TOPOLOGY PROCEEDINGS
Volume 28, No. 1, 2004
Pages 1-26

CROWDING OF FUNCTIONS, PARA-SATURATION OF IDEALS, AND TOPOLOGICAL APPLICATIONS

PETER J. NYIKOS

Abstract. We study a pair of axioms that say $\aleph_2$ functions from ω_1 to ω are crowded in a rather strong sense and discuss a number of topological applications of the weaker one, Axiom F. The axioms are fitted into three different axiom schema: an old one on crowding of functions due to Donder; a modification of a still older one due to P. Erdős and R. Radó; and one on para-saturation of pairs of ideals, which is introduced here. The axioms in the third schema have varying status: the most trivial ones are true in ZFC, while others are false, and still others are independent of ZFC—but some of these are equiconsistent with large cardinal axioms and others are not.

Introduction

The following axiom on crowding of functions was introduced in [11].

Axiom F. Any family of $\aleph_2$ functions from ω_1 to ω has an infinite subfamily that is bounded on a stationary set.

It makes no difference whether "bounded" means "pointwise bounded" or "uniformly bounded by $n \in \omega$": if a function bounds a family on a stationary set S, there exists n so that the function

2000 *Mathematics Subject Classification.* Primary: 03E35, 54A25, 54D05, 54D15, 54D45; Secondary: 03E50, 03E55, 54D30.

Key words and phrases. alignment, club, countable-covering, equiconsistent, hereditarily cwH, ideal, locally compact, ω_1-compact T_5, para-saturated, Type I.

Research partially supported by NSF Grant DMS-9322613.

takes on the value n on a stationary subset of S, because the union of countably many nonstationary sets is nonstationary.

Axiom F^+replaces "is bounded" with "agrees" in the foregoing informal statement of Axiom F. Formal statements of these two axioms are given in section 1 and the relationship between them discussed; in particular, it is shown how Axiom F^+ is implied by Axiom F together with a ZFC-independent axiom involving the concept of para-saturation.

Axioms F and F^+ both have some large cardinal strength, and some steps towards assessing the exact level of that strength are taken in section 2. There we review an axiom schema due to H.-D. Donder [1] which includes Axiom F^+ as one of its members, and we fit Axiom F snugly into it just below F^+ (Theorem 2.2). This helps to explain how both of these axioms imply the consistency of some fairly large cardinals and how both are consistent if there is a supercompact cardinal. Other axioms, mostly having to do with concept of para-saturation of ideals, are discussed there and in section 4 in an effort to make it easier to assess their exact consistency strength. Although axioms F and F^+ do not mention para-saturation of ideals, Axiom F^+ fits nicely between two axioms in a general schema of axioms that do involve the concept.

I am indebted to the referee for numerous improvements.

1. Two basic axioms and the relationship between them

We begin by recalling some elementary facts about subsets of ω_1.

Definition 1.1. A subset of ω_1 that is closed and uncountable ("unbounded") is called a *club.* A subset of ω_1 is *stationary* if it meets every club, and *nonstationary* if it is not stationary. A collection $\mathcal{I}$ of sets is an *ideal* (a *σ-ideal*, resp.) if it is downwards closed (that is, if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$) and the union of finitely (*of countably*, resp.) many members of $\mathcal{I}$ is again in $\mathcal{I}$.

Clearly, the nonstationary subsets of ω_1 form an ideal, denoted $NS(\omega_1)$. The union of countably many nonstationary subsets of ω_1 is nonstationary: this is an immediate consequence of the well-known fact that the intersection of countably many clubs is a club. Thus, it is easy to see that the following formal statement of Axiom F^+ is equivalent to the informal statement given in the introduction.

Axiom $\mathbf{F}^+$. Given any family of functions $\{f_\alpha : \alpha < \omega_2\}$ from ω_1 to ω, there is a stationary set E and an infinite $A \subset \omega_2$ and $k \in \omega$ such that $f_\zeta(\sigma) = k$ for all $\zeta \in A$ and all $\sigma \in E$.

In other words, any family of $\aleph_2$ functions from ω_1 to ω has an infinite subfamily which takes on the same constant value on some common stationary set E. This follows from the same subfamily Z simply agreeing on a stationary set S, because S splits up naturally into subsets S_k such that $f(\sigma) = k$ for all $\sigma \in S_k$ and all $f \in Z$, and at least one of the S_k has to be stationary.

Axiom F can be expressed by writing "$\leq k$" instead of "$= k$," as in Axiom F^+. Obviously, Axiom F^+ implies Axiom F, but I do not know whether the converse is true, nor even whether the two axioms are equiconsistent. Theorem 1.3 below shows that the converse implication does hold if the nonstationary ideal on any stationary subset of ω_1 is $(\omega, n\,;\,\omega)$-para-saturated for all finite n.

Definition 1.2. Let κ be a cardinal number, let E be a stationary subset of ω_1, and let n be a natural number. The ideal $NS(E)$ of nonstationary subsets of E is $(\kappa, n\,;\,\omega)$-*para-saturated* ($(\kappa, \omega\,;\,\omega)$-*para-saturated*, resp.) if, whenever $\{\mathcal{M}_i : i \in \kappa\}$ is a family of partitions of E into n or fewer sets (*into countably many sets*, resp.), then it is possible to choose $M_i \in \mathcal{M}_i$ for infinitely many $i \in \omega$ in such a way that the intersection of the chosen sets M_i is stationary. One can replace the third parameter by a natural number in the obvious way. For instance, $(\kappa, \omega\,;\,2)$-*para-saturation* results from replacing "for infinitely many" with "for a pair of distinct" in the definition of $(\kappa, \omega\,;\,\omega)$-para-saturation.

These concepts generalize straightforwardly to arbitrary ideals on arbitrary sets, and a good theory results if either one is confined to σ-ideals or to families of finite partitions. For instance, substituting "is uncountable" for "is stationary" at the end of Axiom F^+ transfers both definitions from $NS(E)$ to the ideal of countable subsets of E. Further generalizations will be given in section 4.

The axiom that $\mathrm{NS}(E)$ is $(\omega, n\,;\,\omega)$-para-saturated for all n and all stationary subsets E of ω_1 is implied by the axiom $\mathfrak{p} > \omega_1$, which follows from $\mathrm{MA}(\omega_1)$. (See Definition 3.4 and Theorem 4.6 (a).) On the other hand, the axiom fails even for $n = 2$ and $E = \omega_1$ in any model formed by adding uncountably many Cohen reals in the standard way to a model of ZFC; see the discussion following Definition 4.3.

Theorem 1.3. *Axiom F^+follows from Axiom F, together with the axiom that the nonstationary ideal on every stationary subset of ω_1 is $(\omega, n\,;\,\omega)$-para-saturated for all $n < \omega$.*

Proof: Let $\{f_\alpha : \alpha < \omega_2\}$ be a family of functions from ω_1 to ω. Assuming Axiom F holds, let S, k, and Z be as in its conclusion. List Z as $\{\zeta_i : i \in \omega\}$ and for each $i \in \omega$ and $j \leq k$, let $M_i^j = \{\alpha : f_{\zeta_i}(\alpha) = j\}$. Then $\mathcal{M}_i = \{M_i^j : j \leq k\}$ is a partition of S for each $i \in \omega$. Using the para-saturation property, let $M_i^{g(i)}$ be chosen from $\mathcal{M}_i$ for each i in some infinite subset Z' of ω, so that the intersection of all the chosen sets $M_i^{g(i)}$ is stationary. Then there exists $j \leq k$ such that $g(i) = j$ for all i in some infinite subset A of Z', and so Axiom F$^+$ holds. □

Implicit in the foregoing proof is a natural bijection between families $\langle f_\alpha : \alpha < \tau\rangle$ of functions from a set S of ordinals to ω, and $\omega \times \tau$ matrices of sets $M(n, \alpha)$ in which each column is a partition of S. (As is standard in matrix theory, and in contrast to the usual picture of $\mathbb{R}^2$, rows fix the first coordinate and columns fix the second coordinate.) Axiom F$^+$ fails if, and only if, there is a matrix of the following form: $S = \omega_1$, $\tau = \omega_2$, and the intersection of any family of infinitely many sets from the same row is nonstationary. The axiom that $NS(\omega_1)$ is not $(\omega_2, \omega\,;\,\omega)$-para-saturated produces a matrix of the same sort, but the infinitely many chosen sets are even allowed to be taken from different rows, and the only restriction is that no two are allowed to be in the same column. Matrices with similar properties will be considered in the next section, and also section 4, to facilitate discussion of the equiconsistency strength of Axiom F and related axioms.

The comparison in the preceding paragraph makes clear that Axiom F$^+$ implies that $NS(\omega_1)$ is $(\omega_2, \omega\,;\,\omega)$-para-saturated. Is the converse true? At the opposite extreme, in the light of the large cardinal strength of axioms F and F$^+$(see below), the following problem is also unsolved.

Problem 1. *Is it possible to show the consistency of the nonstationary ideal being $(\omega_2, \omega\,;\,\omega)$-para-saturated without using large cardinal axioms?*

Problem 2. *Is Axiom F enough to imply that NS(ω_1) is* $(\omega_2, \omega\,; \omega)$-*para-saturated? to imply Axiom F*$^+$*?*

The remainder of this section gives yet another perspective on axioms F and F$^+$, this time in the partition calculus. We use the natural bijection ψ between families $F = \{f_\alpha : \alpha < \kappa\}$ of functions from λ to ρ and functions from $\kappa \times \lambda$ to ρ given by $\psi(F)(\alpha, \beta) = f_\alpha(\beta)$. Thinking of $\psi(F)$ as a ρ-coloring of the points of $\kappa \times \lambda$, we see that Axiom F$^+$ is equivalent to the following: for every ω-coloring of $\omega_2 \times \omega_1$, there is an infinite $Z \subset \omega_1$ and a stationary $E \subset \omega_1$ such that all points of $Z \times E$ get the same color. Axiom F is equivalent to getting $Z \times E$ to have only finitely many colors.

The following definition, generalizing the above, is a simple modification of one in [5].

Definition 1.4. Let κ, λ, and ρ be cardinal numbers and let $\mathcal{K} \subset \mathcal{P}(\kappa)$, $\mathcal{L} \subset \mathcal{P}(\lambda)$. We write

$$\binom{\kappa}{\lambda} \to \binom{\mathcal{K}}{\mathcal{L}}^{1,1}_{\rho}$$

if for each partition of $\kappa \times \lambda$ into ρ subsets, there are $K \in \mathcal{K}$ and $L \in \mathcal{L}$ such that every element of $K \times L$ is in the same member of the partition.

P. Erdős and R. Radó defined this notation in [5] for cardinals μ and ν in place of $\mathcal{K}$ and $\mathcal{L}$, respectively. Their definition is equivalent to putting $\mathcal{K} = [\kappa]^\mu$ and $\mathcal{J} = [\lambda]^\nu$, and has been extended to ordinals in the obvious way. The status of some of the resulting axioms can be deduced from the discussion in the following section and in section 4.

2. The status of Axioms F and F$^+$

Large cardinal axioms are required for the consistency of Axiom F and hence of F$^+$. This can be seen from the fact that Axiom F fails if there is a Kurepa tree—a tree of height ω_1 in which every level is countable and in which there are more than $\aleph_1$ uncountable chains. (See Theorem 2.2 below.) As is well known, the nonexistence of a Kurepa tree is equiconsistent with the existence of an inaccessible cardinal.

Much better lower bounds on the consistency strength of Axiom F are provided by the following axiom schema, taken from [1].

Definition 2.1. Let $P_\kappa(\rho,\tau)$ ($P^*_\kappa(\rho,\tau)$, resp.) denote the following statement:

There exists a sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ of functions $f_\alpha : \kappa \to \rho$ such that, for all $X \subset \kappa^+$ of order type $\geq \tau$:

$$|\{\xi < \kappa : \forall \alpha, \beta \in X \ f_\alpha(\xi) = f_\beta(\xi)\}| < \kappa$$

$$(\{\xi < \kappa : \forall \alpha, \beta \in X \ f_\alpha(\xi) = f_\beta(\xi)\} \text{ is nonstationary in } \kappa, \text{resp.})$$

Obviously, the axioms in this schema decrease in strength with increasing τ or ρ. In the notation of Erdős and Radó, the negation of $P_\kappa(\rho,\tau)$ is equivalent to

$$\binom{\kappa^+}{\kappa} \to \binom{\tau}{\kappa}^{1,1}_{\rho}$$

and it is left as an exercise for the reader to put the negation of $P^*_\kappa(\rho,\tau)$ into the notation of Definition 1.4.

For regular $\kappa > \omega$, $P_\kappa(\rho,\tau)$ clearly implies $P^*_\kappa(\rho,\tau)$, while the converse is true if τ is a successor ordinal and $\rho^{|\tau|} \leq \rho^+ = \kappa$ [1, Lemma 1]. Thus, $P^*_{\omega_1}(\omega, 2)$ is equivalent to $P_{\omega_1}(\omega, 2)$, for example. The latter axiom has been termed *the transversal hypothesis* and denoted TH(ω_1) in [1] and *the weak Kurepa hypothesis* (wKH) in [7]. Both give fundamental set-theoretic consequences of this axiom, which is clearly equivalent to the existence of an $\omega \times \omega_2$ matrix of subsets of ω_1 in which the columns give partitions of ω_1 and the intersection of any two distinct elements in the same row is countable. This is clearly negated by Axiom F$^+$, which is simply $\neg P^*_{\omega_1}(\omega,\omega)$ in the notation of 2.1. From [3, Theorem 2.16] it also follows that the axioms $P_{\omega_1}(\omega, k)$ are equivalent for all finite $k > 1$. The following theorem fits Axiom F snugly into this schema.

Theorem 2.2. *Each of the following statements implies the next:*

1. *KH ("There is a Kurepa tree.")*
2. *$P_{\omega_1}(\omega, k)$ holds for some finite $k > 1$.*
3. *Axiom F fails.*
4. *Axiom F$^+$fails; in other words, $P^*_{\omega_1}(\omega,\omega)$ holds.*

Proof: (1) implies (2): Given a Kurepa tree T, index each level of the tree by the natural numbers. Let $\{B_\xi : \xi < \omega_2\}$ be a one-to-one indexing of $\aleph_2$ branches of T, and let $M(n, \xi)$ be the set of all $\alpha < \omega_1$ such that B_ξ meets the α-th level of T in the element indexed by n. It is easy to see that the sets $M(n, \xi)$ give exactly the kind of matrix described above, whose existence is equivalent to $P_{\omega_1}(\omega, 2)$.

(2) implies (3): If $k \in \omega$, and E is a stationary subset of ω_1, then any family $f_i (i \leq k^2)$ of $k^2 + 1$ functions from E to k must have a subset of size $\geq k + 1$ that agrees on a stationary set: let $E_0 = E$ and, if E_i is stationary, let E_{i+1} be a stationary subset of E_i on which f_i is constantly equal to some $j_i < k$; by the Pigeonhole Principle, at least $k + 1$ of the f_i must agree on E_{k^2+1}. From this it easily follows that Axiom F implies $\neg P^*_{\omega_1}(\omega, k+1)$, which is equivalent to $\neg P_{\omega_1}(\omega, k+1)$. $\square$

Now it is already quite difficult to negate $\mathrm{TH}(\omega_1)$: Donder and J.-P. Levinski credit Jensen with showing, in effect, that the negation of $\mathrm{TH}(\omega_1)$ implies that $a^\#$ exists for all countable $a \subset \omega_1$ [3]. They give a different formulation of $\mathrm{TH}(\omega_1)$ in [3]:

There exists a sequence $\langle g_\alpha : \alpha < \omega_2 \rangle$ of functions $g_\alpha : \omega_1 \to \omega_1$ such that $g_\alpha(\xi) \leq \xi$ for all ξ, and such that whenever $\alpha \neq \beta$, $\{\xi < \omega_1 : \ g_\alpha(\xi) = g_\beta(\xi)\}$ is countable.

Countability of each ξ makes it trivial to transfer the functions from ${}^{\omega_1}\omega_1$ to ${}^{\omega_1}\omega$, so to speak. For each $\xi < \omega_1$ we let $\varphi_\xi : \xi + 1 \to \omega$ be injective, and replace $g_\alpha(\xi)$ with $\varphi_\xi(g_\alpha(\xi))$; then $\varphi_\xi(g_\alpha(\xi)) = \varphi_\xi(g_\beta(\xi))$ if, and only if, $g_\alpha(\xi) = g_\beta(\xi)$. A similar trick is behind Donder's statement [1] that $\mathrm{TH}(\kappa^+)$ is obviously implied by what he there calls $\mathrm{wCC}(\kappa^+)$ for all infinite κ. Donder and P. Koepke [2] more logically attach the notation $\mathrm{wCC}(\kappa^+)$ to the negations of the respective axioms, since the negation of the one for $\kappa = \omega$ is a weak form of the Chang Conjecture. A combinatorial version of this weak form is as follows:

$\mathrm{wCC}(\omega_1)$: For every sequence $\langle g_\alpha : \alpha < \omega_2 \rangle$ of functions from ω_1 to itself, there are indices $\alpha < \beta$ such that $\{\xi < \omega_1 : g_\alpha(\xi) < g_\beta(\xi)\}$ is nonstationary.

This axiom is equiconsistent with the existence of an almost $< \omega_1$-Erdős cardinal [2]. This, together with the fact [7] that

TH(ω_1) is negated by Chang's Conjecture, gives a fairly tight placement for the consistency strength of $\neg$TH(ω_1), which is further improved in [3].

The consistency strength of the other $P_{\omega_1}(\omega, \tau)$ axioms is a different matter, however. Donder has shown [1] that $\neg P_{\omega_1}(\omega, \omega^\omega)$ implies the existence of an inner model with a measurable cardinal, and leaves open the question whether it implies even stronger axioms. Although axioms F and F$^+$ are formally less strong, we do not seem to have have any better bounds on them than we do on $\neg P_{\omega_1}(\omega, \omega^\omega)$. A (rather high!) upper bound for these three axioms is given by the following remarkable axiom, which is consistent even for $E^* = \omega_1$ if ZF+AD is consistent [16, Corollary 6.82]. As is well known, ZF+AD is equiconsistent with the existence of an inaccessible cardinal above infinitely many Woodin cardinals. This, in turn, is consistent if it is consistent that there is a supercompact cardinal.

Axiom 2.3. The ideal $NS(E^*)$ is $\aleph_1$-*dense* for some stationary subset E^* of ω_1; that is, there is a family $\mathcal{E}$ of $\aleph_1$ stationary subsets of E^* such that, given any stationary $S \subset E^*$, there exists $E \in \mathcal{E}$ such that $E \setminus S$ is nonstationary. ("E is almost inside S.")

The following is well-known, but the proof is so short we include it here.

Theorem 2.4. *If $NS(E^*)$ is $\aleph_1$-dense for some stationary subset E^* of ω_1, then $P^*_{\omega_1}(\omega, \tau)$ fails for all countable ordinals τ. Hence, in particular, $\neg P^*_{\omega_1}(\omega, \omega)$ (= Axiom F$^+$) holds.*

Proof: Let $\langle f_\alpha : \alpha < \omega_2\rangle$ be a family of functions from ω_1 to ω. Let $\mathcal{E}$ be as in Axiom 2.3. For each $\alpha < \omega_2$ let $E_\alpha \in \mathcal{E}$ and $n_\alpha \in \omega$ be such that $E_\alpha \setminus f^{\leftarrow}\{n_\alpha\}$ is nonstationary. Then there exist $E \in \mathcal{E}$ and $n \in \omega$ such that $E = E_\alpha$ and $n = n_\alpha$ for uncountably many (in fact, ω_2-many) α. Let $X \subset \{\alpha : E_\alpha = E, n_\alpha = n\}$ be of order type τ. Since $E \setminus \bigcap\{f_\alpha^{\leftarrow}\{n\} : \alpha \in X\}$ is nonstationary, $\{\xi \in E^* : f_\alpha(\xi) = n$ for all $\alpha \in X\}$ is a stationary subset of ω_1. □

3. Topological applications

Our first topological application is a straightforward but quite specialized application of $\neg$TH(ω_1). We adopt the following terminology:

Definition 3.1. Given a function $f : X \to \omega_1$, a subset of X will be said to be *unbounded* if its image in ω_1 is unbounded. A pair of unbounded subsets of X is [*weakly*] *almost disjoint* if the image of their intersection is countable (*nonstationary*, resp.).

Recall that a space is said to be [*countably*] *compact* if every [countable] open cover has a finite subcover. An elementary fact is that countable compactness implies that every infinite subset has an accumulation point, and that the converse is true for T_1 spaces. All three properties are preserved by continuous images. Compact subsets of Hausdorff spaces are closed, so that the compact subsets of ω_1 are precisely the countable closed subsets. Also, the countably compact subsets of ω_1 coincide with the closed subsets; therefore, if X and f are as in 3.1 and f is continuous, then the image of any unbounded, countably compact subset of X is a club subset of ω_1.

If $f : X \to \omega_1$ is continous, a pair of closed, countably compact subsets is weakly almost disjoint iff it is almost disjoint: the intersection, being closed, is also countably compact, and if it were unbounded, its image would be a club.

Theorem 3.2. [$\neg$ TH(ω_1)] *If $\pi : X \to \omega_1$ is continuous, and each fiber $\pi^{\leftarrow}\xi$ is countable, then X cannot contain a weakly almost disjoint family of $\aleph_2$ unbounded countably compact subspaces.*

Proof: We use the fact that TH(ω_1) is equivalent to $P^*_{\omega_1}(\omega, 2)$. Given a family $\{W_\nu : \nu < \omega_2\}$ of unbounded countably compact subsets, let $C_\nu = \pi^{\rightarrow}W_\nu$ for each ν. For each $\alpha < \omega_1$ let $h_\alpha : \pi^{\leftarrow}\{\alpha\} \to \omega$ be injective, let $g_\nu : \omega_1 \to X$ be such that $g_\nu(\alpha) \in W_\nu$ and $\pi(g_\nu(\alpha)) = \alpha$ whenever $\alpha \in C_\nu$, and let $f_\nu(\alpha) = h_\alpha(g_\nu(\alpha))$ for $\alpha \in C_\nu$ and $f_\nu(\alpha) = 0$ otherwise. By $\neg P^*_{\omega_1}(\omega, 2)$, there is a pair of distinct ordinals $\eta, \nu < \omega_2$ such that $f_\nu(\alpha) = f_\eta(\alpha)$ for all α in a stationary set. This implies W_ν and W_η are not almost disjoint. $\square$

The cardinality of the family in Theorem 3.2 cannot be lowered, as the following simple example shows: Let $Y = \omega_1^2$ with the product topology obtained by putting the usual topology on the first factor and the discrete topology on the second factor. Let X be the subspace of all points below the diagonal; then X and the projection to the first coordinate satisfy the hypotheses of 3.2, and each of the $\aleph_1$ horizontal lines is a copy of ω_1.

The following example shows that the consistency of at least an inaccessible is needed for Theorem 3.2.

Example 3.3. Let X be a Kurepa tree with the interval topology. A base for this topology consists of all sets of the form $(s,t] = \{x : s < x \leq t\}$ together with all singletons of minimal tree elements. Every uncountable branch of T is a copy of ω_1. The function $f : X \to \omega_1$ that takes each element of the αth level to α satisfies the hypotheses of Theorem 3.2 other than $\neg\,\mathrm{TH}(\omega_1)$, but the conclusion fails because any two of the $\geq \aleph_2$ branches are almost disjoint.

In Theorem 3.2, the cardinality condition on the fibers of f is very restrictive. Our next application of Axiom F will replace it with a much weaker condition, but will also assume normality of X. It is very similar to a theorem involving a well-known set-theoretic axiom which requires no large cardinals and which may or may not be compatible with Axiom F. We present this axiom and theorem first.

Definition 3.4. If $\mathcal{F}$ is a collection of subsets of a set A, a *pseudo-intersection* of $\mathcal{F}$ is a subset P of A such that $P \subset^* F$ (that is, $P \setminus F$ is finite) for all $F \in \mathcal{F}$.

The cardinal $\mathfrak{p}$ is the minimum cardinality of a family $\mathcal{F}$ of subsets of ω with no infinite pseudo-intersection, such that the intersection of every finite subfamily of $\mathcal{F}$ is infinite.

Given a collection $\mathcal{F}$ of infinite subsets of ω, a subset R of ω *reaps* $\mathcal{F}$ if $F \cap R$ and $F \setminus R$ are both infinite for all $F \in \mathcal{F}$.

The *reaping number* $\mathfrak{r}$ is the least cardinality of a collection of infinite subsets of ω that cannot be reaped.

It is a well-known fact that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{r} \leq \mathfrak{c}(= 2^\omega)$. These and many other "small uncountable" cardinals are discussed in [15] and [14].

In the following theorem, Ł denotes the subspace of $\omega_1 \times (\omega+1)$ obtained by removing the points $\langle\lambda,\omega\rangle$ from the "top edge," for all limit ordinals λ. Ł is a simple example of a non-normal subspace of a normal space. Let

$$F_0 = \{\langle\alpha,\omega\rangle : \alpha \text{ is a countable successor ordinal }\} \quad \text{and}$$

$$F_1 = \{\langle\gamma,n\rangle : \gamma \text{ is a countable limit ordinal and } n \in \omega\},$$

respectively. If U is an open set containing F_0, then U must contain all but finitely many points in each vertical column of the form $\{\alpha\} \times \omega$ where α is a countable successor ordinal. Consequently, there exists $n \in \omega$ such that $\langle \alpha, n \rangle \in U$ for uncountably many successor α; and this implies that U has points of F_1 in its closure. Similar arguments can be used to show that Ł fails to have quite a number of "nice" properties, some of which will be given in Definition 3.9.

Theorem 3.5. $[\mathfrak{p} > \omega_1]$ *Let Y be a subspace of a normal space X for which there is a continuous $\pi : X \to \omega_1$ and a stationary subset S of ω_1 such that $\pi^{\leftarrow}\{\sigma\} \cap Y$ is sequentially compact for all $\sigma \in S$. If Y contains an infinite family of pairwise disjoint unbounded copies of ω_1, then Y contains a copy of Ł.*

Proof: We will use normality of X to find a relatively discrete family $\{K_n : n \in \omega\}$ of unbounded copies of ω_1 in Y. (By "relatively discrete," we mean that no point of K_n is in the closure of $\bigcup\{K_m : m \neq n\}$.) Once this is accomplished, we cut each K_n down, if necessary, to obtain a copy Ω_n of ω_1 which meets each fiber $\pi^{\leftarrow}\{\xi\}$ in at most one point. A simple way to do the cutting down is to let $f_n : \omega_1 \to X$ be an embedding with range K_n, and to let Ω_n be the image of f_n restricted to the club $C \subset \omega_1$ which is the intersection of the club sets

$$C_n = \{\alpha : \pi(f_n(\xi)) < \alpha \text{ for all } \xi < \alpha \text{ and } \pi(f_n(\alpha)) = \alpha\}.$$

$\vdash$ *Proof that C_n is a club:* It is easy to see that C_n is closed: if $\alpha_i \nearrow \alpha$ and $\alpha_i \in C_n$ for all $i \in \omega$, and $\xi < \alpha$, then $\xi < \alpha_i$ for some i and so $\pi(f_n(\xi)) < \alpha_i < \alpha$; and since $\langle f_n(\alpha_i) : n \in \omega\rangle$ converges to $f_n(\alpha)$, it follows from continuity of π that $\langle \pi(f_n(\alpha_i)) : i \in \omega\rangle = \langle \alpha_i : i \in \omega\rangle$ converges to $\pi(f_n(\alpha)) = \alpha$. The proof that C_n is unbounded is done by a standard leapfrog argument using unboundedness of K_n. To wit, given $\alpha \in \omega_1$, let $\alpha_0 = \alpha$; then, with α_i defined, let $\beta_i = sup\{\pi(f_n(\xi)) : \xi < \alpha_i\}$ and let α_{i+1} be a countable ordinal γ such that both $\pi(f(\gamma))$ and γ are greater than both α_i and β_i. Such a γ exists since K_n meets each set of the form $\pi^{\leftarrow}[0, \eta]$ in a countable set: $K_n \cap \pi^{\leftarrow}[0, \eta]$ is relatively clopen in K_n and so it must be either countable or co-countable in K_n. Let $\delta = sup\{\alpha_i : i \in \omega\}$. Then $\pi(f_n(\alpha_i))$ converges to $\pi(f(\delta))$ and its terms are sandwiched in between the α_i, so it converges to δ; and the other criterion for δ being in C_n is trivially satisfied. $\dashv$

Let D denote the derived set of $C \cap S$ in ω_1 and let $\{\gamma_\xi : \xi < \omega_1\}$ list the club $\overline{C \cap S} = (C \cap S) \cup D$ in its natural order while $\{\alpha_\eta : \eta < \omega_1\}$ lists $(C \cap S) \setminus D$ in its natural order. For each $\xi \in \omega_1$ let x_n^ξ be the unique point of $\Omega_n \cap \pi^{\leftarrow}\{\gamma_\xi\}$ while w_n^ξ denotes the unique point of $\Omega_n \cap \pi^{\leftarrow}\{\alpha_\xi\}$, so that if ν_ξ is the ξth nonlimit ordinal then $w_n^\xi = x_n^{\nu_\xi}$.

Let $Z = \bigcup_{n\in\omega} \Omega_n$ and let A_0 be an infinite subset of ω such that $\{w_n^0 : n \in A_0\}$ converges to some point z_0 of $Y \cap \overline{Z}$. This is made possible by the fact that $Y \cap \pi^{\leftarrow}\{\alpha_0\}$ is sequentially compact. By relative discreteness of $\{\Omega_n : n \in \omega\}$, $z_0 \notin Z$. Suppose infinite subsets A_ξ have been defined for all $\xi < \eta$ in such a way that if $\nu > \xi$ then $A_\nu \subset^* A_\xi$. Let B_η be an infinite subset of ω such that $B_\eta \subset^* A_\xi$ for all $\xi < \eta$. Let A_η be an infinite subset of B_η such that $\{w_n^\eta : n \in A_\eta\}$ converges to some point z_η (of $Y \cap \overline{Z} \setminus Z$). If this has been done for all $\eta \in \omega_1$, we get that $\{z_\eta : \eta < \omega_1\}$ is an uncountable subset of $Y \cap \overline{Z} \setminus Z$ which is discrete in the relative topology because $z_\eta \in \pi^{\leftarrow}\{\alpha_\eta\}$ and because $\alpha_\eta \in (C \cap S) \setminus D$.

Using $\mathfrak{p} > \omega_1$, let A be an infinite subset of ω such that $A \subset^* A_\eta$ for all $\eta < \omega_1$, and let $V_\eta = \{z_\eta\} \cup \{w_n^\eta : n \in A\}$. Then V_η is a copy of $\omega + 1$ for all η and each V_η is a relatively clopen subspace of $L = V \cup R$, where

$$V = \bigcup\{V_\eta : \eta \in \omega_1\} \quad \text{and} \quad R = Z \cap \pi^{\leftarrow}(\overline{C \cap S}).$$

Thanks to the discreteness of $\{\Omega_n : n \in \omega\}$, each $\Omega_n \cap \pi^{\leftarrow}(\overline{C \cap S})$ is a relatively clopen copy of ω_1 in L and hence L is homeomorphic to the space Ł defined above. In fact, if $\{n_i : i \in \omega\}$ lists A in its natural order, then the mapping $\varphi : L \to$ Ł such that

$$\varphi(z_\eta) = \langle \alpha_\eta, \omega \rangle \quad \text{and} \quad \varphi(x_{n_i}^\xi) = \langle \xi, i \rangle$$

is easily seen to be a homeomorphism.

The task that remains is to produce an infinite relatively discrete family of copies of ω_1, given a family $\{F_n : n \in \omega\}$ of disjoint copies of ω_1 satisfying the hypothesis of the theorem.

Claim. There exists an infinite family of disjoint open subsets U_n of X, each of which contains a closed subset K_n of some F_m such that K_n has unbounded image.

Once the claim is proved, the open sets U_n ensure the relative discreteness of $\{K_n : n \in \omega\}$, and each K_n is a copy of ω_1 because every countably compact noncompact subset of ω_1 is a copy of ω_1. (Actually, the K_n we produce will be co-countable subsets of some subfamily of the F_n, but we will not need this fact.) The proof of the claim actually goes through if we begin with any countable family of disjoint closed-in-X countably compact subsets F_n with unbounded image. Even though Y may not be closed in X, any copy of ω_1 with unbounded image will be closed in X. This is immediate from the fact that every countable subset of ω_1 has compact closure, and the fact that any unbounded copy of ω_1 in X meets each $\pi^{\leftarrow}[0, \alpha]$ in a countable set.

Proof of Claim: Using Urysohn's Lemma, let $g_0 : X \to [0, 1]$ be a continuous function sending F_0 to 0 and F_1 to 1. Given any F_n, there exists $r_n \in [0, 1]$ such that, for every neighborhood N of r_n, there are uncountably many $\alpha \in \omega_1$ for which there exist $y_n^\alpha \in F_n \cap \pi^{\leftarrow}\{\alpha\}$ such that $g_0(y_n^\alpha) \in N$. Since F_n is countably compact, it follows that there are also uncountably many γ and $z_n^\gamma \in F_n \cap \pi^{\leftarrow}\{\gamma\}$ such that $g_0(z_n^\gamma) = r_n$. Hence, $F_n^0 = F_n \cap g_0^{\leftarrow}\{r_n\}$ is also closed in X, countably compact, and has unbounded image.

Case 1. $\{r_n : n \in \omega\}$ *is infinite.* In this case, pick a subsequence $\{r_n : n \in A\}$ of distinct reals converging monotonically to some $r \in [0, 1]$ that is distinct from any of the r_n. Then if the sequence is increasing, the open intervals

$U_0 = g_0^{\leftarrow}\left(0, \frac{r_0+r_1}{2}\right)$ and (if $n > 0)U_n = g_0^{\leftarrow}\left(\frac{r_{n-1}+r_n}{2}, \ \frac{r_n+r_{n+1}}{2}\right)$

are the desired disjoint open sets, and one argues analogously if the sequence is decreasing. Letting $K_n = F_n^0$ for all $n \in A$, we obtain the desired families of open sets and closed sets.

Case 2. $\{r_n : n \in \omega\}$ *is finite.* Here, we know at least that both 0 and 1 are in $\{r_n : n \in \omega\}$. We also know that some $g_0^{\leftarrow}\{r_k\}$ meets infinitely many of the sets F_i in a closed unbounded set. Let $s_0 = 0$ if $k \neq 0$; otherwise, let $s_0 = 1$. In either case, let $t_0 = r_k$ and let $K_0 = F_{s_0}^0 = F_{s_0} \cap g_0^{\leftarrow}\{s_0\}$. Let G_0 and H_0 be disjoint open subsets of $[0,1]$ containing s_0 and t_0, so that $K_0 \subset g_0^{\leftarrow} G_0$ and $g_0^{\leftarrow}\{t_0\} \subset g_0^{\leftarrow} H_0$.

Let $U_0 = g_0^{\leftarrow} G_0$. Let I_0 be a closed subinterval of H_0 containing t_0 in its interior. Using the fact that $X_0 = g_0^{\leftarrow} I_0$ satisfies all the hypotheses of this theorem, we obtain a continuous function g_1 from X_0 to $[0,1]$ in the same way we obtained g_0, mutatis mutandis. In particular, we use the set of all the unbounded $F_i \cap g_0^{\leftarrow}\{t_0\}$ in place of $\{F_n : n \in \omega\}$. If the analogue of Case 1 obtains, we are done; otherwise, we obtain a closed unbounded subset K_1 of one of the $F_i \cap g_0^{\leftarrow}\{t_0\}$ along with disjoint open sets G_1 and H_1 analogous to G_0 and H_0, as well as analogues of all other sets with subscripts of 0. Let $U_1 = g_1^{\leftarrow} G_1$. Proceeding in this way through infinitely many steps if necessary, we obtain the desired sets U_n and K_n. $\square$

As remarked in the course of the proof, each set K_n we construct is actually a co-countable subset of one of the F_m. This is because (1) any open subset of ω_1 that contains a closed unbounded subset is co-countable and (2) consequently, every continuous real-valued function on a copy of ω_1 is eventually constant. Thus, all the sets $F_n \cap g_i^{\leftarrow}\{r\}$ are co-countable in F_n if they are unbounded.

Since we did not use normality once the K_n were obtained, we have also shown:

Corollary 3.6. [$\mathfrak{p} > \omega_1$] *Let Y be a space for which there are a continuous $\pi : Y \to \omega_1$ and a stationary subset S of ω_1 such that the fiber $\pi^{\leftarrow}\sigma$ is sequentially compact for all $\sigma \in S$. If Y contains an infinite, relatively discrete family of disjoint unbounded copies of ω_1, then Y contains a copy of L.*

It would be interesting to know whether normality can be eliminated in Theorem 3.5 as well.

The following theorem is called "conditional" because it is not known whether the two axioms used are compatible with each other.

Theorem 3.7 (Conditional)**.** [Axiom F + $\mathfrak{p} > \omega_1$] *Let X be a normal space for which there is a continuous function $f : X \to \omega_1$ with respect to which there is an almost disjoint family of $\aleph_2$ unbounded copies of ω_1, and such that each fiber is locally compact, sequential, and σ-compact. Then X contains a copy of L.*

Proof: Let $\mathcal{W} = \{W_\nu : \nu < \omega_2\}$ be an almost disjoint family of $\aleph_2$ unbounded copies of ω_1. By Theorem 3.5, it suffices to show that X has a subspace Y such that $Y \cap \pi^{\leftarrow}\{\sigma\}$ is sequentially compact for all σ in some stationary subset S of ω_1, and to find an infinite subfamily $\mathcal{W}_0$ of $\mathcal{W}$ such that $W \subset Y$ for all $W \in \mathcal{W}_0$.

By sequentiality, each compact subset of $\pi^{\leftarrow}\{\alpha\}$ is sequentially compact, so $\pi^{\leftarrow}\{\alpha\}$ is the ascending union of compact, sequentially compact sets K^n_α $(n \in \omega)$ for each $\alpha \in \omega_1$, with each K_n in the interior of K_{n+1}. If $\pi^{\leftarrow}\{\alpha\}$ is compact, let it equal K^n_α for all n. Let $C_\nu = \pi^{\rightarrow}W_\nu$ for all $\nu < \omega_2$.

Let $f_\nu : \omega_1 \to \omega$ be defined by letting $f_\nu(\xi)$ be the least n such that $\pi^{\leftarrow}\{\xi\} \cap W_\nu \subset K^n_\xi$ whenever $\xi \in C_\nu$. Such an n exists since $\pi^{\leftarrow}\{\xi\} \cap W_\nu$ is countably compact and so any countable open cover has a finite subcover. If $\xi \notin C_\nu$ let $f_\nu(\xi) = 0$. Axiom F now gives a stationary subset S of ω_1 and an infinite subset Z of ω_2 and $n \in \omega$ such that $f_\nu(\xi) < n$ for all $\nu \in Z$ and all $\xi \in S$. Let $C = \bigcap\{C_\nu : \nu \in Z\}$. Then C is a club subset of ω_1 and $\pi^{\leftarrow}\{\xi\} \cap W_\nu \subset K^n_\xi$ for all $\xi \in S \cap C$ and all $\nu \in Z$. Let

$$Y = \bigcup\{K^n_\xi : \xi \in S \cap C\} \cup \pi^{\leftarrow}[\omega_1 \setminus (S \cap C)].$$

Then Y is as claimed above, and we can let $\mathcal{W}_0 = \{W_\nu : \nu \in Z\}$. $\square$

By being less specific in the kind of subspace we find in X, we can remove our reliance on $\mathfrak{p} > \omega_1$ and obtain a real theorem (modulo large cardinals). A perusal of the proof of 3.5 reveals that $\mathfrak{p} > \omega_1$ was used only to line up the neighborhoods of the points z_η. If we omit all mention of the subset A that is derived from the sets A_ξ and use all of ω in its place, we still obtain a subspace homeomorphic to what will here be called an NN-plank.

Definition 3.8. An *NN-plank* is a space with underlying set $(\omega_1 \times \omega) \cup \{\langle \alpha+1, \omega\rangle : \alpha \in \omega_1\}$ in which the relative topology on $\omega_1 \times \omega$ is the product topology, while $\langle \alpha + 1, \omega\rangle$ is the sole nonisolated point in a clopen set $V_\alpha \subset (\{\alpha + 1\} \times (\omega + 1))$ that is homeomorphic to $\omega + 1$.

Every NN-plank is locally compact, and each V_α meets infinitely many of the copies $\omega_1 \times \{n\}$ of ω_1 in a single point. Using the same sets F_0 and F_1 as before, we can show that an NN-plank is non-normal just as Ł was shown to be non-normal. Thus, we can remove the axiom $\mathfrak{p} > \omega_1$ from among the axioms in 3.5, 3.6, and 3.7 provided we substitute "an NN-plank" for "Ł." For example:

Theorem 3.7′ [Axiom F]**.** Let X be a normal space for which there is a continuous function $f : X \to \omega_1$ with respect to which there is an almost disjoint family of $\aleph_2$ unbounded copies of ω_1, and such that each fiber is locally compact, sequential, and σ-compact. Then X contains a copy of an NN-plank. Hence, X is not hereditarily normal.

Numerous other properties can be put in place of the hereditary normality here, thanks to the "negative" properties of an NN-plank. Many of them follow from its semi-(ω, ω)-antinormality.

Definition 3.9. A space X is *semi-(ω, ω)-antinormal* if it has an infinite discrete collection $\mathcal{D}$ of closed sets such that if $\mathcal{D}'$ is an infinite subcollection of $\mathcal{D}$ and $\{G(D) : D \in \mathcal{D}'\}$ is a collection of G_δ subsets of X such that $D \subset G(D)$, then $\bigcap\{\overline{G(D)} : D \in \mathcal{D}'\} \neq \emptyset$.

A space X is *quasi-paranormal* if for every countably infinite discrete collection $\{D_n : n \in \omega\}$ of closed sets there is a collection $\{U_n : n \in \omega\}$ of open sets, such that $D_n \subset U_n$ for all n, and such that $\bigcap\{\overline{U_n} : n \in \omega\} = \emptyset$.

A space X is *paranormal* (*δ-paranormal*, resp.) if for every countable discrete collection $\{D_n : n \in \omega\}$ of closed sets there is a locally finite family $\{U_n : n \in \omega\}$ of open (G_δ, resp.) sets such that $D_n \subset U_m$ iff $D_n = D_m$.

Properties related to these were introduced in [9]. It is obvious that every paranormal space is both quasi-paranormal and δ-paranormal, and that these properties are incompatible with being semi-(ω, ω)-antinormal. It is also an easy exercise to show that paranormality is implied by both normality and countable paracompactness.

Lemma 3.10. *Ł is semi-(ω, ω)-antinormal.*

Proof: As before, let $F_0 = \{\langle \alpha, \omega \rangle \in \omega_1 \times (\omega+1) : \alpha \text{ is a successor}\}$ and $F_1 = \{\langle \gamma, n \rangle \in \omega_1 \times \omega : \gamma \text{ is a limit ordinal}\}$. This time we focus on subsets of F_1. Let $F_1^n = F_1 \cap (\omega_1 \times \{n\})$. Then $\{F_1^n : n \in \omega\}$ is a discrete collection of closed subsets of Ł and so is $\mathcal{D} = \{D_n : n \in \omega\}$ whenever ω is partitioned into $\{a_n : n \in \omega\}$ and $D_n = \bigcup\{F_1^i : i \in a_n\}$. If a_n is infinite, then every G_δ containing D_n has all but countably many points of F_0 in its closure. Hence, if every a_n is infinite, and $\{G(D) : D \in \mathcal{D}'\}$ is as in 3.9, then all but countably many points of F_0 are in the closure of every $G(D)$. $\square$

For NN-planks we have a slightly weaker result.

Theorem 3.11. *No NN-plank is δ-paranormal. If $\mathfrak{r} > \omega_1$, then every NN-plank is semi-(ω, ω)-antinormal.*

Proof: For the first statement, follow the proof of Lemma 3.10, letting $a_n = \{n\}$. If G_n is a G_δ-set containing D_n, then $\bigcup\{G_n : n \in \omega\}$ has all but countably many members of F_0 in its closure. Hence, $\{G_n : n \in \omega\}$ cannot be locally finite.

For the second statement, let $A_\alpha = \{n \in \omega : V_\alpha \cap (\omega_1 \times \{n\}) \neq \emptyset\}$ and define a_n by induction as follows. Let a_0 be any set that reaps $\mathcal{A}_0 = \{A_\alpha : \alpha \in \omega_1\}$. If $\mathcal{A}_k$ and $a_k \subset \omega$ have been defined, let $\mathcal{A}_{k+1} = \{A_\alpha \setminus a_k : A_\alpha \in \mathcal{A}_k\}$ and let a_{k+1} be any set that reaps $\mathcal{A}_{k+1}$. With D_n and $G(D)$ defined as in 3.10, the reaping nature of the sets a_k insures that all but countably many points of F_0 are in the closure of every $G(D)$, $D \in \mathcal{D}'$. $\square$

The use of $\mathfrak{r} > \omega_1$ cannot be eliminated, as the following example shows.

Example 3.12. Assume there is a free ultrafilter on ω with a base $\{B_\alpha : \alpha < \omega_1\}$ such that $B_\beta \subset^* B_\alpha$ whenever $\alpha < \beta$. (It is easy to construct such an ultrafilter under CH, for example.) Let X be an NN-plank in which the neighborhoods of $\langle \alpha + 1, \omega\rangle$ are defined by letting a base be all sets of the form $\{\alpha + 1\} \times (B_\alpha \cup \{\omega\} \setminus \{0, \dots, n\})$. Every partition of ω has at most one member which is in the ultrafilter, and all the others can meet at most countably many of the B_α in an infinite set. Hence, no matter how a partition $\{a_n : n \in \omega\}$ of ω is chosen, at most one of the sets $\omega_1 \times a_n$ will have more than countably many points of F_0 in its closure.

Every closed subset of $X \setminus F_0$ differs only in a countable set from a countable (perhaps empty) union of sets of the form $C \times \{n\}$, where C is a club subset of ω_1. This makes it easy to expand any (countable) discrete collection of closed subsets of $X \setminus F_0$ to open sets such that the removal of one of these sets results in a locally finite collection. Any countable discrete collection of (closed) subsets of F_0 can be expanded to a locally finite collection of open sets: just have the nth open set miss the first n rows of $X \setminus F_0 = \omega_1 \times \omega$. A tedious but routine argument utilizing these two ingredients shows that X is quasi-paranormal and hence that $\mathfrak{r} > \omega_1$ is needed for the second part of Theorem 3.11.

Here is a fascinating problem closely related to this example.

Problem 3. *Is $\mathfrak{r} = \omega_1$ compatible with the statement that every uncountable family of subsets of ω has an uncountable subfamily that can be reaped?*

A. Dow has observed, in a private communication, that such a model of set theory cannot contain an ultrafilter with a base of cardinality ω_1.

Before going on to the next section, we mention an application of Theorem 3.5 to the following theorem implicit in [12].

Theorem A. [PFA] *If M is a normal, hereditarily collectionwise Hausdorff manifold of dimension > 1, then M is either metrizable or has a subspace X for which there is a continuous $\pi : X \to \omega_1$ such that $\pi^{\leftarrow}\{\alpha\}$ is compact for all $\alpha \in \omega_1$, and in which there is a family of $\aleph_2$ disjoint countably compact subspaces whose image is unbounded.*

Using the PFA a second time gives an application similar to the earlier uses of Axiom F.

Theorem 3.13. [PFA] *If M is a normal, hereditarily collectionwise Hausdorff manifold of dimension > 1, then M is either metrizable or contains a copy of L.*

Proof: The PFA implies $\mathfrak{p} > \omega_1$, and it also figures in the following theorem of Balogh. □

Theorem B. [PFA] *Every first countable, countably compact space is either compact or contains a copy of ω_1.*

For a proof of Theorem B, see [4, Corollary 6.6]. Every manifold is first countable, so if M is not metrizable, theorems A and B give a family of $\aleph_2$ disjoint copies of ω_1 in a subspace X of M. Their images under the map π of Theorem A are unbounded since $\pi^{\leftarrow}[0,\alpha)$ is σ-compact for each $\alpha < \omega_1$, and no first countable, σ-compact Hausdorff space can contain a copy of ω_1. Now apply Theorem 3.5.

Corollary 3.14. [PFA] *If M is a normal, hereditarily collectionwise Hausdorff manifold of dimension > 1, then M is either metrizable or contains a subspace which is neither normal nor countably paracompact, nor even δ-paranormal or quasi-paranormal.*

4. Para-saturation of ideals

The concept of para-saturation that we give here is a generalization of the concept introduced in section 1. It extends the second parameter to all cardinals and adds an extra parameter motivated by the following concept of saturation.

Definition 4.1. An ideal $\mathcal{I}$ of subsets of a set E is $(\kappa\,;\,\lambda,\mu)$-*saturated* if for every collection $\mathcal{Z}$ of κ-many members of $\mathcal{P}(E)\setminus\mathcal{I}$, there is a subcollection $\mathcal{W}$ of $\mathcal{Z}$ such that $|\mathcal{W}| = \lambda$ and such that every subcollection of $\mathcal{W}$ having μ or fewer members has intersection not in $\mathcal{I}$. $\mathcal{I}$ is κ-*saturated* if it is $(\kappa\,;\,2,2)$-saturated.

This kind of saturation was introduced by R. Laver [8], who showed the consistency of an $(\omega_2\,;\,\omega_2,\omega)$-saturated normal ideal on ω_1 assuming the consistency of a huge carkinal. Every $\aleph_1$-dense σ-ideal is easily seen to be $(\omega_2\,;\,\omega_2,\omega)$-saturated. Also, in [13, XIII, 4.3], S. Shelah begins with a stationary, co-stationary subset E of ω_1 in a model with a supercompact cardinal, and defines a semiproper forcing extension in which E remains stationary and the ideal $NS(E) \subset \mathcal{P}(E)$ of nonstationary subsets of E is $(\omega_2\,;\,\omega_2,\omega)$-saturated.

To extend the concept of para-saturation in a fruitful way, we introduce the following concept.

Definition 4.2. If $\mathcal{I}$ is an ideal of subsets of a set S, we say a collection $\mathcal{A}$ of subsets of S is $\mathcal{I}$-*disjoint* if the intersection of any pair of distinct members of $\mathcal{A}$ is in $\mathcal{I}$. A collection $\mathcal{A}$ of sets in $\mathcal{P}(S)\setminus\mathcal{I}$ is *maximal* $\mathcal{I}$-*disjoint* if it is $\mathcal{I}$-disjoint and for each $B \in \mathcal{P}(S)\setminus\mathcal{I}$ there exists $A \in \mathcal{A}$ such that $(B \cap A) \notin \mathcal{I}$.

Definition 4.3. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on a set S. We say *the pair* $\langle\mathcal{I},\mathcal{J}\rangle$ *is* $(\kappa,\iota\,;\,\lambda,\mu)$-*para-saturated* ($(\kappa,<\iota\,;\,\lambda,\mu)$-*para-saturated*, resp.) if, for each family $\mathbb{K}$ of κ maximal $\mathcal{I}$-disjoint collections of sets in $\mathcal{P}(S)\setminus\mathcal{I}$, each of which has $\leq \iota$ (*fewer than* ι, resp.) members, it is possible to choose a subfamily $\mathbb{L}$ of λ collections and a member of each collection in the subfamily $\mathbb{L}$ so that the intersection of any set of μ chosen members is not in $\mathcal{J}$. We say $\mathcal{I}$ *is* $(\kappa, \iota\,;\ \lambda, \mu)$*-para-saturated* if the pair $\langle \mathcal{I}, \mathcal{I} \rangle$ is $(\kappa, \iota\,;\ \lambda, \mu)$-para-saturated.

The concept of $(\kappa, \iota\,;\ \lambda, \mu)$-para-saturation makes sense only if $\kappa \geq \lambda \geq \mu$. On the other hand, there are no restrictions on ι vis-a-vis the other cardinals. Maximality of the members of $\mathbb{K}$ clearly implies that if λ is finite, then every ideal is $(\kappa, \iota\,;\ \lambda, \mu)$-para-saturated. Also, if ι is finite, then any ultrafilter containing the complements of the members of $\mathcal{I}$ is a witness to the fact that $\mathcal{I}$ is $(\kappa, \iota\,;\ \lambda, n)$-para-saturated for all finite n. On the other hand, if μ is infinite than we can generally expect to see only negative results or consistency results. For instance, Laver has observed, in a private communication, that the pair $\langle \mathcal{I}, \mathcal{J} \rangle$ is not $(\kappa, 2\,;\ \omega, \omega)$-para-saturated in a model obtained by adding $\max\{\kappa, \omega_1\}$ Cohen reals, where $\mathcal{I}$ and $\mathcal{J}$ are the nonstationary and countable ideals on ω_1, respectively. Laver's argument for the case $\kappa = \omega_2$ is given in [10] and his argument for the other κ is essentially the same. This is the weakest possible kind of nontrivial para-saturation property involving these two ideals and infinite μ. Since this paper is focused on ideals on ω_1, we will only analyze the case $\mu = \omega$ except for a pair of paragraphs at the end.

The following is immediate from the comments before 2.3 and after 4.1.

4.4. Observation. Every $(\kappa;\ \lambda, \mu)$-saturated ideal is $(\kappa, \iota\,;\ \lambda, \mu)$-para-saturated for all ι. In particular, if it is consistent that there is an inaccessible cardinal above infinitely many Woodin cardinals, it is consistent that $NS(\omega_1)$ is $(\omega_2, 2^{\omega_1}\,;\ \omega_2, \omega)$-para-saturated.

Clearly, the smaller κ is and the larger any of the remaining cardinals or the ideal $\mathcal{J}$ is, the more demanding it is for $\langle \mathcal{I}, \mathcal{J} \rangle$ to be $(\kappa, \iota\,;\ \lambda, \mu)$-para-saturated, and having $\mathcal{I} \subset \mathcal{J}$ is more demanding than having $\mathcal{J} \subset \mathcal{I}$. It is also not hard to see that if $\mathcal{I} \subset \mathcal{I}_0$ and $\mathcal{I}_0$ is ι^+-closed (i.e., closed under unions of cardinality $\leq \iota$), then replacing $\mathcal{I}$ by $\mathcal{I}_0$ gives a stronger condition. Unlike the concepts in section 1, Definition 4.3 is phrased in terms of maximal $\mathcal{I}$-disjoint families, rather than partitions into arbitrary subsets of the set on which $\mathcal{I}$ lives. Nevertheless, $(\kappa, \iota\,;\ \lambda, \lambda)$-para-saturation is easily seen to be equivalent to $(\kappa, \iota\,;\,\lambda)$-para-saturation when $\mathcal{I} = \mathcal{J}$ is ι^+-closed. In particular, this is true if $\mathcal{I} = \mathcal{J}$ and either ι is finite, or ι is countable and $\mathcal{I}$ is a σ-ideal, and section 1 made these restrictions for this very reason.

This change in perspective makes it possible to improve on the observation made near the end of section 1, that Axiom F^+ implies that $NS(\omega_1)$ is $(\omega_2, \omega; \omega, \omega)$-para-saturated. We can now improve the second parameter to ω_1.

Theorem 4.4. *Axiom F^+ implies that $NS(\omega_1)$ is $(\omega_2, \omega_1; \omega, \omega)$-para-saturated.*

Proof: Let $\{\mathcal{M}_\alpha : \alpha \in \omega_2\}$ be a family of maximal $NS(\omega_1)$-disjoint collections of stationary subsets of ω_1, each of cardinality $\leq \omega_1$. Let $\mathcal{M}_\alpha = \{M^\eta_\alpha : \eta < \kappa\}$ where $\kappa \leq \omega_1$. We can assume $\mathcal{M}_\alpha$ is pairwise disjoint by replacing, if necessary, M^η_α by $M^\eta_\alpha \setminus \bigcup\{M^\xi_\alpha : \xi < \eta\}$ for all η. The resulting sets are still stationary and the collection is still maximal $NS(\omega_1)$-disjoint, because $M^\eta_\alpha \cap \bigcup\{M^\xi_\alpha : \xi < \eta\}$ is nonstationary.

Assuming then that $\mathcal{M}_\alpha$ is pairwise disjoint for all α, we let $\{P^\eta_\alpha : \eta < \kappa\}$ list the members of $\mathcal{M}_\alpha$ in order of their least elements, and define functions $g_\alpha : \omega_1 \to \omega_1$ by letting $g_\alpha(\xi) = \eta$ when $\xi \in P^\eta_\alpha$. Then $\eta \leq \xi$ by an easy induction using disjointness of each $\mathcal{M}_\alpha$, and $g^\leftarrow_\alpha\{\eta\} = P^\eta_\alpha$.

Now we use injective transfer functions $\varphi_\xi : \xi + 1 \to \omega$ as in the comments following the proof of Theorem 2.2. That is, let $f_\alpha : \omega_1 \to \omega$ be defined by $f_\alpha(\xi) = \varphi_\xi(g_\alpha(\xi))$. Then $f_\alpha(\xi) = f_\beta(\xi))$ if, and only if, $g_\alpha(\xi) = g_\beta(\xi)$. Also, $P^\eta_\alpha = \{\xi : f_\alpha(\xi) = \varphi_\xi(\eta)\}$

Now Axiom F^+gives gives a stationary set E on which infinitely many f_α's agree, meaning that there is an infinite set of indices $\{\alpha_n : n \in \omega\}$ such that $f_{\alpha_n}(\xi) = f_{\alpha_m}(\xi)$ for all $\xi \in E$. This common value at each $\xi \in E$ equals $\varphi_\xi(\eta_\xi)$ for some $\eta_\xi \leq \xi$. Then $g_{\alpha_n}(\xi) = \eta_\xi$ for all $n \in \omega$.

Let $\Delta = \{\xi : \eta_\xi = \xi\}$. If $\xi \in \Delta$ then $\xi \in P^\xi_{\alpha_n}$ for all n and so Δ meets each member of $\mathcal{M}_{\alpha_n}$ in at most one point. Hence, by NS-maximality of $\mathcal{M}_{\alpha_n}$, Δ is nonstationary and $E \setminus \Delta$ is stationary.

The function $\psi : E \setminus \Delta \to \omega_1$ that takes ξ to η_ξ is regressive. So by Fodor's Lemma, there is a stationary $S \subset E \setminus \Delta$ and $\eta \in \omega_1$

such that $\psi(\xi) = \eta$ for all $\xi \in S$. In other words, $S \subset P^{\eta}_{\alpha_n}$ for all $n \in \omega$. $\square$

The following problems are closely related to Problem 1 and the question preceding it.

Problem 4. *Is it possible to show the consistency of the nonstationary ideal being $(\omega_2, \omega_1\,; \omega, \omega)$-para-saturated without using large cardinal axioms?*

A negative answer would, of course, follow from a positive one to:

Problem 5. *Is the converse of Theorem 4.5 true?*

If we switch the second and third para-saturation parameters in Problem 4, we get an axiom which does require large cardinal axioms and, in fact, implies Axiom F^+.

Theorem 4.5. *If there is a $(\omega_2, \omega\,; \omega_1, \omega)$-para-saturated proper σ-ideal $\mathcal{I}$ on a stationary subset S of ω_1 such that NS(S) $\subset \mathcal{I}$, then $P^*_{\omega_1}(\omega, \tau)$ fails for all $\tau < \omega_1$; hence, there is an inner model with a measurable cardinal, Axiom F^+ holds, and $NS(S)$ is $(\omega_2, \omega_1\,; \omega, \omega)$-para-saturated.*

Proof: Let $\{f_\alpha : \alpha < \omega_2\}$ be a family of functions from S to ω. In the $\omega \times \omega_2$ matrix of sets $M(n, \alpha) = f^{\leftarrow}_\alpha\{n\} \cap S$, the αth column is a partition of S from which we can throw out all members of $\mathcal{I}$ and still have what is left of the αth column be maximal $\mathcal{I}$-disjoint. Let $\{f^{\leftarrow}_\alpha\{n_\alpha\} : \alpha \in W\}$ be a witness to $(\omega_2, \omega\,; \omega_1, \omega)$-para-saturation; in particular, $|W| = \aleph_1$. By the Pigeonhole Principle, some row has uncountably many sets indexed by members of W. The intersection of any subcollection of order type τ witnesses the negation of $P^*_{\omega_1}(\omega, \tau)$. The remaining conclusions follow from the replacement of ω_1 by $NS(S)$ in the statement of Axiom $\mathrm{F}^+$and of Theorem 4.5., and following the proof of the latter with an altered notation. $\square$

In [8], Laver proved a similar theorem, in effect,

Theorem C. *If CH holds and there is an $(\omega_2, \omega\,; \omega_2, \omega)$-para-saturated σ-ideal $\mathcal{I}$ on ω_1, then*

$$\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_1}{\aleph_1}^{1,1}_{\omega}$$

holds, and $[\omega_1]^{\leq\omega}$ is $(\omega_2, \omega\,; \omega_1, \omega_1)$-para-saturated.

Laver's proof implicitly showed the following lemma, from which Theorem C follows easily.

Lemma 4.6. [CH] *If $\mathcal{X}$ is a family of $\aleph_2$ subsets of ω_1 such that every countable subfamily of $\mathcal{X}$ has uncountable intersection, then there exists $\mathcal{X}_0 \subset \mathcal{X}$ such that $|\mathcal{X}_0| = \aleph_1$ and $|\bigcap \mathcal{X}_0| = \aleph_1$.*

The following problem is a variant of Problem 4 which puts a bigger demand on ι but allows the use of the countable ideal.

Problem 6. *Let S be a stationary subset of ω_1. If $\mathcal{I}$ is the nonstationary ideal on S and $\mathcal{J}$ is the countable one, is it possible to show the consistency of $\langle \mathcal{I}, \mathcal{J} \rangle$ being $(\omega_2, 2^{\omega_1}\,;\, \omega, \omega)$-para-saturated without using large cardinal axioms?*

The existence of S for which this para-saturation property holds is called Axiom S in [10]. There, it is used in conjunction with PFA$^+$ to show the consistency of every T_5, hereditarily cwH manifold of dimension > 1 being metrizable. The claim that this conjunction is consistent, modulo large cardinals, turns out to be an unsolved problem: the means used to show its consistency were flawed as explained in [12], the correction to [10]. In [12], one can also read a new proof using PFA alone.

Our final theorem takes care of most of the remaining para-saturation properties where the first parameter is $\leq \omega_1$.

Theorem 4.7. *Let $\mathcal{I}$ be the nonstationary ideal on a stationary subset E of ω_1. Then*

(a) if $\mathfrak{p} > \omega_1$, then $\mathcal{I}$ is $(\omega, n\,;\, \omega, \omega)$-para-saturated for all $n \in \omega$;
(b) $\mathcal{I}$ is not $(\omega, < \omega\,;\, \omega, \omega)$-para-saturated;
(c) if $\mathfrak{p} > \omega_1$, then $\mathcal{I}$ is $(\omega_1, < \omega\,;\, \omega, \omega)$-para-saturated;
(d) $\mathcal{I}$ is not $(\omega_1, \omega\,;\, \omega, \omega)$-para-saturated.

Part (a) will follow from a slightly more general lemma (4.10, below). If we assume (a), then (c) follows quickly: Given a family $\{\mathcal{M}_\alpha : \alpha < \omega_1\}$ of finite maximal $\mathcal{I}$-disjoint collections, there exists n such that uncountably many $\mathcal{M}_\alpha$ are of cardinality $\leq n$, so we can apply (a) to any infinite subfamily of these $\mathcal{M}_\alpha$.

The following example is more than enough to show (b).

Example 4.8. Let $\mathcal{I}$ be any ideal on ω_1 such that $\mathcal{I} \neq \mathcal{P}(X)$ for all $X \subset \omega_1$. Let $S = \{\alpha : \{\alpha\} \in \mathcal{I}\}$ and let $\{f_\alpha : \alpha \in S\}$ be a family of functions from ω to ω, with $f_\alpha(n) \leq n$ for all n, and such that any two agree on only finitely many n. Such a family is easy to construct by transfinite induction: If $F_\alpha = \{f_\beta : \beta < \alpha\}$ has been defined, let $\{g_n : n \in \omega\}$ list F_α and make sure f_α differs from g_n on all integers greater than n. We will now use each f_α to define an $\omega \times \omega$ matrix of subsets of S in a manner "orthogonal" to the method of sections 1 and 2.

For each n and i in ω, let $A_n^i = \{\alpha : f_\alpha(n) = i\}$. Then $\mathcal{A}_n = \{A_n^i : i \leq n\}$ is a partition of S, and if we remove all members that are in $\mathcal{I}$, then what remains is still a maximal $\mathcal{I}$-disjoint family. If $Z \subset \omega$ is infinite and $A_n^{f(n)}$ is chosen from the nth family for each $n \in Z$, then the graph of f can be a subset of the graph of at most one of the f_α, and hence $\bigcap\{A_n^{f(n)} : n \in Z\}$ contains at most one element.

The following example is more than enough to justify (d) of Theorem 4.7.

Example 4.9. Let $\mathcal{I}$ be any σ-complete ideal as in Example 4.8. This time let $\{f_\alpha : \alpha \in S\}$ be a family of functions into ω_1, with $f_\alpha(\xi) \leq \omega + \xi$ for all ξ, such that any two agree on only finitely many ξ. One can construct such a family by transfinite induction as follows. If $F_\alpha = \{f_\beta : \beta < \alpha,\ \beta \in S\}$ has been defined, let $\{g_n : n \in \omega\}$ list F_α and let $\{\xi_n : n \in \omega\}$ list all ordinals less than α and make sure f_α differs from g_n on all ξ_i such that $i > n$. Also let $f_\alpha(\gamma) = \alpha$ for all $\gamma \geq \alpha$. Continue as for Example 4.8, mutatis mutandis. In particular, this time Z is any countably infinite subset of ω_1, and σ-completeness comes into play when we throw out all members of each partition that are not in $\mathcal{I}$ and conclude that the resulting collection is still maximal $\mathcal{I}$-disjoint.

Finally, the following lemma proves part (a) of Theorem 4.7. I am indebted to the referee for the proof given here. It is much simpler than my original proof, which was only given in full for $n = 2$ and incorrectly outlined for $n > 2$.

Lemma 4.10. $[\mathfrak{p} > \omega_1]$ *Suppose* $\{\mathcal{I}, \mathcal{J}\} \subset \{[E]^{\leq\omega}, NS(E)\}$ *where* $E \subset \omega_1$ *is stationary. Then* $\langle \mathcal{I}, \mathcal{J} \rangle$ *is* $(\omega, n\,; \omega, \omega)$*-para-saturated for all finite* n.

Proof: Fix n. First note that by prior observations, it is enough to prove that $\mathcal{I} = NS(E)$ is $(\omega, n\,; \omega, \omega)$-para-saturated. Let $\{\mathcal{A}_j :$

$j \in \omega\}$ be a family of maximal $\mathcal{I}$-disjoint collections $\{A_j^0, \dots, A_j^{n-1}\}$. For each $\alpha \in E$ and $k = 0, \dots, n-1$, define

$$Z_\alpha^k = \{j : \alpha \in A_j^k\}.$$

Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. There exists $k \leq n-1$ such that

$$S = \{\alpha : Z_\alpha^k \in \mathcal{U}\} \text{ is stationary.}$$

By $\mathfrak{p} > \omega_1$ there is an infinite $Z \subset \omega$ with $Z \setminus Z_\alpha^k$ finite for all $\alpha \in S$. Thus, there is a stationary $S_0 \subset S$ and a finite $F \subset Z$ such that $S_0 \subset \bigcap\{A_j^k : j \in Z \setminus F\}$. □

There remain only two open $(\kappa, \iota\,; \lambda, \mu)$-para-saturation problems on $NS(\omega_1)$ where $\kappa \leq \omega_2$ and $\mu = \omega$; even large cardinal axioms have not yet been applied successfully to them.

Problem 7. *Is it consistent that the nonstationary ideal on ω_1 is $(\omega_1, 2\,; \omega_1, \omega)$-para-saturated? that it is $(\omega_1, < \omega\,; \omega_1, \omega)$-para-saturated?*

We conclude with a few remarks on what happens if $\mu > \omega$. The proof of Theorem C does not extend to the substitution of $NS(\omega_1)$ for $[\omega_1]^\omega$ even if $\mathcal{I}$ is a normal ideal, and the following problem seems to be open, even modulo large cardinals.

Problem 8. *Is it consistent for $NS(\omega_1)$ to be $(2^{\omega_1}, 2\,; \omega_1, \omega_1)$-para-saturated, or $(\kappa, \iota\,; \omega_1, \omega_1)$-para-saturated for smaller κ or bigger ι?*

However, if there is such a thing as a model of $2^{\omega_1} = \omega_3$ where the club filter has a base of size ω_2 and $NS(\omega_1)$ is ω_2-dense (we don't even need ω_1-density here), then a 2-step pigeonhole argument shows us that $\mathrm{NS}(\omega_1)$ is $(\omega_3; \omega_3, \omega_3)$-saturated! That is, given any family of ω_3 stationary sets, there is a subfamily of cardinality ω_3 with stationary intersection. This would imply that $NS(\omega_1)$ is $(2^{\omega_1}, 2^{\omega_1}; 2^{\omega_1}, 2^{\omega_1})$-para-saturated, and also that any family of $2^{\omega_1}(= \aleph_3)$ functions from ω_1 to ω has a subfamily of $\aleph_3$ functions that take on the same constant value on the same stationary set.

References

1. H.-D. Donder, *Families of almost disjoint functions,* in: Axiomatic Set Theory, Contemporary Math. **31** AMS (1984), 71–78.

2. H.-D. Donder and P. Koepke, *On the consistency strength of "accessible" Jónsson cardinals and of the weak Chang conjecture,* Ann. Pure Appl. Logic **25** (1983), 233–261.
3. H.-D. Donder and J.-P. Levinski, *Some principles related to Chang's conjecture,* Ann. Pure Appl. Logic **45** (1989), 39–101.
4. A. Dow, *Set theory in topology*, in: Recent Progress in General Topology, M. Hušek and J. van Mill, eds., Amsterdam: North-Holland, 1992, 167–197.
5. P. Erdös and R. Radó, *A partition calculus in set theory,* Bull. Amer. Math. Soc. **62** (1956), 427–489.
6. J. Hirschorn, *Random trees under CH*, Trans. Amer. Math. Soc. **356** (2004), 1281–1290 (electronic).
7. A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory,* in: Higher Set Theory, G. H. Muller and D. S. Scott, eds., Lecture Notes in Math. **669**, Berlin: Springer-Verlag, 1978. 99–275.
8. R. Laver, *An* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated ideal on* ω_1, in: Logic Colloquium '80 (Prague, 1980), Stud. Logic Foundations Math., **108**, Amsterdam-New York: North-Holland, 1982. 173–180.
9. P. Nyikos, *Various smoothings of the long line and their tangent bundles,* Adv. Math. **93** (1992), 129–213.
10. P. Nyikos, *Complete normality and metrization theory of manifolds,* Topology Appl. **123** no. 1 (2002), 181–192.
11. P. Nyikos, *Applications of some strong set-theoretic axioms to locally compact* T_5 *and hereditarily scwH spaces,* Fund. Math. **176** no. 1 (2003), 25–45.
12. P. Nyikos, *Correction to: "Complete normality and metrization theory of manifolds,"* Topology Appl. **138** (2004), 325–327.
13. S. Shelah, *Proper and Improper Forcing*, 2nd edition, Berlin: Springer-Verlag, 1998.
14. J. E. Vaughan, *Small uncountable cardinals and topology*, with an appendix by S. Shelah, Open Problems in Topology. Amsterdam: North-Holland, 1990. 195–218.
15. E. K. van Douwen, *The integers and topology* Handbook of Set-Theoretic Topology. Amsterdam: North-Holland, 1984. 111–167.
16. W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, deGruyter Series in Logic and its Applications, vol. 1, Berlin: deGruyter & Co., 1999.

Department of Mathematics; University of South Carolina; Columbia, SC 29208

E-mail address: `nyikosmath.sc.edu`