COMPLETE NORMALITY AND METRIZATION THEORY OF MANIFOLDS

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Abstract. A manifold is a connected Hausdorff space in which every point has a neighborhood homeomorphic to Euclidean n-space (n is unique). A space is *collectionwise Hausdorff (cwH)* if every closed discrete subspace D can be expanded to a disjoint collection of open sets each of which meets D in one point. There are exactly two examples of 1-dimensional nonmetrizable hereditarily normal, hereditarily cwH manifolds: the long line and the long ray. The main new result is that if it is consistent that there is a supercompact cardinal, it is consistent that every hereditarily normal, hereditarily cwH manifold of dimension greater than 1 is metrizable.

The modern set-theoretic topology of manifolds can be said to begin with the 1975 construction by Mary Ellen Rudin of a nonmetrizable perfectly normal manifold using the set-theoretic axiom $\diamondsuit$. This solved a problem that had been posed by Wilder at the end of his 1949 textbook [11] and thereby made it possible to consistently extend the wealth of algebraic topology techniques used by Wilder beyond the context of metrizable manifolds, at least consistently with the usual axioms of ZFC. Shortly thereafter, with the help of Phillip Zenor [7], Rudin was able to reduce the set-theoretic axiom to the more familiar Continuum Hypothesis (CH).

Then, in 1978, Rudin showed that the existence of perfectly normal nonmetrizable manifolds was independent of the usual axioms of set theory [6], by showing that they do not exist under MA(ω_1). A very natural question is whether "perfectly normal" can be generalized to "hereditarily normal" (= completely normal), especially if one is aware of the old custom of designating perfectly normal spaces as T_6 spaces and completely normal ones as T_5 spaces. However, the long ray and long line show that the straightforward generalization of Rudin's theorem cannot hold. They are linearly ordered (hence hereditarily normal) spaces that are locally like the real line but contain copies of ω_1 and hence are not metrizable. In higher dimensions, however, it is a completely different story, and the following is still unsolved.

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Problem 1. *Is it consistent that every hereditarily normal manifold of dimension greater than* 1 *is metrizable?*

This is a natural question in the light of the fact that $\omega_1 \times (\omega + 1)$ is not hereditarily normal. This is easily shown by removing the points $\langle \lambda, \omega \rangle$ from the 'top edge', where λ is a limit ordinal, and showing that, in the resulting space Ł, what is left of the top edge is a closed set that cannot be separated from the now-closed set $\Lambda \times \omega$ where Λ is the set of all limit ordinals. We will see a similar construction in the process of showing the main result of this paper:

Main Theorem. *If it is consistent that there is a supercompact cardinal, then it is consistent that every every hereditarily normal, hereditarily cwH manifold of dimension greater than* 1 *is metrizable.*

Here "cwH" is the abbreviation for "collectionwise Hausdorff." This is a property which is implied by perfect normality in any locally compact, locally connected space [4]. Since it is easy to show that the long ray and the long line are the only nonmetrizable 1-manifolds, and that they are hereditarily normal and hereditarily cwH, this theorem is a natural extension of Rudin's 1978 independence result, albeit with a much stronger set-theoretic hypothesis. In the final section we will discuss various strategies for reducing or even eliminating the large cardinal hypothesis inherent in it, as well as for eliminating the "hereditarily cwH" condition and thus answering Problem 1 affirmatively. One very natural strategy is to simply find an affirmative answer to a problem that has long been one of Rudin's favorites:

Problem 2. *Is every normal manifold collectionwise Hausdorff?*

Remarkably enough, this problem is unsolved even for locally compact, locally connected spaces. It would be enough to solve it affirmatively for the hereditarily normal ones, as it is an easy exercise to show that if every hereditarily normal, locally compact, locally connected space is cwH, then every one is hereditarily cwH. Preliminary results are encouraging: besides the fact (alluded to above) that the answer is Yes in the perfectly normal case, there is the fact that under the axiom $V = L$ every locally compact normal space is cwH. However, under MA(ω_1) there do exist locally compact normal spaces that are not hereditarily cwH, and both Rudin's metrization theorem and the one of this paper use MA(ω_1). So if Problem 2 has an affirmative answer, the proof will have to make heavy use of local connectedness, and perhaps even of the fact that the space is locally Euclidean.

In a forthcoming paper, we will use the same set-theoretic hypotheses to arrive at some far-reaching structure theorems on locally compact, hereditarily normal, hereditarily cwH spaces, including at least one result that has the Main Theorem of this paper as a quick corollary:

Theorem A. *Under the same hypotheses as in the Main Theorem, every (clopen) component of every locally compact, locally connected, hereditarily normal, hereditarily cwH space is either Lindelöf or has uncountably many cut points.*

Since a manifold M of dimension greater than 1 has no cut points, the Main Theorem follows immediately. Moreover, in the course of proving this latter theorem, we actually construct a copy of the non-normal space Ł inside every hereditarily cwH nonmetrizable manifold of dimension greater than 1. On the other hand, the proof of Theorem A is considerably lengthier than the proof we give here for the

Main Theorem, and our shorter proof may also indicate possibilities for reducing the large cardinal axioms that the other one does not.

Our proof of the Main Theorem will make use of the following 'intuitively obvious' fact about domains (i.e., open connected sets) in $\mathbb{S}^n$, for whose proof I am indebted to David Gauld:

0.1. Theorem. *Suppose that D is a domain in $\mathbb{S}^n$, and $n > 1$. Then for every component C of the complement of D, the frontier ∂C of C is connected.*

This is shown with the help of:

0.2. Lemma. *Let $D \subset \mathbb{S}^n$ be a non-empty domain and $\{C_\alpha : \alpha \in A\}$ be a collection of components of $\mathbb{S}^n \setminus D$. Then $D \cup (\bigcup_{\alpha \in A} C_\alpha)$ is also connected.*

Proof. It suffices to show that $D \cup C$ is connected for one such component, and hence it suffices to show that $\overline{D} \cap C \neq \emptyset$ as C is closed. Choose any $x \in \partial C$ and let U be a ball neighbourhood of x. Then $U \cap C \neq \emptyset$ and since U is connected, this implies $U \subset C$ unless $U \cap D \neq \emptyset$. But as $x \in \partial C$ we have $U \cap (\mathbb{S}^n - C) \neq \emptyset$, so $U \cap D \neq \emptyset$. Therefore, $x \in \overline{D}$. □

Proof of Theorem 0.1. We may assume that D is a proper domain.

Let C be a component of $\mathbb{S}^n \setminus D$. If $\operatorname{int} C = \emptyset$ then there is nothing to prove, so assume that $\operatorname{int} C \neq \emptyset$. Then $\mathbb{S}^n \setminus C$ is also a domain, being open because it is the complement of the closed set C and being connected by Lemma 0.1.

Suppose that ∂C is not connected. Then there are sets $A, B \subset \partial C$ such that $A \cap B = \emptyset$, $A \cup B = \partial C$ and A and B are closed in ∂C, hence in $\mathbb{S}^n$.

Set $X_1 = \mathbb{S}^n \setminus A$ and $X_2 = \mathbb{S}^n \setminus B$. Then $Y = \mathbb{S}^n \setminus (A \cup B) = X_1 \cap X_2$ consists of at least two open components, *viz.* $\mathbb{S}^n \setminus C$ and the components (of which there is at least one) of $\operatorname{int} C$. Note that $X_1 \cup X_2 = \mathbb{S}^n$. Further $\bar{A} \subset \operatorname{int}(\mathbb{S}^n \setminus B)$ as A is closed, $\mathbb{S}^n \setminus B$ is open and $A \cap B = \emptyset$. Thus A may be excised from the pair $(\mathbb{S}^n, X_2)$, ie the inclusion $(X_1, Y) \subset (\mathbb{S}^n, X_2)$ induces an isomorphism of cohomology groups. The same applies to the inclusion $(X_2, Y) \subset (\mathbb{S}^n, X_1)$. Thus the triad $(\mathbb{S}^n; X_1, X_2)$ is excisive [9, p.103].

Hence the Mayer-Vietoris sequence

$$\cdots \to \bar{H}^q(\mathbb{S}^n) \to \bar{H}^q(X_1) \oplus \bar{H}^q(X_2) \to \bar{H}^q(Y) \to \bar{H}^{q+1}(\mathbb{S}^n) \to \cdots$$

for the Alexander-Spanier cohomology of the triad $(\mathbb{S}^n; X_1, X_2)$: is exact. For explications of these notions see [8, 6.4] and [9, Theorem 7.19 and the note at the top of p.125].

Consider the part of the sequence with $q = 0$ and work with coefficients in some group $\mathbb{Z}_p$. Note that X_1 and X_2 are connected so by [8, Corollary 6.4.7] we have $\bar{H}^0(\mathbb{S}^n) \approx \bar{H}^0(X_1) \approx \bar{H}^0(X_2) \approx \mathbb{Z}_p$. Because $n > 1$ we also have $\bar{H}^1(\mathbb{S}^n) \approx 0$. On the other hand, because Y has at least two components we have $\bar{H}^0(Y) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus G$, where G is the sum of as many copies of $\mathbb{Z}_p$ as there are components of $\operatorname{int} C$ in excess of 1. Thus the exact sequence above reduces to:

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus G \to 0.$$

However it is impossible for this sequence to be exact. Thus ∂C must be connected. □

1. The axioms and the forcing model.

Besides MA(ω_1) we will be using two axioms of much more recent vintage, as well as a 1987 application of the Proper Forcing Axiom (PFA) due to Zoltan Balogh (cf. [10]):

1.1. Theorem. [PFA] *Every first countable, countably compact space is either compact or contains a copy of ω_1.*

One of the other axioms was shown consistent in 1997 by Todd Eisworth [2]. It has to do with the following concepts.

1.2. Definition. A subset S of a poset P is *downward closed* if $\hat{s} \subset S$ for all $s \in S$, where $\hat{s} = \{p \in P : p \leq s\}$. A collection of subsets of a set X is an *ideal* if it is downward closed with respect to $\subset$, and closed under finite union. An ideal $\mathcal{J}$ of countable subsets of X is *countable-covering* if $\mathcal{J} \restriction Q$ is countably generated for each countable $Q \subset X$. That is, for each countable subset Q of X, there is a countable subcollection $\{J^Q_n : n \in \omega\}$ of $\mathcal{J}$ such that every member J of $\mathcal{J}$ that is a subset of Q satisfies $J \subset J^Q_n$ for some n.

1.3. Definition. *Axiom* CC_{22} is the axiom that for each countable-covering ideal $\mathcal{J}$ on a stationary subset S of ω_1, either:

(i) there is a stationary subset A of S such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there is a stationary subset B of S such that $B \cap J$ is finite for all $J \in \mathcal{J}$.

Both MA(ω_1) and CC_{22} follow from the strengthening PFA^+ of PFA defined below, but there is an easy construction of a model of MA(ω_1) and CC_{22} and Theorem 1.1 using just the consistency of an inaccessible cardinal. It is the axiom related to the following concept that poses the biggest questions as far as relative consistency strength goes.

1.4. Definition. Let S be a stationary subset of ω_1. We say that *the ideal of nonstationary subsets of S is (κ, λ, μ)-saturated* if for every collection $\mathcal{Z}$ of κ-many stationary subsets of S, there is a subcollection $\mathcal{W}$ of $\mathcal{Z}$ such that $|\mathcal{W}| = \lambda$ and such that every subcollection of $\mathcal{W}$ having μ or fewer members has stationary intersection.

We will be using the fact, due to Shelah, that if it is consistent that there is a supercompact cardinal, it is consistent to have PFA^+ together with the axiom that there is a stationary subset S of ω_1 such that ideal of nonstationary subsets of S is $(\omega_2, \omega_2, \omega)$-saturated. We will refer to this as *Shelah's Supersaturation Axiom,* or SSA for short, in this paper. Actually, we will only make use of $(\omega_2, \omega, \omega)$-saturation, but this does little to ameliorate the large cardinal axiom used. In Section 3 we will address this problem further.

1.5. Definition. Given a poset (partially ordered set) P, *the logicians' wedge topology on P* is the topology whose base is the collection of all sets of the form $V_p = \{q \in P : q \geq p\}$.

The following axiom is known as PFA^+: Given a proper poset P (for the definition of "proper," see [1]), and a collection $\mathcal{D}$ of $\aleph_1$ dense open subsets of P in the logicians' wedge topology, there is a subset G of P such that G is upward-directed and downward closed, and such that

(1) $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$ and

(2) Whenever $\dot{S}$ is the P-name for a stationary subset of ω_1, there is a stationary subset E of ω_1 such that every element of P forces E to be a subset of $\dot{S}$.

The PFA is the same axiom but with (2) missing. Of course, PFA^+ implies PFA, and it is well known that PFA in turn implies $\mathrm{MA}(\omega_1)$, which is just PFA with "c.c.c." substituted for "proper". Also:

1.6. Lemma. PFA^+ *implies* CC_{22}*.*

Proof. We will use a standard cardinal-collapsing trick. Let M be a model of PFA^+. Let $\mathcal{J}$ be a countable-covering ideal on a stationary subset S_0 of ω_1. Let P_0 be the usual countably closed poset for collapsing $\mathfrak{c}$ to $\aleph_1$ and let $M[G_0]$ be the resulting forcing model. If, in $M[G_0]$, there is a stationary $S \subset S_0$ such that every countable subset of S is in $\mathcal{J}$, then we apply (2) in the statement of PFA^+ to conclude that there is a stationary $E \subset S_0$ in M itself such that every countable subset of E is in $\mathcal{J}$.

If there is no such $S \subset S_0$, then since $|\mathcal{J}| = \mathfrak{c}$ in M, and no new countable sets are added, $|\mathcal{J}| = \aleph_1$ in $M[G_0]$. Hence there is a proper poset P_1 which adds a stationary subset S of S_0 such that $S \cap J$ is finite for all $J \in \mathcal{J}$ [2]. Since $P_0 * P_1$ is proper and adds S to M, another application of (2) gives a stationary subset E as before. $\square$

2. The main theorem and its proof.

We begin this section by recalling the definition of cwH and a condition equivalent to being hereditarily cwH.

2.1. Definition. Given a faithfully indexed subset $D = \{d_\alpha : \alpha \in A\}$ of a set X, an *expansion* of D is a family $\{U_\alpha : \alpha \in A\}$ such that $U_\alpha \cap D = d_\alpha$ for all $\alpha \in A$.

A space X is *collectionwise Hausdorff* if every closed discrete subspace expands to a disjoint collection of open sets.

2.2. Lemma. *A space is hereditarily cwH if, and only if, every discrete subspace expands to a disjoint collection of open sets.*

Proof. Sufficiency is clear, so suppose X is hereditarily cwH and D is a discrete subspace of X. Then $F = \overline{D} \setminus D$ is a closed subspace of X, and D is a closed discrete subspace of $X \setminus F$. Any expansion of D to a disjoint collection of open subsets of $X \setminus F$ is also an expansion of D to a disjoint collection of open subsets of X. $\square$

We now restate the main theorem, following the old custom of referring to hereditarily normal spaces as T_5 spaces.

2.3. Theorem. $[\mathrm{SSA} + \mathrm{PFA}^+]$ *Every* T_5*, hereditarily cwH manifold of* $\dim > 1$ *is metrizable.*

The proof starts out essentially the same way as Rudin's proof of her 1978 theorem involving perfectly normal manifolds. That is, we use $\mathrm{MA}(\omega_1)$ to show that the manifolds under consideration are all of Type I:

2.4. Definition. A space X is a *Type I space* if it is the union of an ascending ω_1-sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of open subspaces such that $\overline{X}_\alpha \subset X_\beta$ whenever $\alpha < \beta$ and such that $\overline{X}_\alpha$ is Lindelöf for all α.

2.5. Lemma. [MA(ω_1)] *Every hereditarily cwH manifold is of Type I.*

Proof. Let X be a hereditarily cwH manifold. *Claim 1.* MA(ω_1) *implies that every Lindelöf subset of X has Lindelöf closure.* Assuming this for the moment, let X_0 be any open subset of X that is homeomorphic to $\mathbb{R}^n$. If X_α has been defined for all $\alpha < \beta$, cover the closure Y_β of $\bigcup\{X_\alpha : \alpha < \beta\}$ with countably many copies of the second countable space $\mathbb{R}^n$, using the fact that $\overline{X_\alpha} \subset Y_\beta$ for all $\beta < \alpha$, and let X_β be the union of this cover. By first countability, $\bigcup\{X_\alpha : \alpha < \omega_1\}$ is closed in X and it is clearly open, so it is all of X because X is connected.

Proof of Claim 1. Let A be a Lindelöf subset of X, which we may assume without loss of generality to be open by covering it with countably many copies of $\mathbb{R}^n$. *Claim 2. Every discrete subspace of $\overline{A}$ is countable.* Once Claim 2 is proved, we use Szentmiklóssy's theorem [5] that MA(ω_1) implies that every locally compact space of countable spread (this means: every discrete subspace is countable) is hereditarily Lindelöf to conclude that $\overline{A}$ is Lindelöf.

Proof of Claim 2. Let D be a discrete subspace of $\overline{A}$. Using the hereditarily cwH property of X and Lemma 2.2, let $\mathcal{G} = \{G_d : d \in D\}$ be an open expansion of D. Since A is open and dense in $\overline{A}$, $\{G_d \cap A : d \in D\}$ is a disjoint family of nonempty open subsets of A. But every collection of disjoint open subsets of A is countable since A has a countable cover by copies of $\mathbb{R}^n$. So $\mathcal{G}$ is countable and hence so is D. □

Next comes the only fact for which we need $\dim(M) > 1$.

2.6. Lemma. *If M is a Type I manifold of dimension > 1, then $\{M_\alpha : \alpha < \omega_1\}$ can be chosen as in Definition 2.4 in such a way that each point of $B_\alpha = \overline{M_\alpha} \setminus M_\alpha$ is contained in a compact, connected infinite subset K_α of B_α.*

Proof. Let $M = X$ and let $\{X_\alpha : \alpha < \omega_1\}$ be as in 2.4, with the additional property that X_α is connected for all α. Since every component of $X \setminus \overline{X_\alpha}$ meets the second countable subspace $X_{\alpha+1} \setminus \overline{X_\alpha}$, it follows that $X \setminus \overline{X_\alpha}$ has only countably many components. Hence for each $\alpha < \omega_1$ there exists $\alpha^* < \omega_1$ such that each metrizable (hence second countable) component of $X \setminus \overline{X_\alpha}$ is a subset of X_{α^*}. Next, define Y_α by induction: $Y_0 = X_0$; $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ whenever α is a limit ordinal; and if $Y_\alpha = X_\xi$ then $Y_{\alpha+1} = X_{\xi^*}$. Now let $\{\gamma_\alpha : \alpha \in \omega_1\}$ list the limit ordinals in ω_1 and let $M_\alpha = Y_{\gamma_\alpha}$ for all $\alpha < \omega_1$.

To show that this choice of M_α works, let $p \in B_\alpha$ and let $\varphi : \mathbb{B}^n \to M$ be an embedding of the closed unit n-ball that takes the origin to p, and let $F = \varphi^{\to}\mathbb{B}^n \cap B_\alpha$. Let K be the (compact, connected) component of F containing p. Since F is compact, K is also its quasicomponent containing p. We will be done as soon as we show K is nontrivial.

Suppose $K = \{p\}$. Then p has a base of open neighborhoods whose frontiers miss F. Let U be such a neighborhood of p with the additional property that $\overline{U} \cap \varphi^{\to}\mathbb{S}^{n-1} = \emptyset$. Let D be the component of U containing p. Then M_α meets D but does not contain it, and so M_α is disconnected by ∂D. Now $\varphi^{\to}\mathbb{S}^{n-1}$ is contained in a single component C of $\varphi^{\to}(\mathbb{B}^n \setminus D)$, and by collapsing $\varphi^{\to}\mathbb{S}^{n-1}$ to a single point, we can see from Theorem 0.1 and its proof that the frontier of C in the space $\varphi^{\to}\mathbb{B}^n$ is a connected subset of ∂D. By abuse of language, we denote this frontier by ∂C. Now M_α meets both $\varphi^{\rightarrow}\mathbb{S}^{n-1}$ and D, so it is disconnected by ∂C. If ∂C were a subset of M_α, then since ∂C is compact, it would be a subset of some Y_β where $\beta < \gamma_\alpha$ — recall that γ_α is a limit ordinal and $M_\alpha = Y_{\gamma_\alpha}$. But ∂C separates $D^* = \varphi^{\rightarrow}(\mathbb{B}^n \setminus C)$ from the rest of M, and this would make D^* and its subset D into subsets of $Y_{\beta+1}$, contradicting the assumption that $p \notin M_\alpha$.

So ∂C is not a subset of M_α, but then $B_\alpha = \overline{M_\alpha} \setminus M_\alpha$ meets ∂C since ∂C is connected, and the intersection is a subset of F, contradicting the claim that ∂D does not meet F. □

Now it is time to bring CC_{22} and Balogh's Theorem 1.1 into play. The following theorem is of independent interest since there do exist normal, hereditarily cwH Type I manifolds of dimension > 1 that are nonmetrizable. Easy examples are $L^+ \times \mathbb{R}^n$ for any n and any finite power of L, where L is the long line.

2.7. Theorem. $[\mathrm{CC}_{22} + \mathrm{PFA}]$ *If M is a normal hereditarily cwH manifold satisfying* $\dim(M) > 1$*, then M is either metrizable or contains a family of $\aleph_2$ disjoint copies of ω_1.*

Proof. By Lemma 2.5, M is of Type I. Let $\{M_\alpha : \alpha < \omega_1\}$ be as in Lemma 2.6; in particular, have $M_\gamma = \bigcup\{M_\alpha : \alpha < \gamma\}$ whenever γ is a limit ordinal. Let $B_\alpha = \overline{M_\alpha} \setminus M_\alpha$ for all α. If M is nonmetrizable, then B_α is nonempty for all $\alpha \in \omega_1$. Let $x_\alpha \in B_\alpha$ for all α. Let $\mathcal{I}$ be the ideal of all countable subsets I of ω_1 such that $\{x_\alpha : \alpha \in I\}$ has compact closure. Then $\mathcal{I}$ is countable-covering: the key to this is that every countable subset Z of M is a subset of some M_α and hence has Lindelöf closure; so we can cover $\overline{Z}$ with an ascending chain of countably many open sets $\{U_n : n \in \omega\}$ with compact closures, and then any subset of Z with compact closure is a subset of some U_n. Therefore, if Q is a countable subset of ω_1 and $Z = \{x_\alpha : \alpha \in Q\}$, and we define J_n^Q by letting $\{x_\alpha : \alpha \in J_n^Q\} = U_n \cap \{x_\alpha : \alpha \in Q\}$. then every member of $\mathcal{I} \restriction Q$ is a subset of J_n^Q for some n.

Now, if A is any subset of ω_1 such that $A \cap I$ is finite for all $I \in \mathcal{I}$, it is easy to see that $\{x_\alpha : \alpha \in A\}$ is a closed discrete subspace of M. However, the hereditary cwH property of M prevents any such A from being stationary, because of the Pressing-Down Lemma. Indeed, whenever α is a limit ordinal, and U is an open set containing α, there exists $\xi < \alpha$ such that $U \cap M_\xi \neq \emptyset$; but if A is stationary, and $\mathcal{U}$ is an open expansion of $\{x_\alpha : \alpha \in A\}$, then there is a ξ which works for uncountably many members of $\mathcal{U}$, and this prevents $\mathcal{U}$ from being a disjoint collection since M_α is separable.

Applying CC_{22}, we conclude that there is a stationary subset E of ω_1 such that every countable subset of $\{x_\alpha : \alpha \in E\}$ has compact closure. It follows that the closure of $\{x_\alpha : \alpha \in E\}$ itself is countably compact and hence contains a copy W of ω_1 by Theorem 1.1. It is easy to see that $W \subset \bigcup\{B_\alpha : \alpha \in \omega_1\}$. Pick $p_\alpha \in W \cap B_\alpha$ for all α for which this is possible. Using Lemma 2.6 and the fact that W is scattered, let $q_\alpha \in K_\alpha \setminus W$, where K_α is a compact, connected subset of B_α containing p_α. Now apply CC_{22} to the open, hence locally compact space $M \setminus W$ to produce a stationary subset S_1 of ω_1 such that the closure F_1 of $\{q_\gamma : \gamma \in S_1\}$ in $M \setminus W$ is countably compact. Then F_1 is closed in M itself.

Apply Urysohn's Lemma to obtain a continuous function $f : M \to [0,1]$ taking W to 0 and F_1 to 1. Each fiber $f^{\leftarrow}\{r\}$ is a closed subspace of M, and it meets K_α

for all $\gamma \in C$ by connectedness of K_γ. So we can apply CC_{22} and Theorem 1.1 as before to produce a countably compact subspace F_r inside $f^{\leftarrow}\{r\}$ and a copy W_r of ω_1 inside F_r. Now use the fact that PFA implies $\mathfrak{c} \geq \omega_2$. [Actually, as Todorčević has shown, PFA implies $\mathfrak{c} = \omega_2$ but we will not be needing this fact.] $\square$

One slight modification in the above proof sets the stage for the completion of the proof of our main theorem: that is to make the stationary sets we use in applying CC_{22} be subsets of the stationary set S of Shelah's Supersaturation Axiom. That is, from each $f^{\leftarrow}\{r\}$ we select points $p^r_\sigma \in K_\sigma$ for each $\sigma \in S$ and apply CC_{22} to these points alone. This guarantees that F_r meets K_σ for all σ in some stationary subset S_r of S.

To complete the proof of our main theorem, we apply SSA to $\{S_r : r \in [0,1]\}$. Let $\{r_n : n \in \omega\}$ be any infinite subset of $[0,1]$ such that

$$A = \bigcap \{S_{r_n} : n \in \omega\} \text{ is uncountable.}$$

[We do not even need for the intersection to be stationary.] Let r be a limit point of $\{r_n : n \in \omega\}$. We will do a rough analogue of the proof that $\omega_1 \times (\omega + 1)$ is not hereditarily normal; this will consist of showing that

$$M^* = M \setminus \big(f^{\leftarrow}\{r\} \cap \bigcup \{B_\alpha : \alpha \text{ is a limit point of } A\}\big)$$

is not normal. The two disjoint closed subsets of M^* that we will use to establish non-normality are

$$H = M^* \cap \bigcup \{B_\alpha : \alpha \text{ is a limit point of } A\}$$

and

$$K = M^* \cap f^{\leftarrow}\{r\}.$$

Suppose U is an open subset of M^* containing K. Because K meets each K_α such that $\alpha \in A \setminus A'$, where A' stands for the derived set of A, it follows that for each $\alpha \in A \setminus A'$ there exists $n(\alpha) \in \mathbb{N}$ such that

$$f^{\leftarrow}(r - 1/n(\alpha), r + 1/n(\alpha)) \cap K_\alpha \subset U.$$

Were this not so, we could pick $x_n \in K_\alpha \cap f^{\leftarrow}(r - 1/n, r + 1/n)$ outside of U and these would accumulate at a point of K, contradicting openness of U.

Since $A \setminus A'$ is uncountable, there exists k such that $n(\alpha) = k$ for uncountably many $\alpha \in A \setminus A'$. Pick an ascending sequence $\alpha_i \nearrow \delta$ of members of $A \setminus A'$ such that $n(\alpha_i) = k$ for all i. Then $\delta \in A'$ and $B_\delta \setminus f^{\leftarrow}\{r\} \subset H$. Now if we take r_m such that $0 < |r - r_m| < 1/k$, we can pick $x_i \in F_{r_m} \cap K_{\alpha_i}$ for all i. Then $\{x_i : i \in \omega\}$ has compact closure; but its limit points are all in $B_\delta \cap f^{\leftarrow}\{r_m\}$ since $x_i \in F_{r_m} \subset f^{\leftarrow}\{r_m\}$ for all m; and $B_\delta \cap f^{\leftarrow}\{r_m\} \subset H$. Thus we have points of H in the closure of U, establishing non-normality of M^*. $\square$

3. Room for improvement

We can take off from our Main Theorem in two directions: we can generalize it to the setting of locally compact, locally connected spaces as indicated in the introduction; and we can try to reduce the hypotheses and set-theoretic axioms involved. This section is devoted to the latter topic; the former topic will be dealt with in a paper under preparation.

Our proof of the main theorem is flexible enough to offer good possibilities for lowering the large cardinal axiom involved. The following axiom, coupled with $\mathrm{MA}(\omega_1)$, CC_{22}, and Balogh's theorem, is enough to get the conclusion in the main theorem. Call a family $\mathcal{S}$ of stationary subsets of ω_1 *weakly almost disjoint* if any pair of its members has nonstationary intersection.

Axiom S. There is a stationary subset S of ω_1 such that, for any ω_2-sequence of maximal weakly almost disjoint families $\mathcal{M}_\alpha$ of stationary subsets of S, there is an infinite $I \subset \omega_2$ and a choice of $M_\iota \in \mathcal{M}_\iota$ such that $\bigcap\{M_\iota : \iota \in I\}$ is uncountable.

Of course, Axiom S follows immediately from SSA; on the other hand, for all I know, Axiom S may not require any large cardinals for its consistency. It is not, however, a theorem of ZFC. I am indebted to Richard Laver for showing this by using the usual Cohen model to give a counterexample (Example 3.1 below).

To demonstrate adequacy of Axiom S, begin by replacing each S_r in the proof of Theorem 2.3 by a maximal weakly almost disjoint family $\mathcal{M}_r$ of stationary subsets of S in the following way. Let $S_r^0 = S_r$. If S_r^α has been defined for all $\alpha < \beta$ and $\{S_r^\alpha : \alpha < \beta\}$ is maximal w.a.d., let $\mathcal{M}_r = \{S_r^\alpha : \alpha < \beta\}$. Otherwise let T_r^β be a stationary subset of S that meets every S_r^α in a nonstationary set, and apply CC_{22} as in 2.6 to obtain a stationary $S_r^\beta \subset T_r^\beta$ so that $\{p_r^\sigma : \sigma \in S_r^\beta\}$ has countably compact closure. This process eventually yields a maximal w.a.d. family $\mathcal{M}_r$ as desired. Since $\mathrm{MA}(\omega_1)$ implies $\mathfrak{c} \geq \omega_2$, we can apply Axiom S, letting A equal the uncountable intersection of the M_ι, and continuing as in the last three paragraphs of Section 2.

Problem 3. *Can the consistency of Axiom S be established without using large cardinal axioms? Does it follow from* $\mathrm{MA}(\omega_1)$ *or from* CC_{22}*?*

It might also be interesting to substitute $\mathfrak{c}$ for ω_2 in the statement of Axiom S and ask whether the resulting axiom is consistent with CH, even if large cardinal axioms are assumed. The following example shows that neither this variant nor Axiom S itself is a theorem of ZFC.

3.1. Example. Let P be the poset $\mathrm{Fn}(\omega_1 \times \omega_2, 2)$ of all functions from finite subsets of $\omega_1 \times \omega_2$ to $\{0,1\}$. This makes P a specific example of the usual way of adding $\aleph_2$ Cohen reals to a model of ZFC. Assuming $\mathfrak{c} \leq \omega_2$ in the ground model M, let G be P-generic, so that $\bigcup G$ is a function $f : \omega_1 \times \omega_2 \to \{0,1\}$. For each $\alpha \in \omega_2$ let $\mathcal{M}_\alpha = \{S_\alpha,\ \omega_1 \setminus S_\alpha\}$ where $S_\alpha = \{\xi : f(\xi,\alpha) = 0\}$. The forcing which adds G is c.c.c., so every club subset of ω_1 in the forcing extension $M[G] = M[f]$ contains a club subset in M, the ground model [3, Ch. VII, Exercise H1]. Hence S_α and $\omega_1 \setminus S_\alpha$ are both stationary, by genericity of G. Clearly $\mathcal{M}_\alpha$ is maximal w.a.d. for all $\alpha < \omega_2$.

However, no matter how M_α is chosen from $\mathcal{M}_\alpha$, every infinite subcollection of $\{M_\alpha : \alpha < \omega_2\}$ has countable intersection. To show this, it is clearly enough to consider countable subcollections $\{M_\iota : \iota \in I\}$. We will use the well-known fact

[3, p. 255] that $M[G] = M[G_0][G_1] = M[f_0][f_1]$ where f_0 is any restriction of f to a subset Z of $\omega_1 \times \omega_2$ and f_1 is the restriction of f to the complement of Z. Let $h : I \to \{0,1\}$ indicate which of $S_\iota, \omega_1 \setminus S_\iota$ we are choosing. Then there is a countable Z and associated f_0 as above such that both I and h are in $M[f_0]$, because both I and h are countable [3, p. 256]. If $\{\alpha\} \times \omega_2 \cap Z = \emptyset$, then

$$D_\alpha = \{p : p \in Fn((\omega_1 \times \omega_2) \setminus Z, 2) \wedge \exists \iota \in I \text{ such that } p(\alpha, \iota) \neq h(\iota)\}$$

is dense in $\mathrm{Fn}((\omega_1 \times \omega_2) \setminus Z, 2)$ because I is infinite. Let $p \in G \cap D_\alpha$; then f extends p and so there exists $\iota(\alpha) \in I$ such that $f(\alpha, \iota(\alpha)) \neq h(\iota(\alpha))$, from which it follows that $\alpha \notin M_{\iota_\alpha}$. Genericity of f_1 over $M[f_0]$ thus guarantees that $\bigcap\{M_\iota : \iota \in I\}$ is a subset of the countable set $\pi_1^{\to} Z$.

The foregoing argument takes care of the case $S = \omega_1$ of Axiom S, and can readily be adapted to any stationary subset S of ω_1 by letting $\mathcal{M}_\alpha = \{S \cap S_\alpha,\ S \setminus S_\alpha\}$. □

One may well ask whether we can get by without any such axiom as SSA or Axiom S. After all, each B_α is second countable and the countable union of compact sets, and there are only $\aleph_1$ of these sets altogether, and every copy of ω_1 has to pass through the ones indexed by some club. Moreover, no matter how points are chosen one apiece from the B_α, the other axioms will always give us a 'large' (i.e., indexed by a stationary set) subset of these points with countably compact closure; in a forthcoming paper we will even show that there is a copy of ω_1 containing a 'large' subset. Intuitively speaking, it would seem that these copies of ω_1 are sufficiently crowded together that either something enough like Ł or something enough like H and K could be found. These are, after all, manifolds, and there is often a lot of structure in manifolds that can be brought to light with a little probing; see, for example, [4, Lemma 3.4 and Example 3.5.]. One might also conjecture that the nonexistence of Kurepa trees (which only requires an inaccessible) can somehow cause the $\aleph_2$ disjoint copies of ω_1 to crowd each other strongly enough to make our arguments go through. Nevertheless, I have been unsuccessful in getting the various copies of ω_1 and other countably compact noncompact subspaces to line up properly without at least Axiom S.

We can also hope for some properties of manifolds to allow us to dispense with the hereditarily cwH property, as already suggested in the introduction. Failing that, we can take a closer look at models of ZFC in which every first countable hereditarily normal space is cwH, or try to construct new ones. There is a variety of such models already, but we know of none in which it could even be shown that every perfectly normal manifold is metrizable. Recently, however, Todd Eisworth and I have come up with a model of $2^{\aleph_0} < 2^{\aleph_1}$ where every perfectly normal manifold is metrizable and every T_5 manifold is of Type I. The question of whether they are cwH or not is still open.

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