A CORSON COMPACT L-SPACE FROM A SOUSLIN TREE

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Abstract. The completion of a Souslin tree is shown to be a consistent example of a Corson compact L-space when endowed with the coarse wedge topology. The example has the further properties of being zero-dimensional and monotonically normal.

1. Introduction

In this paper, the coarse wedge topology on trees is used to construct what may be the first consistent example of a Corson compact L-space that is monotonically normal. It is considerably simpler and easier to (roughly!) visualize than the CH example of a Corson compact L-space produced by Kunen [4] or the Corson compact L-space produced by Kunen and van Mill under the hypothesis that $2^{\omega_1}$ with the product measure is the union of a family $\aleph_1$ nullsets, such that every nullset is contained in some member of the family [5].

Corson compact L-spaces cannot be constructed in ZFC alone, because $MA_{\omega_1}$ implies there are no compact L-spaces at all. This is one of the earliest applications of $MA_{\omega_1}$ to set-theoretic topology, and one of the few that uses its topological characterization, viz., that a compact ccc space cannot be the union of $\aleph_1$ nowhere dense sets [3], [9, 6.2], [10, p. 16].

Recall that a Corson compact space is a compact Hausdorff space that can be embedded in a $\Sigma$-product of real lines, viz., the subspace of a product space $\mathbb{R}^\Gamma$ (for some set $\Gamma$) consisting of all points which differ from the zero element in only countably many coordinates. Corson compact spaces play a role in functional analysis, especially through their spaces of continuous functions, the Banach space $\langle C(K), \| \cdot \|_\infty \rangle$ and $C_p(K)$, the space of real-valued continuous functions with the relative product topology.


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Recall that a topological space is *separable* iff it has a countable dense subset, and *Lindelöf* iff every open cover has a countable subcover. The following terminology is now standard:

**Definition 1.1.** An L-space is a regular, hereditarily Lindelöf space which has a non-separable subspace.

For about four decades, one of the best known unsolved problems of set-theoretic topology was whether there is a ZFC example of an L-space. This was solved in an unexpected manner by Justin Tatch Moore, who constructed one with the help of a deep analysis of walks on ordinals [6]. The following problem, motivated by our main example, may still be unsolved:

**Problem 1.2.** Is there a ZFC example of an L-space which embeds in a $\Sigma$-product of real lines?

**Definition 1.3.** A space $X$ is *monotonically normal* if there is a function $U(E,F)$ defined on pairs of disjoint closed sets $\langle E,F \rangle$ such that: (1) $U(E,F)$ is an open set; (2) $E \subset U(E,F)$ and $U(E,F) \cap U(F,E) = \emptyset$; and (3) if $E \subset E'$ and $F \supset F'$, then $U(E,F) \subset U(E',F')$.

A neat feature of our main example is that it is being monotonically normal, and is thus the continuous image of a compact orderable space [11] — and yet every linearly orderable Corson compact space is metrizable [1]. One natural question is whether the main example is actually the continuous image of a compact orderable L-space: such spaces exist iff there is a Souslin tree/line. A much more general pair of contrasting questions may be open:

**Problem 1.4.** Is the existence of a monotonically normal compact L-space equivalent to the existence of a Souslin tree?

**Problem 1.5.** Is there a ZFC example of a monotonically normal L-space?

2. Trees and the coarse wedge topology

The purpose of this section is to make this paper as self-contained as reasonable, and to show that trees with the coarse wedge topology have a property even stronger than being monotonically normal. Readers with a good understanding of trees might try omitting it on a first reading.

**Definition 2.1.** A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements $x < y$ of a poset, we say $x$ is a *predecessor* of $y$ and $y$ is a *successor* of $x$.]
Definition 2.2. If a tree has only one minimal member, it is said to be rooted and the minimal member is called the root of the tree. A chain in a poset is a totally ordered subset. An antichain in a tree is a set of pairwise incomparable elements. Maximal members (if any) of a tree are called leaves, and maximal chains are called branches.

Definition 2.3. If $T$ is a tree, then $T(0)$ is its set of minimal members. Given an ordinal $\alpha$, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T \upharpoonright \alpha = \bigcup\{T(\beta) : \beta < \alpha\}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T \upharpoonright \alpha$. The set $T(\alpha)$ is called the $\alpha$-th level of $T$. The height or level of $t \in T$ is the unique $\alpha$ for which $t \in T(\alpha)$, and it is denoted $\ell(t)$. The height of $T$ is the least $\alpha$ such that $T(\alpha) = \emptyset$.

The following example illustrates some fine points of associating ordinals with trees and their elements.

Example 2.4. The full $\omega$-ary tree of height $\omega + 1$ is the set $T$ of all sequences of nonnegative integers that are either finite or have domain $\omega$, and in which the order is end extension. Each chain of order type $\omega$ consists of finite sequences whose union is an $\omega$-sequence on level $\omega$. Since this is the last nonempty level of the tree, the tree itself is of height $\omega + 1$. The subtree $T \upharpoonright \omega$ is the full $\omega$-ary tree of height $\omega$.

Definition 2.5. A tree is chain-complete [resp. Dedekind complete] if every chain [resp. chain that is bounded above] has a least upper bound. A tree is complete if it is rooted and chain-complete.

Definition 2.6. For each $t$ in a tree $T$ we let $V_t$ denote the wedge $\{s \in T : t \leq s\}$. The coarse wedge topology on a tree $T$ is the one whose subbase is the set of all wedges $V_t$ and their complements, where $t$ is either minimal or on a successor level.

Because of the way trees are structured, the nonempty finite intersections of members of the subbase are “notched wedges” of the form

$$W_t^F = V_t \setminus \bigcup \{V_s : s \in F\} = V_t \setminus V_F$$

where $F$ is a finite set of successors of $t$.

If $t$ is minimal or on a successor level, then a local base at $t$ is formed by the sets $W_t^F$ such that $F$ is a finite set of immediate successors of $t$. If, on the other hand, $t$ is on a limit level, then a local base is formed by the $W_s^F$ such that $s$ is on a successor level below $t$.

It is easy to see that a tree is Hausdorff in the coarse wedge topology iff it is Dedekind complete. In particular, if $C$ is a chain that is bounded above but has no supremum, then it converges to more than one point.

A corollary of the following theorem is that every complete tree is compact Hausdorff in the coarse wedge topology.
Theorem 2.7. [7, Corollary 3.5] A tree is compact Hausdorff in the coarse wedge topology iff it is chain-complete and has only finitely many minimal elements.

Theorem 2.8. A complete tree is Corson compact in the coarse wedge topology iff every chain is countable.

Proof. A necessary and sufficient condition for a compact space being Corson compact is that it have a point-countable $T_0$ separating cover by cozero sets—equivalently, open $F_\sigma$-sets [1]. If the complete tree has an uncountable chain, then it has a copy of $\omega_1 + 1$, which does not have a point-countable $T_0$-separating open cover of any kind, thanks in part to the Pressing-Down Lemma (Fodor’s Lemma).

Conversely, if every chain is countable, then the clopen sets of the form $V_t$ clearly form a $T_0$-separating, point-countable cover, and $T$ is compact Hausdorff by Theorem 2.7. □

Hausdorff trees with the coarse wedge topology have a property even stronger than monotone normality; it is the property that results if “clopen” is substituted for “open” in Definition 1.3:

Definition 2.9. A space $X$ is monotonically ultranormal if there is a function $U(E, F)$ defined on pairs of disjoint closed sets $\langle E, F \rangle$ such that: (1) $U(E, F)$ is a clopen set; (2) $E \subset U(E, F)$ and $U(E, F) \cap U(F, E) = \emptyset$; and (3) if $E \subset E'$ and $F \supset F'$, then $U(E, F) \subset U(E', F')$.

The property in the following theorem is named with the Borges criterion [see below] for monotone normality in mind.

Theorem 2.10. [8, Theorem 2.2] Every Hausdorff space satisfying the following property is monotonically ultranormal.

Property $B+$. To each pair $\langle G, x \rangle$ where $G$ is an open set and $x \in G$, it is possible to assign an open set $G_x$ such that $x \in G_x \subset G$ so that $G_x \cap H_y \neq \emptyset$ implies either $x \in H_y$ or $y \in G_x$.

The Borges criterion puts $H$ for $H_y$ and $G$ for $G_x$ in the part of Property $B+$ after “implies.”

The question of whether every monotonically ultranormal Hausdorff space satisfies Property $B+$ was posed in [8] and is still open.

Theorem 2.11. Every Hausdorff tree with the coarse wedge topology has Property $B+$.

Proof. For each point $t$ and each open neighborhood $G$ of $t$, there exists $s \leq t$ for which there is a basic clopen set $W^F_s$ such that $t \subset W^F_s \subset G$, and for which $F \subset V_t$. [If $t$ is on a successor level we can let $s = t$, while if $t$ is on a limit level we first find some $s' < t$ on a successor level and finite $F' \subset V_s$ for which $t \subset W^F_{s'}$; then let $F = F' \cap V_t$ and,
using Dedekind completeness, choose $s$ such that $s' \leq s < t$ and all elements of $F' \setminus F$ are incomparable with $s$. Then $W_s^F$ is as desired.]

Now for each $x \in F$ let $x'$ be the immediate successor of $t$ below $x$ and let $F^* = \{x' : x \in F\}$.

**Claim.** Letting $G_t = W_s^F$ for each $t$, $G$ as above produces an assignment witnessing Property $B^+$.  

**Proof of Claim.** The notched wedges $W_t^F$ clearly have the property that the intersection of any two contains the minimum point of one of them. Let $G_x \cap H_y \neq \emptyset$. Assume that the minimum point $t$ of $G_x$ is in $H_y$; in particular, $t \geq s$. Let $H_y = W_s^{F^*}$.

Case 1. $y < t$. Then $G_x \subset V_t \subset H_y$, because $t$ is not in $V_{z'}$ for any $z' \in F^*$.

Case 2. $y$ and $t$ are incomparable. Then $t > s$, and we again have $G_x \subset V_t \subset H_y$.

Case 3. $t \leq y$. Then if $x$ and $y$ are incomparable, we clearly have $s < x \in H_y$. This also holds if $x \leq y$. Finally, if $x > y$, we must have $y \in G_x$. \hfill \square

**Corollary 2.12.** Every Hausdorff tree is monotonically normal in the coarse wedge topology.

### 3. The main example

The following construction is utilized in the main example of this paper.

**Example 3.1.** For any tree $T$, we call a tree a completion of $T$ if it is formed by adding a supremum to each downwards closed chain that does not already have one. Formally, we define the completion $\hat{T}$ of $T$ as follows. If $T$ is not rooted, we let $\hat{T}$ be the collection of downwards closed chains (called “paths” by Todorčević), ordered by inclusion. If $T$ is rooted, we only put the nonempty paths in $\hat{T}$.

We identify each $t \in T$ with the path $P_t = \{s \in T : s \leq t\}$. Completeness of $\hat{T}$ follows from rootedness of $\hat{T}$ and from the easy fact that the supremum of a chain $C$ of $\hat{T}$ is the same as the supremum of $C \cap T$. In particular, if $C$ is a path in $\hat{T}$ then $C \cap T$ is downwards closed in $T$.

Todorčević called the set of characteristic functions of the paths of $T$ the path space of $T$ when endowed with the topology inherited from the product topology on $2^T$. Gary Gruenhage [2] showed that this topology is the coarse wedge topology of $\hat{T}$.

Recall that a Souslin tree is an uncountable tree in which every chain and antichain is countable. Let us call a tree uniformly $\omega$-ary if every nonmaximal point has denumerably many immediate successors. [For instance, Example 2.4 is a uniformly $\omega$-ary tree.]

As is well known, every Souslin tree has a subtree $T$ in which every point has more than one successor at every level above it. Thus every point of $T$ has denumerably many successors on the next limit level above it. And so, a uniformly $\omega$-ary Souslin tree results when we take the subtree $S$ of all points on limit levels of $T$. 

Theorem 3.2. The completion \( \hat{\mathcal{S}} \) of a uniformly \( \omega \)-ary Souslin tree \( \mathcal{S} \) is an L-space in the coarse wedge topology.

Proof. Since \( \hat{\mathcal{S}} \upharpoonright \alpha + 1 \) is closed for all \( \alpha < \omega_1 \), \( \hat{\mathcal{S}} \) is not separable. In the proof that \( \hat{\mathcal{S}} \) is hereditarily Lindelöf, uniform \( \omega \)-arity plays a key role: if the tree were finitary, every point on a successor level would be isolated.

We make use of the elementary fact that a space is hereditarily Lindelöf if (and only if) every open subspace is Lindelöf. Let \( W \) be an open subspace of \( \hat{\mathcal{S}} \), and let \( W_0 \) be the set of points \( t \in W \) such that \( V_t \subset W \). If \( t \in W_0 \) is on a limit level, there is also \( s < t \) such that \( V_s \) is clopen and \( s \in W_0 \); see the first paragraph in the proof of Theorem 2.11, and note that here, \( F = \emptyset \). Let \( A = \{ a \in W_0 : a \) is minimal in \( W_0 \} \). Then \( W_0 \) is the disjoint union of the clopen wedges \( V_\alpha \ (a \in A) \), and \( A \) is countable by the Souslin property.

If \( x \in W \setminus W_0 \), then there is a basic clopen subset of \( W \) of the form \( W_t^F \) where \( F \neq \emptyset \) and \( F \subset V_x \); see the first paragraph in the proof of 2.11 again. There are no more than \( |F| \) immediate successors of \( x \) below some element of \( F \), and if \( s \) is one of the other immediate successors of \( x \), then \( V_s \subset V_x \setminus V_F \), so \( s \in W_0 \). But then \( s \in A \) also, since any \( V_z \) containing \( V_s \) properly must also contain \( x \), contradicting \( x \in W \setminus W_0 \). So \( W \setminus W_0 \) is countable, and we have countably many basic clopen sets whose union is \( W \). \( \square \)

The following is now immediate from 2.8, 2.12, and 3.3.

Corollary 3.3. If there is a Souslin tree, there is a Corson compact, monotonically normal L-space.

References


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