathcal{B}_0$ and $\mathcal{W}_0^n = \mathcal{B}_0^n$. For greater self-similarity in defining $\mathcal{W}_k$ for all k below, we also write $W_0(\overline{s}) = B_0(\overline{s})$.

Next we define $\mathcal{W}_1^0$. Let σ be a bijection from $[0,1]$ to the set Σ of all one-to-one sequences in $[0,1]$. We designate $\sigma(r)$ as σ_r for convenience. Each set of the form $B_1(r)$ "grabs" (i.e., is associated with) the set of open balls $\{W_0(\sigma_r(n)) : n \in \omega\}$. From each of these balls it "ropes" (i.e., attaches to itself) the ball $W_0(\sigma_r(n)), r)$ on the next level, so that $\mathcal{W}_1^0$ is the collection of all sets of the form

$$W_1(r) = B_1(r) \cup \bigcup_{i=0}^{\infty} W_0(\sigma_r(i), r)$$

where $r \in [0,1]$. Of course, we could have written $B_0(\sigma_r(i), r)$ instead of $W_0(\sigma_r(i), r)$, but this does not work for $\mathcal{W}_k^0$ if $k > 1$. So, in general, if $\mathcal{W}_{k-1}^0$ has been defined for $k > 0$, we let $\mathcal{W}_k^0$ be the collection of all sets of the form

$$W_k(r) = B_k(r) \cup \bigcup_{i=0}^{\infty} W_{k-1}(\sigma_r(i), r)$$

We next define $\mathcal{W}_k^n$ to be the collection of all sets of the form

$$W_k(r_0, \ldots, r_n) = B_k(r_0, \ldots r_n) \cup \bigcup_{i=0}^{\infty} W_{k-1}(\sigma_{r_0}(i), r_0, \ldots r_n).$$

Let $\mathcal{W}_k = \bigcup_{n\in\omega} \mathcal{W}_k^n$. In other words, $\mathcal{W}_k = \{W_k(\overline{s}) : B_k(\overline{s}) \in \mathcal{B}_k\}$. Let $\mathcal{W} = \bigcup_{k=0}^{\infty} \mathcal{W}_k$.

We let the topology τ_k on V_k be the one whose base is the Boolean algebra generated by $\mathcal{W}_k$, via the elementary operations of binary union and intersection, and complementation. The topology τ on V is the one whose base is the Boolean algebra generated by $\mathcal{W}$. Since the operations for generating a subalgebra of a Boolean algebra are finitary, and $\mathcal{W}$ is the ascending union of the $\mathcal{W}_k$, $\bigcup\{\tau_k : k \in \omega\}$ is a base for τ. The following lemma simplifies this picture further.

Lemma 4.1. *The topology τ_k on V_k is the relative τ_{k+1}-topology on V_k. Hence τ_k is the τ-relative topology on V_k.*

Proof. If $k = 0$, then each $W_1(r_0, \ldots r_n) \in \mathcal{W}_1$ meets V_0 in a disjoint family of members of $\mathcal{W}_0$. So, because of the way Boolean algebras are generated, there are no new basic clopen sets added to τ_0 by the contribution of members of τ_1. This proves the case $k = 0$ of the first sentence.

If $k > 0$, and τ_{k-1} is the τ_k-relative topology on V_{k-1}, then the same argument (with subscripts k and $k+1$ in place of 0 and 1) applies to show the first sentence of the lemma.

A slightly different induction shows that τ_k is the relative τ_ℓ-topology on V_k for all $\ell > k$, and this will complete the proof of the "Hence" portion, by the remarks immediately preceding the lemma.

Since τ_ℓ traces the $\tau_{\ell-1}$ topology on $V_{\ell-1}$, and $\tau_{\ell-1}$ traces the $\tau_{\ell-2}$ topology on $V_{\ell-2}$, etc., the rest follows by transitivity of the relation, "$\mathcal{T}$ traces the $\mathcal{T}'$-topology on $\bigcup \mathcal{T}'$." $\square$

In the Appendix, there is a detailed picture of how the tracings propagate backwards, and of the structure of the members of the $\mathcal{W}_k$ themselves. But this picture will not be needed any earlier in this paper.

The following lemma is the key to much of what we show below.

Lemma 4.2. *Disjoint members of each $\mathcal{B}_k$ expand to disjoint members of $\mathcal{W}_k$.*

Proof. In other words, if $B_k(\overline{r})$ and $B_k(\overline{s})$ are disjoint, then so are $W_k(\overline{r})$ and $W_k(\overline{s})$.

We proceed by induction on k. For $k = 0$ this is true by definition. So let $k > 0$. If $\overline{r} = (r_0, \dots r_m)$ and $\overline{s} = (s_0, \dots s_n)$, then $B_k(\overline{r})$ and $B_k(\overline{s})$ are disjoint iff they are distinct iff $r_i \neq s_i$ for some $i \leq min\{m, n\}$.

If $i = 0$ then $W_k(\overline{r})$ and $W_k(\overline{s})$ are disjoint because of the induction hypothesis on k and because the attached $W_{k-1}(\sigma_{r_i}(j), r_i)$ and $W_{k-1}(\sigma_{s_i}(j'), s_i)$ differ in the second coordinates of their indices. If $i > 0$, the induction hypothesis works even more simply. □

Definition 4.3. A space X is 0-*dimensional* if it is Hausdorff and $ind(X) = 0$; that is, X has a base of clopen sets.

Theorem 4.4. *The space (V, τ) is* 0*-dimensional, hence regular.*

Proof. An easy exercise shows that if a space with a base of clopen sets is T_0, then it is Hausdorff. The base given above for τ is a Boolean algebra, so all its members are clopen.

To show V is T_0, suppose $x \in X_k$ and $y \in X_\ell$ be distinct points and $k \leq \ell$. If $k = \ell$ then we use Lemma 4.2 and metrizability of X_k. Otherwise, let $x \in W_k^0$. Clearly, $y \notin W_k^0$. □

5. Additional basic properties of (V, τ)

The quick definition of $\mathcal{W}$ and then of τ has its drawbacks, including the difficulty of visualizing the neighborhoods of points, and of proving that each V_k is metrizable.

To address the first difficulty, we shrink each $W_k(r_0, \dots r_n)$ when $k > 0$ to a subset $G_k(r_0, \dots r_n)$ and use these shrinkages as basic neighbornoods in V_k for the points of X_k.

For each $W = W_k(r_0)$, let $\mathcal{Z}(W)$ be the set of all members of $\mathcal{W}_0^0 \cup \dots \cup \mathcal{W}_{k-1}^0$ that meet W. An easy backwards induction shows that $\mathcal{Z}(W)$ is countable: each of the countably many $W_{k-1}(\sigma_{r_0}(i), r_0)$ that is "roped" into W by $B_k(r_0)$ is in one member, $W_{k-1}(\sigma_{r_0}(i))$, of $\mathcal{W}_{k-1}^0$, and meets only countably many members of $\mathcal{W}_0^0 \cup \dots \cup \mathcal{W}_{k-2}^0$, etc. Of course, none of the members of $\mathcal{Z}(W)$ meets $B_k(r_0)$.

List $\mathcal{Z}(W)$ bijectively as $\{Z_n^W : n \in \omega\}$. Let $\mathcal{G}_k^n$ be the collection of all sets of the form

$$G_k(r_0, \dots, r_n) = W_k(r_0, \dots, r_n) \setminus (Z_0^W \cup ... \cup Z_n^W).$$

As with $\mathcal{W}_k$ and $\mathcal{W}$, let $\mathcal{G}_k = \bigcup_{n=0}^\infty \mathcal{G}_k^n$ and let $\mathcal{G} = \bigcup_{k=0}^\infty \mathcal{G}_k$. Because of the bijective "shrinkages," each $\mathcal{G}_k^n$ is a disjoint (by Lemma 4.2) collection of clopen sets.

The Appendix has some (optional) details about the effect that the Z_n^W have on members of $\mathcal{W}_k$. For now, we just note that

$$W_k(r_0) \setminus (Z_0^W \cup ... \cup Z_n^W) = \bigcup \{G_k(r_0, \dots, r_n) : r_i \in [0, 1] \text{ whenever } 0 < i \leq n\}.$$

. This is because of the way Z_j^W depends only on $W = W_k(r_0)$, and the way $W_k(r_0)$ is the disjoint union of all the $W_k(r_0, \dots, r_n)$ for any n.

Lemma 5.1. *Let $x = \langle k; r_0, \dots, r_m, \dots \rangle$. Then $\{G_k(r_0, \dots r_n) : n \in \omega\}$ is a base for the neighborhoods of x in (V_k, τ_k).*

Proof. Let $U = W_k(r_0)$ and let $U^* = W_k(r_0, \dots r_n)$. The subalgebra of τ_k generated by the sets in $\mathcal{W}_k^n$ has the members of $\mathcal{W}_k^n$ as its atoms: by Lemma 4.2, each $W \in \mathcal{W}_k^n$ has no proper subset in this subalgebra besides $\emptyset$. Similarly, if $m < n$, then the only member of $\mathcal{W}_k^m$ that meets U^* is $W_k(r_0, \dots r_m) \supset U^*$.

Now let $\ell < k$. No member of $\mathcal{W}_\ell$ contains x. If $W \in \mathcal{W}_\ell$ meets U^*, let W_0 be the unique member of $\mathcal{W}_\ell^0$ that contains W. Then $W_0 = Z_i^U$ for some i, and $U^* \setminus W$ contains $U^* \setminus Z_i^U$, which in turn contains $G_k(r_0, \dots r_p)$ for a sufficiently large p. In fact, $p = max\{i, n\}$ will do if $W \in \mathcal{W}_\ell^n$.

From these particulars, it easily follows that every member of the base of τ_k generated by $\bigcup\{\mathcal{W}_\ell^j : \ell \leq k, j \leq n\}$ that contains x will also contain $G_k(r_0, \dots r_q)$ for a sufficiently large q. ☐

Theorem 5.2. *For all $k > 0$, X_k is nowhere dense in V_k.*

Proof. Equivalently, V_k is dense in V_{k+1} (and hence in V itself). Let $x = \langle k; r_0, r_1, \dots \rangle \in X_k$. Let $x_n = \langle k-1; \sigma_{r_0}(n), r_0, r_1, \dots \rangle \in X_{k-1}$. Then, for all $j \in \omega$, $x_j \in G_k(r_0, \dots r_n)$ unless $W_{k-1}(\sigma_{r_0}(j)) = Z_i^{W_k(r_0)}$ for some $i \leq n$. Therefore, $G_k(r_0, \dots r_n)$ contains all but finitely many terms of $\langle x_j : j \in \omega \rangle$, which therefore converges to x. ☐

Now we move on to the proof that V_k is metrizable for each k. We will actually prove that V_k is homeomorphic to $\mathfrak{B}(\mathfrak{c})$ after a pair of preliminary lemmas.

Lemma 5.3. *For each $k > 0$, $\bigcup \mathcal{W}_k^0$ is a proper subspace of V_k, and $\bigcup \mathcal{G}_k^0$ is a proper subspace of $\bigcup \mathcal{W}_k^0$.*

Proof. The only member of $\mathcal{W}_k^0$ that could meet a given $W = W_{k-1}(r, r_0)$ is $W_k(r_0)$, but that can only happen if r is one of the $\sigma_{r_0}(i)$. This leaves all but countably many $W_{k-1}(r, r_0)$ out of $\bigcup \mathcal{W}_k^0$ for each choice of r_0. The second conclusion is immediate from a much stronger one by way of Lemma 4.2: each $G_k(r_0) \in \mathcal{G}_k^0$ is a proper subspace of $W_k(r_0) = W$ because Z_0^W has been subtracted off. ☐

Lemma 5.4. *For each $k, n \in \omega$, $\bigcup \mathcal{G}_k^n$ is clopen in V_k, and so is $\bigcup \mathcal{G}_k$ for all k.*

Proof. Clearly, $\bigcup \mathcal{G}_k^j \subset \bigcup \mathcal{G}_k^i$ whenever $i \leq j$, so $\bigcup \mathcal{G}_k = \bigcup \mathcal{G}_k^0$. If $p \in X_k$, then $p \in \bigcup \mathcal{G}_k^n$ for all n. So let $p \in X_\ell$ for some $\ell < k$. If $\ell = k-1$, then $W_k(r_0, \dots r_n)$ meets X_ℓ in the balls of the form $B_\ell(r_{-1}, r_0, \dots, r_n)$, where $r_{-1} = \sigma_{r_0}(i)$ for some $i \in \omega$. By induction on ℓ [see the Appendix for further details], $W_k(r_0, \dots r_n)$ meets X_ℓ in all balls of the form $B_\ell(r_{\ell-k}, \dots, r_{-1}, r_0, \dots, r_n)$ where, for $i > 0$, $r_{-i} = \sigma_{r_0}(i)$ for some $i \in \omega$.

It follows that $\bigcup \mathcal{G}_k^n$ meets X_ℓ in a family $\mathcal{D}$ of balls of the form $B_\ell(r_{\ell-k}, \dots, r_{-1}, r_0, \dots, r_n)$ There is a unique ball $B = B_\ell(s_{\ell-k}, \dots, s_{-1}, r_0, \dots, r_n)$ in $\mathcal{B}_\ell$ for which $p \in B$. If $p \notin \bigcup \mathcal{G}_k^n$, then $B \cap \bigcup \mathcal{G}_k^n = \emptyset$. Then, by Lemma 4.2, B expands to a member of $\mathcal{W}$ (hence of $\mathcal{G}$) disjoint from all expansions of members of $\mathcal{D}$. So $p \notin c\ell_{V_k}(\bigcup \mathcal{G}_k^n)$. ☐

Before proving that each V_k is a copy of $\mathfrak{B}(\mathfrak{c})$, we recall some elementary facts about $\mathfrak{B}(\mathfrak{m})$, generically defined as D^ω where D is an infinite discrete space of cardinality $\mathfrak{m}$.

(1) *The product of a copy of* $\mathfrak{B}(\mathfrak{m})$ *and any nonempty discrete space of cardinality* $\leq \mathfrak{m}$ *is a copy of* $\mathfrak{B}(\mathfrak{m})$. Replace one D factor with $D_0 \times D$, a discrete space of cardinality $\mathfrak{m}$.

(2) *If* C *is a nonempty clopen subset of* $\mathfrak{B}(\mathfrak{m})$, *then* C *is homeomorphic to* $\mathfrak{B}(\mathfrak{m})$. If $C = [x]_n$ this is trivial: if $p = \langle r_i : i \in \omega \rangle \in C$, let $\varphi(p) = \langle r_{n+i} : i \in \omega \rangle$. Otherwise, for $x \in C$, let $n(x) = min\{n : [x]_n \in C\}$. Then, $\mathcal{P}_C = \{[x]_{n(x)+1} : x \in C\}$ is a partition of C into $\mathfrak{m}$-many clopen copies of $\mathfrak{B}(\mathfrak{m})$, so that C itself is homeomorphic to $\mathfrak{B}(\mathfrak{m})$.

Criterion (3) A space X is a copy of $\mathfrak{B}(\mathfrak{m})$ if, and only if,

(i) there is a sequence of partitions $\mathcal{P}_n$ of X into $\mathfrak{m}$ clopen sets, and

(ii) each $\mathcal{P}_{n+1}$ partitions each member of $\mathcal{P}_n$ into $\mathfrak{m}$ clopen sets, and

(iii) If $P_n \in \mathcal{P}_n$ for all $n \in \omega$, and $P_{n+1} \subset P_n$ for all n, then $\bigcap P_n = \{x\}$ for a unique $x \in X$.

(4) If X has a base which is a union of partitions $\mathcal{P}_n$ as in (3), then every open cover $\mathcal{U}$ of X can be refined to a partition into members of that base. Instead of defining $n(x)$ as in (2), let

$n(x) = min\{n : \mathcal{P}_n(x) \subset U \text{ for some } U \in \mathcal{U}\}$,

where $\mathcal{P}_n(x)$ is the unique member of $\mathcal{P}_n$ that contains x.

Theorem 5.5. *Each* V_k *is a copy of Baire's* 0*-dimensional space* $\mathfrak{B}(\mathfrak{c})$ *of weight* $\mathfrak{c}$. *Moreover, each* V_k *has a base that is the union of partitions into basic open neighborhoods of the form* $G_\ell(r_0, \ldots r_n)$, *where* $\ell \leq k$.

Proof. When $k = 0$, $V_k = X_k$ and $\mathcal{G}_0^n = \mathcal{B}_0^n$ for all n, and the partitions $\mathcal{G}_0^n$ for V_0 are as described.

Assume the theorem is true for $k-1$. Let $\mathfrak{P} = \{\mathcal{P}_n^{k-1} : n \in \omega\}$ be a sequence of partitions of V_{k-1} into members of $\mathcal{G}_{k-1}$, with notation as in Criterion (3) except for the superscript. By the preceding two Lemmas, $V_k \setminus \bigcup \mathcal{G}_k^0$ is a nonempty clopen subset of V_{k-1}. Use Lemmas 4.2 and 5.4 and fact (2) above to get a partition $\mathcal{P}_0'$ of $V_k \setminus \bigcup \mathcal{G}_k^0$ into $\mathfrak{c}$-many clopen sets from $\bigcup\{\mathcal{P}_n^{k-1} : n > 0\}$ and let $\mathcal{P}_0^k = \mathcal{P}_0' \cup \mathcal{G}_k^0$. If $\mathcal{P}_{n-1}$ has been defined, let $\mathcal{P}_n'$ partition $\bigcup \mathcal{G}_k^{n-1} \setminus \bigcup \mathcal{G}_k^n$ (which is also a clopen subset of V_{k-1}) into clopen sets from $\bigcup\{\mathcal{P}_m^{k-1} : m \geq n\}$ Let $\mathcal{P}_n''$ partition each member of $\mathcal{P}_n' \cup \mathcal{P}_{n-1}''$ into $\mathfrak{c}$-many clopen sets from $\bigcup \mathfrak{P}$. Let $\mathcal{P}_n^k = \mathcal{P}_n'' \cup \mathcal{G}_k^n$.

Clearly, the P_n^k satisfy (i) and (ii) of Criterion (3). To show (iii), consider the two possibilities for a sequence $\langle P_n \in \mathcal{P}_n^k : n \in \omega \rangle$, with $P_{n+1} \subset P_n$ for all n.

Case 1. $P_n \in \mathcal{G}_k^n$ *for all* n.

Let $P_n = G_k(r_0, \ldots, r_n)$. Then $P_{n+1} \subset P_n$ iff $P_{n+1} = G_{k+1}(r_0, \ldots, r_n, r_{n+1})$ for some $r_{n+1} \in [0, 1]$. Clearly, the P_n close down on the unique $x \in X_k$ whose nth coordinate after the initial k is r_n.

Case 2. There exists n *such that* $P_n \in\ P_n''$.

Let n_0 be the least such n, then $P_{n_0} \in \mathcal{P}_m^{k-1}$ for some $m \geq n_0$. Then, with $\mathcal{P}_m^{k-1}$ replacing $\mathcal{P}_n$ in Criterion (3) (iii), it is immediate that (3)(iii) is witnessed by $\{P_n : n \in \omega\}$. □

The foregoing proof works even if $\mathcal{P}'_n$ is empty for some n, but this cannot happen for all n, because $\bigcap_{n=0}^{\infty} \bigcup \mathcal{G}^n_k = X_k$. In fact, it can be shown that $\mathcal{P}'_n$ is never empty, but we do not need this fact anywhere in this paper.

6. The subspace ∇

We now introduce a subspace of V which is the key to the proof in the next section that V is not countably metacompact (and, alas, the proof that it is not normal).

Definition 6.1. *For $k \in \omega$, $\nabla(k) = X_k \cap \bigcap_{n=k}^{\infty} \bigcup \mathcal{W}^0_n$, and $\nabla = \bigcup_{k=0}^{\infty} \nabla(k)$.*

The proof of Lemma 5.3 shows that already $\bigcup \mathcal{W}^0_1$ leaves out a big subspace of $X_0 = V_0$, so it is perhaps noteworthy that $\nabla(k)$ is not only nonempty, but is a copy of $B(\mathfrak{c})$ for each k. The following lemmas show that $\nabla(k)$ is quite robust, and also give some useful elementary facts about the basic open subsets of V.

Lemma 6.2. *Let $k < m \in \omega$. Let $W_n(s_n) \in \mathcal{W}^0_n$ when $k \leq n \leq m$, and suppose $W_k(s_k) \cap \cdots \cap W_m(s_m) \neq \emptyset$. Then $W_k(s_k) \cap \cdots \cap W_m(s_m) = W_k(s_k, \ldots s_m)$, and $s_n = \sigma_{s_{n+1}}(j_n)$ for some $j_n \in \omega$ whenever $k \leq n < m$.*

Proof. It is clear from the definition of $W_k(r)$ that $W_k(r)$ meets $W_{k-1}(r')$ if, and only if, $r' = \sigma_r(i)$ for some $i \in \omega$, and that $W_k(r) \cap W_{k-1}(r') = W_{k-1}(r', r)$. A shift in notation shows this for all pairs $\langle W_n(s_n), W_{n+1}(s_{n+1})\rangle$, $k \leq n < m$.

To show that the intersection of all the $W_n(s_n)$ is $W_k(s_k, \ldots s_m)$, we use induction on $\ell = m - k$. We have it for $\ell = 1$. Suppose it is true for $\ell = n$. A shift in notation gives $\bigcap_{i=k+1}^{n+1} W_i(s_i) = W_k(s_{k+1}, \ldots s_{n+1})$. From Lemma 4.2, it quickly follows that if $W_k(s_k, \ldots s_n)$ meets $W_{k-1}(s)$, then it meets it in $W_{k-1}(s, s_k, \ldots s_n)$. Another shift in notation ($s_i \to s_{i+1},\ k \to k+1,\ s \to s_k$) gives us

$$W_k(s_{k+1}, \ldots s_{n+1}) = W_{k+1}(s_{k+1}, s_{n+1}) \cap W_k(s_k) = \bigcap_{i=k}^{n+1} W_i(s_i).$$

So now we have it for $\ell = n + 1$. $\square$

The converse is trivial; in fact:

Lemma 6.3. *Let $k < m \in \omega$, and suppose there exists $n \in [k, m) \cap \omega$ such that $s_n \notin \{\sigma_{n+1}(i) : i \in \omega\}$. Then $\bigcap_{j=k}^{n} W_j(s_j) = \emptyset$.*

Proof. Take the first two sentences in the preceding proof. $\square$

The following criterion plays a key role in the following theorem, and also in the next section.

Lemma 6.4. Let $x = \langle k; r_0, \ldots, r_n, \ldots \rangle$. Then $x \in \nabla(k)$ *iff for each $n, r_n = \sigma_{r_{n+1}}(j_n)$ for some $j_n \in \omega$.*

Proof. Clearly, $x \in \nabla(k)$ iff for each $\ell \in \omega$, there is a (unique, by Lemma 4.2) member $W_{k+\ell}(s_\ell)$ of $\mathcal{W}^0_{k+\ell}$ such that $x \in W(s_\ell)$. Now use Lemmas 6.2 and 6.3. $\square$

Theorem 6.5. *For each k, $\nabla(k)$ and $\nabla \cap V_k$ are homeomorphic to the Baire space $\mathfrak{B}(\mathfrak{c})$.*

Proof. We will use partitions of X_k and V_k into basic clopen subsets $B_k(r_0, \ldots, r_n)$ and $G_k(r_0, \ldots, r_n)$ in their relative topologies, identifying the ones that meet ∇, and show that their intersections with ∇ satisfy the conditions of Criterion (3) leading up to Theorem 5.5.

Let $\mathcal{N}_k^n$ be the set of all members of $\mathcal{B}_k^n$ that meet $\nabla(k)$. It follows from Lemma 6.4 that $B_k(r_0, \ldots, r_n) \in \mathcal{N}_k^n$ if, and only if, $r_{n-i} = \sigma_{n-i+1}(j_i)$ for some (unique) j_i and $0 < i \leq n$. In particular, every member of $\mathcal{B}_k^0$ meets ∇.

If $B_k(r_0, \ldots, r_n) \in \mathcal{N}_k^n$, then there are $\mathfrak{c}$-many $t \in [0, 1]$ such that $r_n = \sigma_t(i_n)$ for a (unique) i_n: they are the ones for which r_n is in the range of σ_t. By induction, one obtains a tree [2] of sequences $(r_0, \ldots, r_n, t_{n+1}, \ldots, t_{n+m})$ for each $m \in \omega$ with each choice of such t for $n + 1$ and their counterparts up to $n + m$. The union of any ω-sequence of such extensions that agree on common domains gives a (unique) point of $\nabla(k)$ by Lemma 6.4.

One conclusion is that $\mathcal{N}_k^0 = \mathcal{B}_k^0$. Another is that each member of $\mathcal{N}_k^n$ splits into $\mathfrak{c}$-many members of $\mathcal{N}_k^{n+1}$. Finally, each nested sequence of $B_n \in \mathcal{N}_k^n$ as $n \to \infty$ closes down on a singleton of $\nabla(k)$.

Therefore, if we let $\mathcal{P}_n = \{B \cap \nabla : B \in \mathcal{N}_k^n\}$, then $\{\mathcal{P}_n : n \in \omega\}$ satisfies all three conditions in criterion (3) of the proof of Theorem 5.5, and thus $\nabla(k)$ is homeomorphic to $\mathfrak{B}(\mathfrak{c})$.

In the proof of Theorem 5.5, the $\mathcal{P}_n^k$ can take the place of $\mathcal{B}_k^n$ above in defining $\mathcal{N}_k^n$, and the proofs go through with obvious changes in notation. The key thing to observe is that if $G = G_\ell(r_0, \ldots r_n)$ meets ∇, and G is split into $\mathfrak{c}$-many basic clopen sets, then $\mathfrak{c}$-many of those clopen sets also meet ∇. Hence $V_n \cap \nabla$ is a copy of $\mathfrak{B}(c)$. $\square$

7. X is neither normal nor countably metacompact

The first lemma in this section features a family $\mathcal{D}$ of closed sets that is discrete in V_k and remains discrete all the way down through V_{k+n}. But then it acquires so many limit points in V_{k+n+1} that it leads to a theorem that V is strongly almost Dowker, and gives part of the proof that V fails to be normal.

Lemma 7.1. *Let $\mathcal{D}$ be a denumerable subfamily of $\mathcal{B}_k^n$ for some fixed k and n. For each subfamily $\mathcal{D}'$ of $\mathcal{D}$, let $C(\mathcal{D}') = c\ell_V(\bigcup \mathcal{D}' \cap \nabla)$. Suppose that $D \cap \nabla \neq \emptyset$ for all $D \in \mathcal{D}$, and that no two members of $\mathcal{D}$ meet the same member of $\mathcal{W}_{k+\ell}^0$, $0 \leq \ell \leq n$. If $\mathcal{D}_0$ and $\mathcal{D}_1$ are infinite subcollections of $\mathcal{D}$, then $C(\mathcal{D}_0) \cap C(\mathcal{D}_1)$ meets every set of the form F_m.*

Proof. Let $\{D_j : j \in \omega\}$ be a 1-1 listing of $\mathcal{D}$, and let $\mathcal{D}_i = \{D_j : j \in A_i\}$ for $i = 0, 1$.

<u>*Case 1.* $n = 0$,</u> *so that $D_j = B_k(r_0^j)$ and $r_0^i \neq r_0^j$ if $i \neq j$.* In this case the "no two of which..." clause is vacuously true, and the proof does not involve ∇.

Let $\{\sigma_{s_\alpha} : \alpha \in \Gamma\}$ list an infinite family of sequences, with $s_\alpha \neq s_\beta$ when $\alpha \neq \beta$, such that $ran(\sigma_{s_\alpha}$ includes infinitely many r_0^j for which $j \in A_i (i = 0, 1)$. Given j, $D_j = B_k(\sigma_{s_\alpha}(h_j))$ for some $h_j \in \omega$, and $W_{k+1}(s_\alpha)$ meets infinitely many D_j in $B_k(\sigma_{s_\alpha}, s_\alpha) = B_k(r_0^j, s_\alpha)$.

[2]The set-theoretic definitions of "tree" and "branch" are given in the Appendix [Definition A.1.]

More generally, $W_{k+1}(s_\alpha, r_1, \ldots, r_m)$ meets infinitely many members of both $\mathcal{D}_0$ and D_1 for each choice of $(s_\alpha, r_1, \ldots, r_m)$; and $G_{k+1}(s_\alpha, r_1, \ldots, r_m)$ meets all but finitely many of these members. Therefore, every point of $B_{k+1}(s_\alpha, r_1, \ldots, r_m)$ and hence of $B_{k+1}(s_\alpha)$ is in the closure of both $\bigcup D_0$ and $\bigcup D_1$.

At this point, we have up to $\mathfrak{c}$-many balls $B_{k+1}(s_\alpha) \in \mathcal{B}^0_{k+1}$ at our disposal, and any denumerable subfamily can be treated just like $\mathcal{D}$, with $k+1$ in place of k. However, there is no need to split the subfamily up, because every member is in $C(\mathcal{D}_0) \cap C(\mathcal{D}_1)$. So a simpler argument produces $\mathfrak{c}$-many members of $\mathcal{B}^0_{k+2}$ inside $C(\mathcal{D}_0) \cap C(\mathcal{D}_1)$, and then, by induction, $\mathfrak{c}$-many members of $\mathcal{B}^0_{k+m}$ for all m.

Case 2. $n > 0$. In this case, the following notation is helpful. As in the proof of Lemma 5.4, each $D \in \mathcal{D}$ can be written as $B_k(r_{-n}, \ldots, r_{-1}, r_0)$, where $r_{-i} = \sigma_{r_{-i+1}}(\ell_i)$ for some $\ell_i \in \omega$. (This is where $D \cap \nabla \neq \emptyset$ comes in.) The condition "no two members of $\mathcal{D}$ ..." ensures that no infinite subfamily $\mathcal{D}'$ of $\mathcal{D}$ meets only finitely many members of $\mathcal{W}_{k+\ell}$ for some $\ell \leq n$. (That would imply that $C(\mathcal{D}')$ does not meet X_m for $m > k+\ell$.) Let $D_j = B_k(r^j_{-n}, \ldots, r^j_{-1}, r^j_0)$. This same condition is equivalent to $r^j_0 \neq r^i_0$ for $i \neq j$, together with $r^j_{-i} = \sigma_{r_{-i+1}}(h^i_j)$ for a *unique* h^i_j whenever $0 < i \leq n$.

Now if $W \in W^0_{k+\ell}$, and $W \cap D_j \neq \emptyset$, then $W \cap X_k = B_k(r^j_{\ell-n}, \ldots, r^j_{-1}, r_0)$, and $W = W_{k+\ell}(r^j_{\ell-n})$. Since $D_j \cap \nabla \neq \emptyset$, D_j meets at least one member of $\mathcal{W}^0_{k+\ell}$ for all $\ell \in \omega$. When $\ell = n$, $D_j = W_{k+n}(r^j_0) \cap X_k$. Now the induction can proceed as in Case 1, with the various $B_k(r^j_0)$ replaced by $B_{k+n}(r^j_0)$ with subscripts appropriately shifted. These balls are no longer equal to the D_j, so it is good to write D'_j instead and use $\mathcal{D}'$ instead of $\mathcal{D}$ etc. $\square$

Lemma 7.2. *If $\mathcal{U}_0$ and $\mathcal{U}_1$ are uncountable families of disjoint members of $\mathcal{B}_k$, such that each member of $\mathcal{U}_0 \cup \mathcal{U}_1$ meets ∇, and no two members of $\mathcal{U}_0 \cup \mathcal{U}_1$ are in the same member of $\mathcal{B}^0_k$, then there are denumerable subsets $\mathcal{D}_i$ of $\mathcal{U}_i$ such that $C(\mathcal{D}_0) \cap C(\mathcal{D}_1)$ meets every set of the form F_m.*

Proof. We may assume that there exists n such that $\mathcal{U}_0 \cup \mathcal{U}_1 \subset \mathcal{B}^n_k$. Indeed, by cutting down on $\mathcal{U}_i$ if necessary, we have n_0 and n_1 such that $\mathcal{U}_0 \subset \mathcal{B}^{n_0}_k$ and $\mathcal{U}_1 \subset \mathcal{B}^{n_1}_k$. Let $n = max\{n_0, n_1\}$. For each $U \in U_0 \cup U_1$ let $H(U) \subset U$ be a member of B^n_k that meets ∇, and let $\mathcal{U}'_i = \{H(U) : U \in \mathcal{U}_i\}$ for $i = 0, 1$. Then if Lemma 7.2 is witnessed by $\mathcal{U}'_0$ and $\mathcal{U}'_1$, then it is clearly witnessed by $\mathcal{U}_0$ and $\mathcal{U}_1$.

Arbitrarily choose $D_0 = B_k(r^0_0, \ldots, r^0_n) \in \mathcal{U}_0$. As can be seen from the proof of Lemma 6.2, the unique member of $\mathcal{W}_{k+n}$ that meets D_0 is $W_{k+n}(r^0_n)$. Then $W_{k+n}(r^0_n) \cap X_k$ is the union of all sets of the form $B_k(s_0, \ldots, s_{n-1}, r^0_n)$ where, as usual, each term but the last is one of $\aleph_0$ possibilities for each possible term to its immediate right. So we can choose $D_1 \in \mathcal{U}_1$ outside this set of possible $B_k(s_0, \ldots, s_{n-1}, r^0_n)$.

In general, if $D_i = B_k(r^i_0, \ldots, r^i_n)$ has been chosen from $\mathcal{U}_0 \cup \mathcal{U}_1$ for all $i < j$, and j is even, there are only countably many sets of the form $B_k(s_0, \ldots, s_{n-1}, r^i_n)$ altogether, and we can choose $D_j \in \mathcal{U}_1$ from outside this set. Similarly choose D_j from $\mathcal{U}_0$ if j is odd. Let $\mathcal{D} = \{D_j : j \in \omega\}$. Then $\mathcal{D}$ is as in Lemma 7.1 and $\mathcal{D}_i = \{D_j : j \equiv i \text{ mod } 2, j \in \omega\}$ is as desired. $\square$

Remark 7.3. It takes very little extra work to continue the building up of $\mathcal{D}$ to the point where $|\mathcal{D}| = |\mathcal{B}|$ as long as $\mathcal{B} \subset \mathcal{B}_k^n$ for fixed k and n. (Otherwise, singular cardinals of countable cofinality complicate the proof.) But the special case where $|\mathcal{D}| = \aleph_0$ is all that is needed in this paper.

Theorem 7.4. *V is not countably metacompact.*

Proof. Let U_n be an open set containing $F_n = V \setminus V_n$. We show that $X_0 \cap \bigcap_{n=0}^{\infty} U_n \neq \emptyset$, and this will complete the proof.

Suppose the intersection is empty. By the Baire Category Theorem, $\nabla(0) \setminus (U_0 \cap \cdots \cap U_n)$ has nonempty interior in its relative topology for some n. In fact, there exists n and uncountably many $s_\alpha \in [0,1]$ such that $\nabla(0) \cap W_0(s_\alpha) \setminus (U_0 \cap \cdots \cap U_n)$ has nonempty relative interior. This gives uncountably many members of $\mathcal{B}_0$ that satisfy the hypotheses of Lemma 7.2, contradicting the hypothesis that F_n is a subset of the open set $U_0 \cap \cdots \cap U_n$. □

Corollary 7.5. *V is strongly almost Dowker.*

Proof. By Theorems 5.2 and 7.4. □

Lemma 7.6. *There is a closed discrete subspace of $X_0 \cap \nabla$ of cardinality $\mathfrak{c}$ such that no two are in the same member of $\mathcal{W}_0^0$.*

Proof. We define the closed discrete $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ by induction, with a subsidiary induction on each coordinate. Let $d_0 \in W_0(0) \cap \nabla$ be arbitrary. If d_α has been chosen for all $\alpha < \beta$, let $d_\alpha(k)$ stand for r_k^α in $d_\alpha = \langle 0; r_0^\alpha, r_1^\alpha, \ldots \rangle$. By the argument in Lemma 6.2, $d_\alpha(k) = r_k^\alpha$ identifies the unique $W \in \mathcal{W}_k^0$ that contains d_α: $W = W_k(r_k^\alpha)$. Let

$$A_k^\beta = \bigcup \{ran(\sigma_{d_\alpha(k)}) : \alpha < \beta\}$$

and let $d_\beta(0) \notin A_1^\beta$. Since $|A_1^\beta| < \mathfrak{c}$ this is easy.

If $d_\beta(k)$ has been chosen, let $C_k^\beta = \{d_\alpha(k) : \alpha < \beta\}$. Let $d_\beta(k+1) = r$ satisy (1) $r \notin A_{k+2}^\beta$; (2) $ran(\sigma_r) \cap C_k^\beta = \emptyset$ and (3) $d_\beta(k) = \sigma_r(i)$ for some $i \in \omega$. All this is easily done: the countable range of σ_r need only avoid C_k^β and contain $d_\beta(k)$, and even though r is outside A_{k+2}^β, there are $\mathfrak{c}$-many possible r.

Obviously, D is discrete: if $\alpha \neq \beta$, then $d_\alpha(0) \neq d_\beta(0)$, which puts d_α and d_β into different members of $\mathcal{W}_0^0$.

To show that D is closed, let $p = \langle k; r_0, \ldots, r_n, \ldots \rangle \in X_k$. Then $p \in W_k(r_0)$. If $d_\alpha \in W_k(r_0)$, then

$$d_\alpha = \langle 0; r_{-k}, \ldots, r_{-1}, r_0, \ldots, r_n, \ldots \rangle \text{ and } r_i^\alpha = r_{i+k} \text{ for all } i.$$

where $r_{-i} \in ran(\sigma_{r_{-i+1}})$, because $d_\alpha \in \nabla$ and because of the inductive construction of $W_k(r_0)$. But then $r_1 = d_\alpha(k+1)$, and this can be true of at most one α. If such an α exists, let $W = W_k(r_0)$ and let $W_0(r_{-k}) = Z_n^W$. Then d_α is not in the neighborhood $G_k(r_0, \ldots, r_n)$ of p. □

Definition 7.7. An *open expansion* of a subset A in a space X is a family $\{U_a : a \in A\}$ of open sets such that $U_a \cap A = \{a\}$. Let κ be a cardinal number. A space X is *strongly κ-collectionwise Hausdorff (*abbreviated *strongly κ-cwH)* if every closed discrete subspace D of cardinality $\leq \kappa$ has a discrete open expansion. A space X is *strongly collectionwise Hausdorff (strongly cwH)* iff it is strongly $|X|$-cwH.

The definitions of *κ-collectionwise Hausdorff (κ-cwH)* and *collectionwise Hausdorff (cwH)* are obtained by replacing "discrete open expansion" with "disjoint open expansion". G. M. Reed [21] and others have shown that every normal Reed space is cwH (and, by an elementary exercise using normality, strongly cwH). On the other hand, none of Examples 2.14 through 2.16 is cwH. I do not know whether V is cwH, but:

Theorem 7.8. *V is neither strongly $\aleph_1$-cwH nor normal.*

Proof. This follows routinely from Lemma 7.2 and the proof of Lemma 7.6, where D was chosen so that no two points of D are in the same member of $\mathcal{W}_m^0$ for all $m \in \omega$. Let D' be an $\aleph_1$-cardinality subset of the closed discrete subspace of a D of this special sort. Let $\mathcal{U} = \{U_d : d \in D'\}$ be a disjoint open expansion of the singletons of D', that is, $U_d \cap D' = \{d\}$. We may assume that $\mathcal{U} \subset \mathcal{W}_0 = \mathcal{G}_0 = \mathcal{B}_0$. Then, letting $\mathcal{U} = \mathcal{U}_0 = \mathcal{U}_1$ in Lemma 7.2, one can see how badly $\mathcal{U}$ fails to be discrete.

To show that V is not normal, let E_0 and E_1 be uncountable disjoint subsets of D', and let G_0 and G_1 be open subsets of V_0 containing E_0 and E_1 respectively. Without loss of generality, we may assume that each G_i is the union of a disjoint expansion $\mathcal{U}_i$ of E_i into basic open subsets of V_0. Then $\mathcal{U}_0$ and $\mathcal{U}_1$ are as in Lemma 7.2. So, if $\mathcal{D}_i \subset \{U_d : d \in E_i\}$, it follows that $\overline{G_0} \cup \overline{G_1}$ meets every set of the form F_n, contradicting normality of V. $\square$

8. Notes on "almost Dowker" spaces

Countable metacompactness is such a weak property that Brian M. Scott coined the term "almost Dowker space" for a regular space that is not countably metacompact [24]. However, he overlooked several published examples, including two by J. Chaber [4], one of which (called Y_4) is a Reed space [3]. Two more "almost Dowker" spaces [5, Examples 3.2 and 3.3] were found by Gary Gruenhage in the same year. The second was shown to have a σ-locally countable base in [10] and by G.M. Reed [21] to be a Reed space.

Like most published non-normal "almost Dowker" spaces, Chaber's Y_4 and Gruenhage's two examples in [5] each have a countable discrete collection of closed subsets C_n such that if $F_k = \bigcup_{n=k}^\infty C_n$, and G_k is an open set containing F_k, then $\bigcap_{k=0}^\infty G_k \neq \emptyset$ even though, obviously, $F_k \downarrow \emptyset$. Hence, by Lemma 2.6 they are "almost Dowker." However, they fail very badly to be Dowker: were there disjoint open subsets $U_n \supset C_n$, it would follow that $F_k \subset G_k = \bigcup_{n=k}^\infty U_n$ and $G_k \downarrow \emptyset$, a contradiction. So one might call such spaces "barely almost Dowker."

Strongly almost Dowker spaces, like V, are at the opposite extreme. These include all Dowker spaces of which I am aware:

[3] In the description of Y_4 in [4], the sets $P^* \cup A_q$ are metrizable for each $q \in \mathbb{Q}$, and open in Y_4. However, the union of any two of these subspaces is not normal: A_q and $A_{q'}$ are disjoint closed subsets of Y_4 if $q \neq q'$, but cannot be expanded to disjoint open subsets.

Problem 6. Is every Dowker space strongly Dowker?

Other examples of strongly almost Dowker spaces include the published examples of linearly Lindelöf spaces and of (consistent) weakly first countable, not first countable compact spaces. For surveys of these spaces, see [19] and [15], respectively. Here, the following theorem is of immediate interest:

Theorem 8.1. *Every nonmetrizable normal monotonic Reed space is strongly Dowker.*

Proof. Let $X = \bigcup_{k=0}^{\infty} M_k$, where each M_k is an open metrizable subspace of X, and $M_k \subset M_{k+1}$ for all k. The hypothesis that X is nonmetrizable is equivalent to the statement that $\{F_k = X \setminus M_k : k \in \omega\}$ is a witness to X being Dowker. This is clear from Theorem 2.5, Corollary 2.7, and Remark 2.8.

We will replace each M_k by an open metrizable subspace M_k^* containing M_k, such that M_0^* is dense in X, and such that $\{F_k^* = X \setminus M_k^* : k \in \omega\}$ is a witness to X being strongly Dowker. This will complete the proof of Theorem 8.1 and also justify the comments after Definition 3.1.

For $k > 0$, let $W_k = M_k \setminus \overline{M_{k-1}}$, where the overhead bar means closure in X. Let $M_0^* = M_0 \cup \bigcup_{k=1}^{\infty} W_k$. Including M_0, this is the union of a collection of disjoint open metrizable subsets of X, so M_0^* is metrizable. It is also dense in X. To show this, let $M_k^1 = M_k \setminus (W_1 \cup \cdots \cup W_k)$ for all $k > 0$. Then $F_k \cap M_k = \emptyset$, and for $k > 0$, $X \setminus M_O^* F_k \cap M_{k+1}$ is the disjoint union of M_{k+1}^1 and W_{k+1}, because $F_k \cap W_j = \emptyset\}$ for all $j \leq k$. Let $F_k^1 = F_k \setminus M_0^*$ for all $k > 0$. Then

$$F_k^1 = F_k \setminus \bigcup_{\ell=k}^{\infty} W_\ell = \bigcup_{\ell=k}^{\infty} M_k^1.$$

Let $F_0^* = F_0 \setminus M_0^* = F_1^1$. An easy induction shows that $F_k^1 \cap M_{k+1}$ is nowhere dense in M_{k+1}, and so F_0^* is nowhere dense in X, and M_0^* is a dense open subspace of X.

The closed subspaces F_k^1 are enough to witness that X is almost Dowker. Indeed, if H_k is an open set containing F_k^1 for each k, and $\bigcap_{k=1}^{\infty} H_k = \emptyset$, then $G_k = H_k \cup \bigcup_{j=k+1}^{\infty} W_j$ is an open set containing F_k. Since the W_k are disjoint, $\bigcap_{k=1}^{\infty} G_k = \emptyset$, a contradiction.

A subtle point here is that $X \setminus M_0^*$ meets F_k for every k (and hence $F_k^1 \neq \emptyset$ for all k. Indeed, were $(X \setminus M_0^*) \cap F_k$ empty for some k (hence for all $n > k$), then X would be the union of the open metrizable subspaces M_0^* and M_k, making X metrizable by Corollary 2.13. [This is the only place where normality of X comes in at this stage, but it also shows up in analogous places in the treatment of M_k^* for $k > 0$.]

The process that produced M_0^* and the subspaces F_k^1 (including $F_1^1 = F_0^*$) can be applied to F_0^* in place of X, giving relatively open subspaces $W_k^1 = M_k^1 \setminus \overline{M_{k-1}}$ $(k > 1)$, and $M_1^* = M_0^* \cup \bigcup_{k=2}^{\infty} W_k^1$. Also, $F_k^2 = F_k^1 \setminus M_1^*$, and $F_2^2 = F_1^* = F_1^1 \mathcal{M}_1^*$. This sets the pattern for stage n, with W_k^n and M_n^* and F_k^{n+1} and F_n^*.

However, there are new complications already with $n = 1$. This is because F_0^* is nowhere dense in X. So, although the W_k^1 are disjoint relatively clopen sets in F_0^*, and $\bigcup_{k=2}^{\infty} W_k^1$ (the complement of M_0^* in M_1^*) is thus metrizable, they are a disjoint family of relatively closed, nowhere dense sets in M_1^*. This is reminiscent of the "barely almost Dowker" spaces. However, the placement of the various W_k^1 makes the following claim go through.

<u>*Claim.*</u> *M_1^* is metrizable.*

Once the claim is proved, the inductive definition of M_n^* can proceed for all n, with the proof of metrizability of M_1^* modified for later n only by obvious changes in subscripts and superscripts.

Proof of Claim. Each W_k^1 is a relatively closed subspace of M_1^*, and a subset of M_k. It is therefore disjoint from the relatively closed subspace $M_{k+1} \setminus M_k$ of M_{k+1}. Hence we can put W_k^1 and $M_1^* \cap (M_{k+1} \setminus M_k$ into disjoint open subsets of $M_1^* \cap (M_{k+1}$. We use induction on $k \geq 2$.

Let G_2 and H_2 be disjoint open subsets of M_3 containing W_2^1 and $M_3 \setminus M_2$ respectively, with $G_2 \subset M_1^*$. Next, W_3^1 is a subset of $M_3^1 \setminus M_2^1 = M_3 \setminus M_2 \cup W_2$; in particular, it is disjoint from M_2. Let G_3 and H_3 be defined like G_2 and H_2, but with the additional condition that $G_3 \subset H_2$. This sets the pattern for the remaining G_n and H_n, with obvious subscript changes.

Now the G_n are disjoint open subsets of M_1^*, and they are a discrete family, because each point of X has a neighborhood meeting only finitely many of the M_n, hence finitely many of the G_n. Thus M_1^* has a locally finite open cover $\{M_0^*\} \cup \{G_n : n \geq 2\}$, and each member of the cover has a σ-locally finite open base, and so it is metrizable.

The proof for going from M_n^* to M_{n+1}^* is a straightforward generalization of the foregoing proof, if one keeps working with the original M_n where they appeared above. $\square$

An important feature of this proof is that the M_n^* are metrizable. This ensures that any normal monotonic Reed space fits the description in Section 3 that implied the three bullet points in the $Ind(X) = 0$ case. It is not necessary to know *a priori* whether the space is metrizable or not.

The proof of Theorem 8.1 can be easily adapted to show a more general theorem.

Theorem 8.2. *If a normal space X is the ascending union of an ω-sequence of normal, countably paracompact open subspaces, then X is either countably paracompact or strongly Dowker.*

Outline of proof. The chief modification is to shrink the open G_m to open sets G'_m such that $W_k^m \subset G_m$ and $c\ell_{M_{m+1}}(G'_m) \subset G_m$. In this way, the open covers $\{M_{n-1}^*\} \cup \{G_m : m > n\}$ of M_n^* are not only discrete, they can be used to show that every countable open cover of M_n^* has a locally finite open refinement. Details are left to the reader. $\square$

For this theorem, it is not necessary for M_n^* itself to be normal. In other words, the new ascending open cover need not itself satisfy the hypotheses of the theorem. Whether it can actually fail to do so is an open question.

9. Future directions

Recall that $\mathcal{M}$ stands for the class of monotonic Reed spaces, and $\mathcal{ZM}$ for the subclass of the $X \in \mathcal{M}$ that satisfy $Ind(M_n) = 0$ for all M_n in an ascending union of open metrizable spaces whose union is X.

It is too early to tell how representative $\mathcal{ZM}$ is of the whole class $\mathcal{M}$, but the fact that Theorem 8.1 referred to all spaces in $\mathcal{M}$ is encouraging, as is Morita's classic theorem that every metrizable space of infinite weight $\mathfrak{m}$ is a perfect image of a subspace of Baire's zero-dimensional space $\mathfrak{B}(\mathfrak{m})$. [8, Exercise 4.4.J].

The main example V is a good starting point for future research on $\mathcal{ZM}$. This research must contend with the many features that the spaces in $\mathcal{ZM}$ have in common with V, as well as with the extra features of V that made normality fail so badly.

Turning to the common features first, the proof of Theorem 8.1 shows that all three bullet points in Section 3 hold for normal spaces in $\mathcal{ZM}$. We revert to the notation used in the bullet points, but we can also adopt some later notation if it helps to compare various spaces in $\mathcal{ZM}$ to V. For example:

(A) Each X_k can have a simple expression as a subspace of $\{k\} \times D^\omega$. The canonical partitions $\mathcal{P}'_k(n)$ [see the third bullet point] can be labeled as $\mathcal{B}^n_k$ and their union as $\mathcal{B}_k$, a base for the relative topology on X_k.

(B) If $|D| \leq \mathfrak{c}$, D can be identified with a subset of $\mathbb{R}$ with the discrete topology, and the notation $B_k(r_0, \dots, r_n)$ used for members of $\mathcal{B}^n_k$. Of course, sometimes a slightly different choice is more natural, as in Example 9.3 below.

(C) By the first bullet point, any open set G containing any $B \in \mathcal{B}^n_k$ meets every X_ℓ, $0 \leq \ell < k$ in a relatively open set which is not relatively closed since it has limit points in X_k. By the second bullet point, it can be defined as a union of members of $\mathcal{B}_\ell$.

Other standard properties of subspaces of $\mathfrak{B}(\mathfrak{m})$ give us additional structure.

(D) Once M_{k-1} has been identified with a subspace of $\mathfrak{B}(\mathfrak{m})$, the members of $\mathcal{B}_k$ are closed in the relative topology of M_k, and each $\mathcal{B}^n_k$ is a discrete collection there. Then there is an expansion of $\mathcal{B}_k$ to a discrete collection of clopen subsets of M_k, while the rest of M_k is given a partition as in Theorem 5.5. Careful choice of these clopen sets, and of partitions of the complement of the union of each one, can then produce the $\mathcal{P}_k(n) \cap M_k$ of the third bullet point.

The variation between members of $\mathcal{ZM}$ is due to in part to which subspaces of D^ω the X_k are copies of, but even more to the way the analogues of σ influence the way the members $\mathcal{B}^n_k$ attach members of the $\mathcal{B}_\ell$ for which $\ell < k$.

Where subspaces are concerned, the following theorem of G.M. Reed puts a consistent constraint on the cardinality of a counterexample to Problem 1 (the Normal Reed Space Problem).

Theorem 9.1. [21] *If $\mathfrak{b} = \mathfrak{c}$, then every normal Reed space of cardinality $< \mathfrak{c}$ is metrizable.*

The cardinal $\mathfrak{b}$ is one of many "small uncountable" cardinal numbers $\leq \mathfrak{c}$ studied in depth in [6] and [27]. Martin's Axiom (MA) implies that $\mathfrak{b} = \mathfrak{c}$, and there are a wide variety of forcing models in which it holds, and where $\mathfrak{c}$ can be "arbitrarily large," including models of MA.

The technique of producing attachments in the main example V is among the simplest: σ depended only on the members of B^0_k. The way the earlier X_ℓ were met by the attachments was determined by the way the attached members of $\mathcal{B}^1_{k-1}$ expanded to a collection of relatively clopen subsets of V_{k-1}. The $0 \to 1$ transition was the simplest way to ensure that the members of $\mathcal{B}^0_{k-1}$ would get "tied together" while keeping the expansions of distinct members of $\mathcal{B}^0_k$ disjoint.

There is plenty of room for variation on this technique, including the use of separate σ's for each $\mathcal{B}^n_k$ as k and n vary. The high degree of self-similarity of V greatly simplified that technique, and the taking of the Boolean generated by the (already tight) way that σ "tied together" the $\mathcal{B}^0_k$ as tightly as metrizability of the V_k allowed. The technique can easily be varied in the way expansions of the $\mathcal{B}^n_k$ split up the expansions of the $\mathcal{B}^0_k$ that they refined, and the way those depended on the expansions of earlier $\mathcal{B}^{n+\ell-k}_\ell$.

As it stands, σ ties together the $\mathcal{B}^0_k$ far more tightly than is needed to destroy countable metacompactness, yet not tightly enough to prevent the existence of closed discrete subspaces of cardinality $\mathfrak{c}$ as critically placed as the points of D in the proof of Lemma 7.6: compare the first sentence in the proof of Theorem 7.8 with the hypothesis of Lemma 7.1.

One strategy for obtaining normality would be to impede the process that produced D in Lemma 7.6 by making the counterparts of most σ_r into $\mathfrak{c}$-sequences rather than ω-sequences. But then, for M_k to be metrizable, the members of $\mathcal{B}^0_k$ have to be G_δ-sets, and this complicates the process. A simpler process, at the outset at least, is to tie things together more loosely in hopes of defining expansions of the singletons of D that give a discrete collection of open sets. There are a number of axioms of set theory that require that normal first countable spaces to be collectionwise Hausdorff (cwH), and cwH + normal $\implies$ strongly cwH. So this is an important first step in any effort to define a nonmetrizable normal Reed space just from the ZFC axioms.

One method of loosening the grip of σ, so to speak, is to use the following concept, which has a long history of use in set theory and in set-theoretic topology.

Definition 9.2. Given a limit ordinal α, a *ladder at* α is a strictly ascending sequence of ordinals less than α whose supremum is α. Given an ordinal θ, a *ladder system on* θ is a family

$$\mathcal{L} = \{L_\alpha : \alpha \in \theta,\ \alpha \text{ is a limit ordinal of countable cofin ality}\}$$

where each L_α is a ladder at α.

We will use the symbol Λ for the set of countable limit ordinals.

The best known ladder systems have $\theta = \omega_1$, partly because then all limit ordinals $< \theta$ can be used, but more so because there are many ZFC-independent axioms that influence what can be done with them.

Example 9.3. For each $k \in \omega$ let $X_k = \{k\} \times (\Lambda^{(k-1)})^\omega$, where $\Lambda^(-1) = \omega_1$, and otherwise the superscript (k) denotes the kth Cantor-Bendixson derivative, with $\Lambda^{(0)} = \Lambda$, and $\Lambda^{(k+1)}$ as the set of nonisolated points in the relative topology of $\Lambda^{(k)}$. The restriction as k grows is to ensure that X_k is nowhere dense in $Y_k = \bigcup_{i=0}^k X_i$ under the following topology whenever $k > 0$. Let $\mathcal{L}$ be a ladder system on ω_1, with the restriction that L_ν is a subset of $\Lambda^{(k-1)}$ for all $\nu \in \Lambda^{(k)}$. Similarly to V, let $B_k(\nu_0, \dots, \nu_n) = \{x \in X_k : x(i) = \nu_i \text{ for all } i \leq n\}$ and let

$$W_1(\nu) = B_1(\nu) \cup \bigcup_{i=0}^{\infty} W_0(L_\nu(i), \nu)$$

and, in general,

$$W_k(\nu_0, \dots, \nu_n) = B_k(\nu_0, \dots, \nu_n) \cup \bigcup_{i=0}^{\infty} W_{k-1}(L_\nu(i), \nu_0, \dots, \nu_n)(i).$$

Thus L_ν plays the role that σ_r played for V. As with V, a base for the topology on $Y = \bigcup_{i=0}^\infty X_k$ is the Boolean algebra generated by these $W_k(\nu_0, \dots, \nu_n)$ as k and n range over ω.

The looseness of Example 9.3 can lead to very different kinds of behavior, depending on the ladders chosen and on which axioms beyond ZFC (if any) are chosen. But before going further, a cautionary note: if weight could be substituted for cardinality in Theorem 9.1, this would mean that Example 9.3 cannot be a nonmetrizable normal Reed space under $\mathfrak{b} = \mathfrak{c}$ unless we assume CH. Of course, one could modify Example 9.3 by making $|\theta| = \mathfrak{c}$ instead of $\theta = \omega_1$.

The following well-known axiom has the effect of Example 9.3 being "too tight":

Axiom 9.4. *Axiom* $\clubsuit$ states that there is a ladder system $\mathcal{L}$ on ω_1 such that, for any uncountable subset S of ω_1, there is $L_\alpha \in \mathcal{L}$ such that $L_\alpha \subset S$.

Any $\mathcal{L}$ as in $\clubsuit$ actually has the stronger property: $L_\alpha \subset S$ for a stationary (hence uncountable) set of α's. This makes Example 9.3 strongly almost Dowker, but it will also be non-normal: the arguments for Lemma 7.2 and Lemma 7.6 go through with relatively minor modifications. Details will appear in a forthcoming paper.

This is not to say that a modification of Example 9.3 under $\clubsuit$ cannot produce a nonmetrizable Reed space. A more generous cutting down of the $\mathcal{W}_k(\nu_0)$ in the refinements $W_k(\overline{\nu})$ to produce $G_k(\overline{\nu})$ than the one in Example 9.3 may be worth pursuing. Given the failure of even Rudin and Balogh to find a counterexample under the less demanding Problem 2, under the much stronger axiom $\diamondsuit^+$, a consistent counterexample under $\clubsuit$ even to Problem 2 would be a remarkable achievement.

Other axioms make Example 9.3 much looser, and may give a better chance at a counterexample for Problem 1. Looseness is obvious under MA_{ω_1}. A well known consequence of MA_{ω_1} is that, for each ladder system $\mathcal{L}$ on ω_1, there is a partition $\mathcal{P}$ of ω_1 into countably many subsets such that each one meets each ladder in a finite set. This has the effect of making $\mathcal{W}_0^0$ a σ-discrete collection in M_1. However, in M_2, the discreteness of many of the $P_j \in \mathcal{P}$ will be lost. All it takes to destroy discreteness in M_2 is one ladder $L_\alpha = \{\alpha_n : n \in \omega\}$ for which some P_j meets infinitely many of the L_{α_n}. As of this writing, the issue of whether

Example 9.3 must be metrizable under MA_{ω_1} is still unsettled. This is also the case with the stronger axiom PFA, despite [1, Theorem 3.6], which has the following immediate corollary:

Corollary 9.5. [PFA] *For each ladder system $\mathcal{L}$ on ω_1, there is a club subset C of ω_1 such that $C \cap L_\alpha$ is finite for all $\alpha \in C$.*

As James Baumgartner put it in a workshop lecture at SETOP (1980), "This doesn't just kill ♣, it stomps it into the ground."

Other axioms may have implications for other kinds of spaces in $\mathcal{ZM}$. The choice of $[0,1]_D$ in V was motivated by the following measure. Let $\mathcal{W} \subset W^0_k$. Let

$$\mu(\mathcal{W}) = \big(\frac{1}{2^k}\big)\tilde{\mu}(\{r : W_k(r) \in \mathcal{W}\})$$

where $\tilde{\mu}$ is an extension of Lebesgue measure on $[0,1]$ to all subsets of $[0,1]$. Such an extension is equiconsistent with that of an uncountable measurable cardinal number. Extend μ to subsets of $\mathcal{W}^n_k$ of the form $\mathcal{R} = \{W_k(r_0, \dots r_n) : r_i \in A_i\}$ ("rectangular polytopes"), by letting $\mu(\mathcal{R}) = (1/2^k)\Pi^n_{i=0}\tilde{\mu}(A_i)$. Then extend μ to all subcollections of $\mathcal{W}$ and thence to all subsets of X_k using what is known as"Fischer's axiom," which states that all countably additive measures on a set can be extended to a countably additive measure for all subsets of X. Fischer's axiom is equiconsistent with that of a strongly compact cardinal number. This suggests the use of μ instead of the Baire Category Theorem (as in the proof of Theorem 7.4) for refuting countable metacompactness for various spaces in $\mathcal{ZM}$.

Appendix: Trees for $\mathcal{W}$ and $\mathcal{G}$

Each basic open set in $\mathcal{W}_k$ and $\mathcal{G}_k$ meets the sets X_ℓ ($\ell \le k$) in subsets that form a tree of height $k+1$ in a natural way.

Definition A.1. A *tree* is a partially ordered set $(T, \le)$ in which the predecessors of each element are well-ordered. The minimal elements of a tree T constitute *level* 0, denoted $T(0)$. Given $T(\alpha)$ for all $\alpha < \beta$, *level* β consists of the minimal level of $T \setminus \bigcup\{T(\alpha) : \alpha < \beta\}$. The *height* of T, denoted $h(T)$, is the least γ such that $T(\gamma)$ is empty. If $h(T)$ is a successor $\alpha + 1$, then a *leaf of* T is an element of $T(\alpha)$.

Since our trees will be of finite height, they will always have a maximum nonempty level.

Given a basic open set of the form $W_k(r_0)$, we define its associated tree $\mathcal{T}_k(r_0)$ as follows. Writing $\mathcal{T}$ for $\mathcal{T}_k(r_0)$, the 0th level $\mathcal{T}(0)$ is the singleton $\{B_k(r_0)\}(= \{W_k(r_0) \cap X_k\}$. Then

$$\mathcal{T}(1) = \{B_{k-1}(\sigma_{r_0}(i), r_0) : i \in \omega\}.$$

Clearly, $\bigcup \mathcal{T}(1) = W_k(r_0) \cap X_{k-1}$. In general, given $\ell < k$,

$$\mathcal{T}(k-\ell) = \{B_\ell(r_{\ell-k}, \dots, r_{-1}, r_0) : r_j = \sigma_{r_{j+1}}(i_j) \text{ for some } i_j \in \omega\},\ \ell - k \le j < 0\}. \qquad (1)$$

The order $\le$ on $\mathcal{T}_k(r_0)$ is that of "roping": $B_k(r_0)$ ropes all the $B_{k-1}(\sigma_{r_0}(i), r_0)$, which are thus its immediate successors in $\mathcal{T} = \mathcal{T}_k(r_0)$. Given $i \in \omega$, let $r_{-1} = \sigma_{r_0}(i)$. Then if $k > 1$, the immediate successors of $B_{k-1}(r_{-1}, r_0)$ are the sets of the form $B_{k-1}(r_{-1}, r_0)$, and so forth. The top nonempty level of $\mathcal{T}$ is $\mathcal{T}(k)$, and $\bigcup \mathcal{T}(k-\ell) = W_k(r_0) \cap X_\ell$ for all $\ell \le k$.

Definition A.2. A *branch* of a tree is a maximal chain, i.e., totally ordered subset.

Each branch of $\mathcal{T}_k(r_0)$ is determined by its leaf (topmost node), $B_0(r_{-k}, \dots r_{-1}, r_0)$. $\mathcal{T}_(r_0)$ is order-isomorphic to the full ω-ary binary tree of height $k+1$. The immediate successors of each node indexed by $(r_h, \dots, r_0)$ are the nodes indexed by $(\sigma_{r_h}(i), r_h, \dots, r_0)$.

The tree that goes with $G_k(r_0)$ is obtained by a process of "clipping" $\mathcal{T}_k(r_0)$ at various nodes. If B is a removed node, then all nodes B' above B are removed along with it. The clipped nodes are determined by Z_0^W, where $W = W_k(r_0)$. The simplest case is where $Z_0^W \in \mathcal{W}_{k-1}$, and one could make $Z_0^W = W_{k-1}(\sigma_{r_0}(0))$. Then Z_0^W meets $W_k(r_0)$ in $W_{k-1}(\sigma_{r_0}(0), r_0)$, and then this basic open set is removed to give $G_k(r_0) = W_k(r_0) \setminus Z_0^W$.

If instead we let $Z_0^W \in \mathcal{W}_{k-\ell}$ for $\ell > 1$, we may find ourselves clipping $\mathcal{T}_k(r_0)$ in infinitely many places, as in the following example.

Example A.3. Let $k = 2$ and $W = W_2(r_0)$. Let $\sigma_{r_0}(i) = 2^{-i}$ for all $i \in \omega$. For each i, let $\sigma_{2^{-i}}(0) = 0$. Then each $B_0(0, 2^{-i}, r_0) = W_0(0, 2^{-i}, r_0)$ is a "leaf" of $\mathcal{T}_2(r_0)$. If we make $Z_0^W = W_0(0)$, then all these leaves are clipped away in $W_2(r_0) \setminus Z_0^W = G(r_0)$.

More generally, when $k > 2$ and we let $Z_0^W = W_{k-2}(0)$, and σ_{r_0} and $\sigma_{2^{-i}}$ are as before, the clipped nodes are the $B_{k-2}(0, 2^{-i}, r_0)$. These nodes no longer equal $W_{k-2}(0, 2^{-i}, r_0)$, which are the basic clopen sets that are actually removed from $W_k(r_0) = W$ by Z_0^W to produce $G_k(r_0)$.

The tree $\mathcal{T}_k(r_0)$ is easily generalized to $\mathcal{T}_k(r_0, \dots, r_n$ for all n and is left as an exercise for the reader. The only really new feature is dealing with Z_0^W through Z_n^W, not just Z_0^W alone.

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