CLASSIC PROBLEMS III

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Abstract. This is a survey of four problems that are "classics" in many different senses of the word, and of several related problems associated with each one. The numbering of the Classic Problems picks up where that of a similar article left off about four decades ago:

IX. *Is every point of* ω^* *a butterfly point?*

X. *Is there a nonmetrizable perfectly normal, locally connected continuum?*

XI. *Is there a normal space with a* σ*-disjoint base that is not paracompact?*

XII. *Is there a regular symmetrizable space with a non-*G_δ *point?*

Several related problems are given for each classic problem. Consistency results are summarized, and there is a discussion of each problem that explains various implications among the related problems and justifies calling certain problems equivalent. For each classic problem, an appendix goes deeper into some implications and/or includes reminisces.

There is a purely set-theoretic problem related to Classic Problem IX. Call a filter on a set D *nowhere maximal* if it does not trace an ultrafilter on any subset of D.

Related Problem D. *Is every free ultrafilter the join of two nowhere maximal filters?*

It is shown that the special case of ultrafilters on ω is actually equivalent to Classic Problem IX.

1. Preamble

For my talk at the 2017 Auburn conference in honor of Gary Gruenhage's 70th birthday, I decided to revive an old series of articles on "Classic Problems" which was discontinued after the first two installments. These were in the first two volumes of the journal *Topology Proceedings*, which covered the 1976 and 1977 Spring Topology Conferences [32], [33]. Each of the two articles featured four classic problems and numerous related problems.

A quarter of a century later, there appeared "Classic Problems — 25 Years Later," in two installments [37], [38]. These articles summarized the progress that had been made on these problems and many of the related problems. This included the final solutions to Classic Problems II, III, V, VI, and VIII. There has been relatively little progress on the other problems since then, mostly on Classic Problem I (see below).

In the last three decades, there have appeared two books on open problems in topology, [30] and its sequel [41]; the latter is referred to here as OPIT II. In particular, Part 2 of OPIT II featured in-depth studies of tightly focused groups of related problems, written at

2010 *Mathematics Subject Classification.* Primary 54-02, 54A35, 54D05, 54D15, 54D20, 54D30, 54D45, 54E99. Secondary 01A70, 03E99, 46L35, 54D80, 54E35, 54F20.

Key words and phrases. butterfly point, ultrafilter, perfectly normal (T_6), continuum, metrizable, locally compact, locally connected, σ-disjoint, Dowker, paracompact, countably metacompact, symmetrizable.

the invitation of Michael Hrušak and Justin Tatch Moore. Their introductory article to Part 2 gave a list of 20 "classical problems" [24], with the preamble,

> "Every healthy mathematical discipline needs a short and concise list of its central problems to maintain its focus. These problems are presumably hard to solve and indicative of the major directions in the field. Ideally, the problems themselves form these directions. In the case of set-theoretic topology, such problems have always been there. However, over the course of the years these problems may have shifted out of focus."

Three of the articles in Part 2 of OPIT II were centered on Efimov's Problem (Classic Problem I in [32]), the M3 - M1 Problem (Classic Problem IV in [32]) and the Small Dowker Space Problem (Classic Problem VII in [33]). These problems and related ones were updated by Klaas Pieter Hart [22], by Gary Gruenhage, [18] and by Paul Szeptycki [47], respectively. The only major progress since then on any of the Classic Problems of [32] [33] was a result of Alan Dow and Saharon Shelah: the axiom $\mathfrak{b} = \mathfrak{c}$ implies the existence of an Efimov space (that is, an infinite compact Hausdorff space which does not contain a copy of either ω_1 or $\beta\omega$) [10]. This answered Question 1 in the K.P. Hart article [22], "Does MA $+ \neg$CH (or PFA) imply that a compact Hausdorff space without convergent sequences contains a copy of $\beta\mathbb{N}$?" in an unexpected direction.

2. Introduction

This paper largely follows the format of [32] and [33]: four sections devoted to one problem apiece, along with problems equivalent to the main one, a list of related problems, consistency results on all the problems, and further discussion including the implications between the problems. One little addition to the usual format is that a definition is provided in each section prior to the statement of the respective Classic Problem, to enhance its readability. A more substantial change is the addition of appendices to give proofs of some pivotal theorems or to give additional information more loosely connected with the problems.

The main problem of each section is a "classic" in just about every sense of the word: of considerable interest for at least four decades; tackled at length by some of the best researchers in set-theoretic topology and general topology; a final solution directly implying solutions of many other interesting problems; stimulating major discoveries even while a final solution continues to elude everyone; and dealing with elementary properties of topological spaces that are easily explained to anyone who has had a semester of point-set topology.

Moreover, there is not much overlap with Part 2 of OPIT II. One of the problems discussed here did get some attention there: Classic Problem X in the article by Gruenhage and Moore [19]; it was also treated in the article by Karasev, Tuncali and Valov [26]. Two others, Classic Problems XI and XII, were briefly mentioned in the article by Szeptycki [47]. On the other hand, I could find no mention of Classic Problem IX anywhere in OPIT II, even though an equivalent problem on ultrafilters makes it an outstanding open problem in pure set theory. This is not meant as criticism: there are simply too many good unsolved problems in topology for even a book as long as [30] and [41] to do justice to them all.

This paper uses the following conventions.

1. No separation axioms are assumed unless stated. However, "regular" and "normal" are both understood to include "Hausdorff."

2. When a problem asks for an example, it is understood to be an example whose existence follows just from the usual (ZFC) axioms of set theory.

3. Similarly, if it asks a question like the one in the following classic problem, an affirmative solution that follows from just the ZFC axioms is to be understood. Also, "a model for Problem ____" is used as shorthand for "a model in which the answer to Problem ____ is affirmative."

4. The expression "final solution" means either a ZFC solution in one direction or the other, or a proof of ZFC-independence, with an example in one model of ZFC, and a proof that no such example exists in some other model.

One reason for conventions 2. and 3. is to avoid the misconception that a ZFC-consistent answer is sought, if one is already known. However, in the case of Classic Problems XI and XII, and in some other cases, even consistency results are lacking, and would be very welcome!

3. Classic Problem IX

Definition 3.1. A point x of a space X is a *butterfly point* if there are closed sets F_0 and F_1 such that $F_0 \cap F_1 = \{x\}$ and x is a nonisolated point in the relative topology of both F_0 and F_1.

Classic Problem IX. Is every point of ω^* a butterfly point?

As has become customary, ω^* refers to the Stone-Čech remainder $\beta\omega \setminus \omega$ of the discrete space ω of natural numbers. This notation is also used for subsets of ω and is extended in this paper to subsets of discrete spaces, as follows. If D is a discrete space and $A \subset D$, then $A^* = c\ell_{\beta D} A \setminus A$.

Equivalent Problems:

(1) Is every nonisolated point of $\beta\omega$ a butterfly point of $\beta\omega$?

(2) Is $\beta\omega \setminus \{p\}$ non-normal for every nonisolated point $p \in \beta\omega$?

(3) Is every free ultrafilter on ω the join of two nowhere maximal filters?

Definition 3.2. A filter $\mathcal{F}$ on a set D is *free* if $\bigcap \mathcal{F} = \emptyset$, and is *nowhere maximal* if it does not trace an ultrafilter on any subset of D.

The *join* of two filters is the smallest filter containing both of them.

An elementary exercise is that if $\mathcal{F}_0$ and $\mathcal{F}_1$ are filters on D, then the join $\mathcal{F}_0 \vee \mathcal{F}_1$ is the filter on D whose base is $\{F_0 \cap F_1 : F_0 \in \mathcal{F}_0 \text{ and } F_1 \in \mathcal{F}_1\}$.

Personal Note. My fascination with Classic Problem IX was greatly increased by the realization that it is equivalent to (3) and thus that (3) is an open problem. Moreover, it naturally suggests the removal of "on ω" — see Related Problem D below.

The equivalence of (1), (2), (3) and Classic Problem IX will be shown in Appendix 1. The following definition is for Related Problem C, below.

Definition 3.3. A *P-point* in a space is a point p such that every G_δ containing p has p in its interior.

Related Problems:

A. Is $\omega^* \setminus \{p\}$ non-normal for every $p \in \omega^*$?

B. Does $\beta\omega$ have a normal, countably compact, noncompact subspace?

C. (*Szymanski*) Is every point of ω^* a nonisolated P-point in some (wlog closed) subspace?

D. Is every free ultrafilter the join of two nowhere maximal filters?
Equivalently: is every nonisolated point of βD a butterfly point for every discrete space D?

E. Is every point of $\beta D \setminus D$ a non-normality point for every discrete space D? (In other words, is $\beta D \setminus (D \cup \{p\})$ non-normal for each nonisolated $p \in \beta D$?)

F. Is there an infinite compact Hausdorff space which does not have a butterfly point?

Consistency results:

The axiom $\mathfrak{r} = \mathfrak{c}$ implies an affirmative answer to Classic Problem IX. If in addition $2^{<\mathrm{cf}(\mathfrak{c})} = \mathfrak{c}$, then Yes to Related Problem A also. In particular, both axioms hold if $\mathfrak{p} = \mathfrak{c}$ (hence if MA), which also implies that $\mathfrak{c}$ is regular.

Analogous results hold for the more general Related Problems D and E, respectively [6]. The axioms used are $\mathfrak{r}_\kappa = 2^\kappa$ and also, for E, $\sup\{2^\lambda : \lambda < \mathrm{cf}(2^\kappa)\} = 2^\kappa$ for all $\kappa \leq |D|$. In particular, the Generalized Continuum Hypothesis (GCH) implies an affirmative answer to Related Problem D!

I know of no consistency results either way for Related Problem B. The consistency results I know for Related Problem C all come from consistency results for IX itself. The axiom $\diamondsuit$ implies an affirmative answer to Related Problem F: there are no butterfly points in Fedorchuk's example [12] of a hereditarily separable, hereditarily normal Efimov space. *Note.* the axiom Φ used in [12] is equivalent to $\diamondsuit$.

Discussion:

Any model for Related Problem A is also a model for (2) and hence for Classic Problem IX. This is because ω^* is the (closed) set of nonisolated points of $\beta\omega$, hence normality is preserved in going from $\beta\omega \setminus \{p\}$ to $\omega^* \setminus \{p\}$. On the other hand, it is not known whether Classic Problem IX and Related Problem A are equivalent; the claim in [23] that they are equivalent was unfounded.

A consistent negative answer to Related Problem A would settle it, and would also give a consistent example for B, because ω^* has no nontrivial convergent sequences, and so $\omega^* \setminus \{p\}$ is countably compact.

Any model for Classic Problem IX is one for Related Problem C. More generally, a butterfly point p in ω^* has to be a P-point in at least one of F_0 or F_1 when these subsets are as in Definition 3.1. To see this, suppose that there is a collection $\{W_n : n \in \omega\}$ of open neighborhoods of p in the relative topology of F_0, such that $p \notin \mathrm{Int}(\bigcap\{W_n : n \in \omega\})$. Then $\{F_0 \setminus W_n : n \in \omega\}$ is a sequence of compact subsets of $\beta\omega$ whose union is $F_0 \setminus \{p\}$. Now in $\beta\omega$, any two disjoint F_σ-subsets have disjoint closures [11, Exercise 3.6G(c)][15, Theorem 14.27], so p has to be a P-point in F_1.

If the answer to Related Problem F is affirmative in some model, then there is an Efimov space in that model. This is because the closure of every infinite subset of a witnessing example fails to have a copy of $\omega + 1$ and also of $\beta\omega$, inasmuch as ZFC is enough to show that $\omega + 1$ and $\beta\omega$ have butterfly points. (For $\beta\omega$, see below.)

By the same reasoning, an affirmative answer to Related Problem F would imply a final solution to the elusive Classic Problem I, which asked whether there is a ZFC example of an Efimov space. The converse is an open problem: there are (consistent !) examples of Efimov spaces with butterfly points. In fact, the space obtained from two copies of an Efimov space by picking copies of the same point from each and identifying them, is obviously an Efimov space with a butterfly point.

The issue in Classic Problem IX is whether *all* the points of ω^* are butterfly points. This is not the case with Related Problem F: the last point of $\omega_1 + 1$ is nonisolated and is not a butterfly point.

Many kinds of points of ω^* are known to be butterfly points. For example, $\omega^* \setminus \{p\}$ is nonnormal if p is an accumulation point of a countable discrete subspace [7]—in other words, if p is not Rudin-Frolík minimal. Any point q of ω^* which is the limit of a convergent free ω_1-sequence is a butterfly point, and such points q exist in ZFC, and indeed in any compact space of uncountable tightness, by a powerful theorem of Juhász and Szentmiklóssy [25]. Also, if a point p of a compact space is the limit of a free sequence of cofinality κ and its character exceeds κ, then p is a butterfly point.

Also, in any model of $\mathfrak{b} = \mathfrak{c}$, any point of ω^* that is not a P-point is a point of non-normality in ω^*; however, even $\mathfrak{d} = \mathfrak{c}$ implies the existence of P-points, by Ketonen's theorem [43, p. 40].

The 0-dimensional case of Classic Problem F has a translation similar to that of Equivalent Problem (3). Recall that ideals of a Boolean algebra B are subsets $J \subset B$ defined analogously to ideals in a ring: they are subsets of B that are closed under finite joins and have the property that the meet of any member of J with any member of B is itself in J. Filters of Boolean algebras are defined dually, via the unary operation of complementation.

The following definition is more localized than Definition 3.2, for reasons that will be explained in Appendix 1. It can however be applied to free ultrafilters, using the fact that a free ultrafilter *on* a set D is the same thing as a non-principal ultrafilter *in* or *of* $\mathcal{P}(D)$.

Definition 3.4. Let U be an ultrafilter of the Boolean algebra B. A sub-filter F of U is *essentially non-maximal at* U if there is no $b \in U$ such that U is generated by $F \cup \{b\}$.

Here is a Boolean algebra translation of the 0-dimensional case of Classic Problem F.

($*$) *Does every infinite Boolean algebra have a non-principal ultrafilter that is the join of two filters that are essentially non-maximal at it?*

Another translation is dual to this one, using a dual definition:

Definition 3.5. Let M be a maximal ideal of Boolean algebra B. A sub-ideal J of M is *essentially non-maximal at* M if there is no $b \in M$ such that M is generated by $J \cup \{b\}$.

(†) *Does every infinite Boolean algebra have a non-principal maximal ideal that is generated by two ideals that are essentially non-maximal at it?*

For compact Hausdorff spaces in general, the possibilites for "translation" are more complicated; see Appendix 1.

4. Classic Problem X

Definition 4.1. A space X is *perfectly normal* if X is normal and every closed subset of X is a G_δ.

Perfect normality is a hereditary property. This is because every closed G_δ in a normal space is a zero-set, *i.e.,* the preimage of $\{0\}$ under a suitable continuous real-valued function.

For convenience, "T_6" is used for "perfectly normal" and "T_5" for "hereditarily normal" below, as well as "compactum" for "compact Hausdorff space" and "continuum" for "connected compactum". By the preceding remarks, every T_6 space is T_5.

Classic Problem X. Is there a nonmetrizable T_6 locally connected continuum?

This represents a slight shift from my Auburn conference talk, where this was Related Problem A, and vice versa. As it now reads, Classic Problem X features prominently in the OPIT II article [19], where one learns that two powerful axioms, *if consistent,* would give a final solution to this problem by showing the consistency of a negative answer. [There do exist consistent examples of such continua, explained below.] I have taken the liberty of naming these axioms after the people who originated them; see the section on Consistency Results below.

Equivalent Problems:

(1) Is there a T_6 locally connected continuum that is not monotonically normal?

(2) Is there a T_6 locally connected continuum that is not the continuous image of a linearly orderable continuum?

Related Problems:

A. Does there exist a T_6 locally connected, locally compact space that is not metrizable?

B. Does there exist a T_5 locally connected, locally compact space that is not monotonically normal?

C. (*Alexandroff*) Is there a T_6 nonmetrizable generalized manifold in the sense of Čech?

D. (*Wilder*) Is there a T_6 nonmetrizable generalized manifold in the sense of [50]?

E. Is every T_6 locally connected continuum of weight $\leq \aleph_1$?

F. Is there a T_6, locally connected continuum that (a) is not arcwise connected, or (b) contains no arc?

G. Is there a T_6 locally connected, locally compact space that is not rim-metrizable?

Consistency results:

A Souslin continuum (Definition 4.2 below) is a consistent example for Classic Problem X and also for Related Problems C and F(a). Removal of all points with metrizable neighborhoods from a Souslin continuum gives a consistent example for F(b) as well.

Filippov [13] showed that a Luzin set gives a rim-metrizable example for Classic Problem X. This, together with the theorem that MA + $\neg$CH implies every T_6 locally compact, locally connected space is paracompact (due to Dianne Lane), plus a few standard tricks,[1] shows that Classic Problem X and Related Problem A are equivalent under MA. Gary Gruenhage generalized Lane's result by showing that every T_6, collectionwise Hausdorff, locally compact space is paracompact under MA + $\neg$CH [16]. The interplay between Classic Problem X and Related Problem A is a rich one, further explored in Appendix 2.

Mary Ellen Rudin showed [46] that CH implies that there is a T_6 nonmetrizable Euclidean manifold, and Gary Gruenhage [3] improved this to the existence of a Luzin set. Assuming $\diamondsuit$, Rudin also found an example of a countably compact noncompact (hence nonmetrizable) T_6 Euclidean manifold, described in [34]. These are consistent examples for Related Problem A and Related Problem D. On the other hand, no Euclidean manifold can work for Classic Problem X, because every compact (indeed, Lindelöf) Euclidean manifold is metrizable, by Urysohn's Metrization Theorem. Nor can one work for Related Problem C, since generalized manifolds in the sense of Čech are compact.

Every generalized manifold, in either sense, is T_6, locally compact, and locally connected, so a negative answer to Classic Problem X would also settle Related Problem C, the 1935 problem of Alexandroff [1].

A negative answer to Related Problem A would settle both Related Problems C and D.

Consistent examples for Related Problem B include all the ones for Classic Problem X and for all the other related problems (for Related Problem E, this means a counterexample). But there are many other consistent examples that are not T_6, even vector bundles over the long line under the assumption of $\diamondsuit$ [35, Examples 10 and 12].

Every perfectly normal compactum is of weight at most $\mathfrak{c} = 2^{\aleph_0}$, so the answer to Related Problem C is Yes if CH. A Souslin continuum has weight $\aleph_1$, and every Luzin set is of cardinality $\aleph_1$. Filippov's example of a perfectly normal, locally connected continuum from a Luzin set [13] is obtained by doing a Sierpinski carpet construction using a Luzin subset of the sphere, and is thus of weight $\aleph_1$.

Gary Gruenhage showed that CH implies there is a T_6 locally connected continuum that is rim-metrizable but not arcwise connected [18], giving a consistent example for Related Problem F(a) that does not depend on there being a Souslin continuum.

If there is a model where Related Problem A has an affirmative answer but Classic Problem X does not, then the existence of a space as in Related Problem G is ZFC-independent: see Theorem 8.8 and the ensuing discussion in Appendix 2.

Gary Gruenhage and David Fremlin formulated hypotheses *which are not known to be consistent,* but which would finally solve Classic Problem X and hence Alexandroff's 1935 problem. I call them axioms, in the spirit of Axiom Ω of [39]:

[1]These tricks are displayed in the proof of Theorem 8.4 in Appendix 2.

Gruenhage's Axiom. Every uncountable regular, first countable space contains either (i) an uncountable discrete space, (ii) an uncountable subspace of $\mathbb{R}$, or (iii) an uncountable subspace of the Sorgenfrey line.

Gruenhage's Axiom implies:

Fremlin's Axiom Every T_6 compactum admits a continuous, at-most-two-to-one map onto a metric space.

Under the PFA, Fremlin's axiom is equivalent to the restriction of Gruenhage's axiom to T_6 compacta. The proof of this is outlined in [17].

Discussion:

For many years, it was mistakenly believed that Wilder's 1949 and Alexandroff's 1935 problems were both settled by Mary Ellen Rudin's CH examples on the one hand and her 1978 theorem [44] that MA $+ \neg$CH implies T_6 (Euclidean) manifolds are metrizable on the other hand. This was due to a misunderstanding of the term "generalized manifold." For more details on this, see Section 5 of [36].

The classic Hahn-Mazurkiewicz theorem states that a Hausdorff space is the continuous image of $[0, 1]$ if it is a locally connected, metrizable continuum. [The converse is elementary.] Classic Problem X asks, in effect, whether one can consistently put "perfectly normal" in place of "metrizable," thanks to its equivalent formulation (2).

The equivalence of (1) and (2) is a corollary of a beautiful generalization of the Hahn-Mazurkiewicz theorem which was the culmination of over four decades of first-rate research, recounted in a thorough and excellent article by Sibe Mardešić [28], which was published just a year before his death in 2016. The generalization states:

A Hausdorff space is the continuous image of a linearly orderable continuum if it is a locally connected, monotonically normal continuum.

[As with the original, the converse is elementary.]

The equivalence of (1) with Classic Problem X makes use of a trick which will be explained in Appendix 2, along with a similar trick which shows that any model for Classic Problem X is a model for Related Problem G.

Related Problem B was also inspired by this generalization. A negative solution to it in some model would make it possible to replace "monontonically normal" in the generalized Hahn-Mazurkiewicz theorem with the much more general "hereditarily normal" in that model. Of course, it would also provide a final solution to Related Problem B along with Classic Problem X and all the other listed problems related to it.

Related Problem F was inspired by a lemma which plays a key role in the proof of the Hahn-Mazurkiewicz theorem itself: every locally connected metrizable continuum is arcwise connected. Unfortunately, the rest of the proof is heavily dependent on second countability, so a consistent negative answer to Related Problem F(a) — that is, a theorem in some model that every T_6 locally connected continuum is arcwise connected — may still be a far cry from a consistent negative answer and thus a final solution to Classic Problem X.

In [40], there has been much progress towards reducing Related Problem B to the existence of countably tight examples and even first countable examples in MM(S)[S] models. There is even hope of reducing Related Problem B to Classic Problem X. This is partly because MM(S)[S] models share with models of MA the feature that Classic Problem X is equivalent to Related Problem A in them. But perhaps the most formidable obstacle to a final solution of Related Problem B will be Classic Problem X itself.

There is some discussion of Classic Problem X and related problems in [26]. Some of the related problems there have to do with the special cases of rim-metrizable and Suslinian continua that are T_6 and locally connected.

Definition 4.2. A space is *rim-metrizable* if it has a base in which every member has a metrizable boundary. A *Souslin continuum* is a linearly orderable continuum which is of countable cellularity (that is, every disjoint collection of open sets is countable) but not metrizable — equivalently, not homeomorphic to [0,1]. A *Suslinian continuum* is a continuum in which every disjoint collection of non-singleton subcontinua is countable.

Suslinian continua enjoy many strong properties, such as all of them being of weight $\leq \aleph_1$; and if there is no Souslin continuum, then all Suslinian continua are metrizable [5]. For more on locally connected rim-metrizable continua, see Appendix 2.

5. Classic Problem XI

As is usually (but not always) the case, the σ in the next classic problem designates a countable union:

Definition 5.1. A collection of sets is *σ-disjoint* [resp., *σ-point-finite*] if it is the union of countably many disjoint [*resp.,* point-finite] collections.

Classic Problem XI. Is there a normal space with a σ-disjoint base that is not paracompact?

The following definition is for Related Problems A and B below.

Definition 5.2. A *Dowker space* is a normal space that is not countably paracompact (equivalently, not countably metacompact). A space is *countably metacompact* resp., *countably paracompact* if every countable open cover has a point-finite *resp.,* locally finite refinement.

Related Problems:

A. Is there a Dowker space with a σ-point-finite base?

B. Is there a Dowker space with a point-countable base?

C. (*Reed*) Is there a normal nonmetrizable space which is the union of countably many open metrizable subspaces?

D. Is there a normal, countably paracompact space with a σ-point-finite base that is not paracompact?

E. Is there a normal, countably paracompact space with a point-countable base that is not paracompact?

Consistency results:

None for Classic Problem XI, nor for Related Problems A, B and C.

In contrast, the answer to Related Problems D and E is affirmative if if $\mathfrak{p} > \aleph_1$, and also if the Covering Lemma holds over the Core Model. This is because any normal metacompact Moore space is countably paracompact and has a σ-point-finite base, and non-paracompact examples do exist if either of these axioms hold [43] [14]. This shows that a negative answer to Related Problems D and E would require the consistency of some very large cardinals, including a proper class of measurable cardinals.

Discussion:

In 1955, Nagami showed that a space is paracompact if, and only if, it is normal, countably paracompact, and *screenable*, meaning that every open cover has a σ-disjoint open refinement. Since spaces with σ-disjoint bases are screenable, Problem XI asks for a Dowker space, and so A and B are successive weakenings.

On the other hand, Related Problem C is a strengthening: a countable cover by open metrizable spaces implies a σ-disjoint base by Bing's Metrization Theorem.

Classic Problem XI was already posed in [32] in "opposite" form: "Is a normal space with a σ-disjoint base paracompact?" as a related problem to Classic Problem III, which asked whether every normal screenable space is paracompact. Zoltán Balogh solved Classic Problem III by producing a ZFC counterexample, recounted in [37]. At one point, Balogh thought he had an example for Classic Problem XI itself: see Appendix 3.

Mike Reed posed Related Problem C in "opposite form," at a conference in memory of Zoltán Balogh [42]. For a Yes answer to Related Problem C as stated, one would have to use open subspaces of uncountable weight, by Urysohn's Metrization Theorem. Reed wrote [42] that it is consistent "under various set-theoretic assumptions" that there are no counterexamples of size $< \mathfrak{c}$. Reed showed [unpublished] that $\mathfrak{b} = \mathfrak{c}$ is one such hypothesis.

6. Classic Problem XII

Definition 6.1. A *symmetric* on a set X is a "distance function" $d : X^2 \to X$ such that:

(i) $d(x, y) = 0$ if, and only if, $x = y$ for all x, y in X.

(ii) $d(x, y) = d(y, x)$ for all x, y in X.

A *weak base* for a topological space is a system of filterbases $\{\mathcal{B}(x) : x \in X\}$ such that a subset W of X is open if, and only if there exists, for each $x \in W$, a member B of $\mathcal{B}(x)$ such that $B \subset W$. A topological space X is *symmetrizable* if there is a symmetric d such that the filterbases $\{B_n(x, d) : n \in \omega\}$ form a weak base, where $B_n(x, d) = \{y \in X : d(x, y) < 2^{-n}\}$.

Classic Problem XII. Is there a regular symmetrizable space with a non-G_δ point?

Related Problems:

A. Is there a normal symmetrizable space with a closed subset that is not a G_δ? [In other words, is there a normal symmetrizable space which is not T_6 (perfectly normal)?]

B. Is there a normal symmetrizable space which is not subparacompact?

C. Is there a normal symmetrizable space with a non-G_δ point?

D. Is there a symmetrizable Dowker space?

Consistency Results: None.

Discussion:

Obviously, any consistent or ZFC example for Related Problem C would also provide one for A and also for Classic Problem XII itself. Since every subparacompact space is countably metacompact, Related Problem D bears the same relationship to Related Problem B.

Classic Problem XII and Related Problem A were posed in a letter from E. Michael to A. V. Arhangel'skiĭ, who posed Related Problem B, all in the 1960's, but no separation axioms were specified. The best separation that has been achieved so far on these problems is by a regular symmetrizable space due to Gary Gruenhage that is not subparacompact and has a non-G_δ closed subset, and by a modification of Gary's space by a standard technique to produce a Hausdorff symmetrizable space with a non-G_δ point. [9]

The standard technique was brought to my attention by Sheldon Davis while the research for our joint paper with Gary [9] was ongoing. As explained in Appendix 4, it adds a non-G_δ point ∞ to any topological space X that is not countably metacompact. When X is symmetrizable, or normal, so is $X \cup \{\infty\}$. Therefore, an affirmative solution to Related Problem D implies one to C as well. Unfortunately, the technique often does not preserve regularity; see Appendix 4 for details.

The reverse direction behaves in an almost opposite manner: if p is a non-G_δ point of a regular symmetrizable space, then $X \setminus \{p\}$ is not countably metacompact — see below — but is, of course, regular. On the other hand, there seems to be no good reason why $X \setminus \{p\}$ should be normal even if X is normal.

The key to $X \setminus \{p\}$ not being countably metacompact is a well-known characterization that is usually more useful than Definition 5.2 itself.

Theorem 6.2. *A topological space X is countably metacompact* [resp. *countably paracompact] if, and only if, for each descending sequence of closed sets $\langle F_n : n \in \omega \rangle$ with empty intersection, there is a sequence of open sets U_n such that $F_n \subset U_n$ and $\bigcap_{n=0}^{\infty} U_n = \emptyset$* [resp. $\bigcap_{n=0}^{\infty} \overline{U_n} = \emptyset$].

Now, if p is a point of the regular symmetrizable space X, the closures of the sets $B_n(p,d)$ form a descending sequence of closed sets with empty intersection. Clearly, p is a non-G_δ point iff whenever $B_n(p,d) \subset U_n$ and U_n is open, then $\bigcap_{n=0}^{\infty} U_n \neq \emptyset$.

7. Appendix 1

Ultrafilters are so important to mathematical logic, including set theory and model theory, as well as to various branches of algebra, analysis, and topology, that it is quite remarkable that such a fundamental problem as Related Problem D of Classic Problem IX seems to have escaped notice until now.

It is true that the very existence of non-principal ultrafilters requires some fairly strong form of the Axiom of Choice (AC). In models of ZF in which there is no non-principal ultrafilter on ω, Classic Problem IX is vacuously true, and Related Problem D is vacuously true in any model in which there are no non-principal ultrafilters at all.

But for almost all set-theoretic and general topologists, Theorem 7.3 below is what really matters. We begin with a topological observation.

Lemma 7.1. *Let D be an infinite discrete space and let F be an infinite closed subset of βD. The nonisolated points in the relative topology of F form a dense-in-itself subspace.*

Proof. Let E be the set of relatively isolated points of F. Then E is a discrete subspace of βD. The closure of every denumerable subset Z of E is homeomorphic to $\beta\omega$ by a homeomorphism taking Z to ω [15, Exercise 6O.6, p. 97]. So $c\ell(Z) \setminus Z$ is a closed dense-in-itself subset of F. Every closed neighborhood of a point $p \in F \setminus E$ contains such a subset and so p cannot be isolated in the relative topology of $F \setminus E$. $\square$

Corollary 7.2. *If p is a butterfly point in βD, then this is witnessed by sets F_i which have no isolated points in their relative topology. In particular, they are closed subsets of $D^*(= \beta D \setminus D)$.* $\square$

Next we recall some well-known natural bijections.

1. Subsets of ω correspond to clopen subsets of $\beta\omega$: the clopen subsets of $\beta\omega$ are all of the form $c\ell_{\beta\omega}A$ for subsets A of ω.

2. Points of $\beta\omega$ are associated with ultrafilters on ω, with the points of $\omega \subset \beta\omega$ corresponding to the ultrafilters that are fixed on them.

3. Ultrafilters *on* ω are ultrafilters *in* (or *of*) the Boolean algebra $\mathcal{P}(\omega)$. As an algebra, $\mathcal{P}(\omega)$ is in turn isomorphic to the Boolean algebra $CO(\beta\omega)$ of clopen subsets of $\beta\omega$ via Stone duality; the isomorphism is given in 1. above.

Theorem 7.3. *Let p be a nonisolated point of $\beta\omega$. The following are equivalent.*

(0) *p is a butterfly point in ω^*.*

(1) *p is a butterfly point in $\beta\omega$.*

(2) *$\beta\omega \setminus \{p\}$ is non-normal.*

(3) *As an ultrafilter, p is the join of two nowhere maximal filters.*

Proof. (1) $\iff$ (0): ω^* is the subspace of nonisolated points of $\beta\omega$, and ω^* is closed in $\beta\omega$, so if p is a butterfly point in ω^*, it is also one in $\beta\omega$. Conversely, if F_0 and F_1 witness that p is a butterfly point in ω^*, let $W_i = F_i \cap \omega$. The Stone-Čech remainders $W_i^* = c\ell_{\beta\omega}(W_i) \setminus \omega$ are disjoint clopen subsets of ω^*, so both miss p. Thus $F_0 \setminus W_0^*$ and $F_1 \setminus W_1^*$ witness that p is a butterfly point in ω^*.

(1) $\iff$ (2): If $\beta\omega \setminus \{p\}$ is non-normal, it has disjoint closed subsets C_0 and C_1 with p in their closure in $\beta\omega$. Then $F_i = C_i \cup \{p\}$ is as in the definition of a butterfly point. Conversely, if p is a butterfly point witnessed by F_0 and F_1, then letting $F_i \setminus \{p\} = C_i$ gives a pair of disjoint closed subsets of $\beta\omega \setminus \{p\}$. Let U_0 and U_1 be open subsets of $\beta\omega \setminus \{p\}$ containing C_0 and C_1 respectively. Then U_0 and U_1 are also open in $\beta\omega$. But $\beta\omega$ is extremally disconnected [11, Corollary 6.2.29], which means that disjoint open sets have disjoint closures [11, Theorem 6.2.26]. Hence $U_0 \cap U_1 \neq \emptyset$, and $\beta\omega \setminus \{p\}$ is non-normal.

(1) $\iff$ (3): This uses a correspondence based on 1. through 3. above.

If $\mathcal{F}$ is a filter *on* ω (in other words, a filter *of* $\mathcal{P}(\omega)$, then $\mathfrak{F} = \{c\ell_{\beta\omega} F : F \in \mathcal{F}\}$ is a filter *of* the algebra $CO(\beta\omega)$, which in turn corresponds to the closed subset $C_{\mathcal{F}} = \bigcap \mathfrak{F}$ of $\beta\omega$.

Also, if F_i is the closed subspace of $\beta\omega$ that corresponds to the filter $\mathfrak{F}_i$ in $CO(\beta\omega)$, then $F_0 \cap F_1$ corresponds to the join $\mathfrak{F}_0 \vee \mathfrak{F}_1$.

By Corollary 7.2, F_0 and F_1 can be taken to be dense in themselves. Closed dense-in-themselves subsets of $\beta\omega$ correspond bijectively to nowhere maximal filters on ω. This follows easily from 1. through 3. above.

Finally, if F_0 and F_1 are dense in themselves and $F_0 \cap F_1 = \{p\}$, then the ultrafilter that corresponds to p is the join of the nowhere maximal filters $\mathcal{F}_0$ and $\mathcal{F}_1$, where $\mathcal{F}_i = \{A \subset \omega : F_i \subset c\ell_{\beta\omega} A\}$. □

The proofs of equivalence in Theorem 7.3 extend to βD and $\beta D \setminus D$ for all infinite discrete D, and the three correspondences preceding it also extend. The proofs of consistency for Related Problem D in [6] use the spaces $U(\lambda)$ of uniform ultrafilters[2] on λ for all infinite $\lambda \leq |D|$. The following elementary lemma explains this.

Lemma 7.4. *Let D be an infinite discrete space. Every nonisolated point of βD is a butterfly point if, and only if, every point of $U(\lambda)$ is one for all $\lambda \leq |D|$.*

Proof. If $\mathcal{U} \in \beta D$ and $A \in \mathcal{U}$, then $c\ell_{\beta D} A$ is a clopen neighborhood of $\mathcal{U}$ homeomorphic to $\beta(|A|)$. If A is a member of minimal cardinality in $\mathcal{U}$, then $\mathcal{U} \restriction A$ is a uniform ultrafilter in the copy of $\beta(|A|)$. Clearly, $\mathcal{U}$ is then a butterfly point in βD if, and only if, $\mathcal{U} \restriction A$ is a butterfly point in $U(A)$. □

The proof that the 0-dimensional case of Related Problem F is equivalent to the Boolean algebra condition $(*)$ is done similarly to that of (1) $\iff$ (3) above, with two differences. One is that there is no complication of a third structure like the noncompact dense subspace ω. The other is that there is nothing like Lemma 7.1 for 0-dimensional compact spaces in general, and so we must hew to the more general concept of "F_i is essentially non-maximal at U". This is equivalent to the point associated with U in the Stone space $\mathcal{S}(B)$ being non-isolated in the closed subset associated with F_i. By a general Boolean algebraic principle, $(\dagger)$ is also a translation.

For compact Hausdorff spaces in general, one natural substitute for the class of Boolean algebras is that of commutative C*-algebras. Each is of the form $C(K)$ for a unique compact Hausdorff space K, just as each Boolean algebra is $CO(K)$ for a unique compact 0-dimensional space K. Then the algebraic structure of $C(K)$ is enough to determine K

[2]As usual, an ultrafilter on a set A is said to be *uniform* if all its members are of cardinality $|A|$.

by the Banach-Stone theorem [15, 4.9]. This theorem associates the maximal ideals of $C(K)$ with the points p of K in a natural fashion, the simpler direction being given by $\mathbf{M}^p = \{f \in C(K) : f(p) = 0\}$.

There are a number of other ways of recapturing K from $C(K)$. For instance, the ideal $\mathbf{O}^p$ of continuous functions that vanish in a neighborhood of p is enough to determine p. It is the smallest such ideal: if an ideal I is contained in a unique maximal ideal $\mathbf{M}^p$, then $\mathbf{O}^p \subset I \subset \mathbf{M}^p$ [15, Theorem 7.13]. This illustrates how the correspondence of closed sets of K to ideals of $C(K)$ is not a bijection as it was for Boolean algebras. More generally, the ideal of functions that vanish in a neighborhood of a closed set $F \subset K$ is enough to determine F, but so is the ideal of all functions that vanish on F. This suggests that there may be a variety of useful algebraic and analytic "translations" of Related Problem F, some of more use to functional analysts than others.

8. Appendix 2

A handy piece of information about perfectly normal [abbreviated T_6] spaces is that a regular space is hereditarily Lindelöf if, and only if, it is T_6 and Lindelöf. This helps to streamline the proofs of Theorems 8.4 and 8.8 below.

The following theorem links Classic Problem X with Equivalent Problem (1):

Theorem 8.1. *The following are equivalent:*

(0) *There is a T_6 locally connected continuum that is not metrizable.*

(1) *There is a T_6 locally connected continuum that is not monotonically normal.*

Proof. (0) $\Longrightarrow$ (1): The product of a T_6 continuum X with $[0,1]$ is obviously a continuum, and it is perfectly normal by the general theorem of Morita [31] that the product of a perfectly normal space and a metrizable space is perfectly normal. If X is locally connected, so too is $X \times [0,1]$. Finally, if X is nonmetrizable, then $X \times [0,1]$ is not monotonically normal: Treybig showed [48] that if the product of two infinite compact spaces is monotonically normal, then both are metrizable.

(1) $\Longrightarrow$ (0): This follows by contrapositive: every metrizable space is monotonically normal. □

Remarks 8.2. Treybig had "the continuous image of an ordered compact space" rather than "[compact and] monotonically normal," but Mary Ellen Rudin showed that the two are equivalent in an extraordinarily deep paper [45] which was the final great link to the generalization of the Hahn-Mazurkiewicz theorem. In a similar way, Mardešić [27] showed, in effect, that *every compact, monotonically normal space is rim-metrizable.* Additional connections between monotone normality and rim-metrizability are explored below, beginning with a theorem and its proof that bring out resemblances in their behavior.

Theorem 8.3. *If there is a T_6 locally connected continuum that is not metrizable, then there is one that is not rim-metrizable.*

Proof. Any space of the form $K \times [0,1]$, where K is any T_6 locally connected, nonmetrizable continuum serves as an example. This is because of the poor preservation properties of rim-metrizability in products, similar to the ones for monotone normality. In fact, if X is a rim-metrizable continuum, then $X \times [0,1]$ is rim-metrizable iff X is metrizable. More generally, no rim-metrizable continuum can contain a non-metric product of non-degenerate[3] continua [49]. □

Every Souslin continuum is rim-metrizable, giving another interesting connection:

Theorem 8.4. *There is a monotonically normal, locally compact, locally connected T_6 space that is not metrizable if, and only if, there is a Souslin continuum.*

Proof. A Souslin continuum is monotonically normal, as is every linearly orderable space, and it is locally compact and locally connected. But it is not metrizable.

For the converse (actually, the inverse), we begin with the fact [4] that every perfectly normal, monotonically normal space is paracompact. Every locally compact, paracompact space is the topological direct sum of Lindelöf (clopen) subspaces [11, 5.1.27], which in a locally connected space can be taken to be the components. It is easy to see that the one-point compactification of a Lindelöf, locally compact, locally connected T_6 space has all these properties. It is also monotonically normal if the original space is [8]. So by the generalization of the Hahn-Mazurkiewicz theorem, the one-point compactification of each component is the continuous image of a linearly orderable continuum.

By the remark at the beginning of this Appendix, each component has countable cellularity. Countable cellularity is referred to in [29] as "the Suslin property," and Corollary 6 in that paper is that the nonexistence of a Souslin continuum is equivalent to every countable cellularity continuous image of a linearly ordered continuum being metrizable. Hence, if there is no Souslin continuum, then each component of the resulting space is metrizable; and so is the original space. □

Problem 8.5. *Is it consistent that every rim-metrizable T_6 locally connected continuum is monotonically normal, yet not all are metrizable?*

By Theorem 8.4, this calls for a model which has a Souslin continuum, but *not* a Luzin set: Filippov's example [13] is rim-metrizable, but not monotonically normal.

Remarks 8.6. The statements in Theorem 8.4 are strictly stronger than the ones in Theorem 8.1. For instance, there are models of CH in which there are no Souslin trees (equivalently, no Souslin continua), but CH implies the existence of Luzin sets, and hence of Filippov's example and Gary Gruenhage's manifold mentioned in Section 4. In contrast, the statements in the following theorem are (at least formally) weaker than the ones in Theorem 8.1, and yet Theorem 8.4 plays a role in showing that they are equivalent.

Theorem 8.7. *The following are equivalent:*

(2) *There is a T_6 locally compact, locally connected space that is not monotonically normal.*

(3) *There is a T_6 locally compact, locally connected space that is not metrizable.*

[3] This means both factors are "non-degenerate" in the sense of continua theory: consisting of more than one point.

Proof. (2) implies (3) for the same reason that (1) implies (0) in Theorem 8.1: every metrizable space is monotonically normal.

(3) implies (2): If (2) fails but (3) holds, then any space that witnesses (3) is monotonically normal, but then Theorem 8.4 implies that there is a Souslin continuum, whence Theorem 8.1 gives a contradiction. □

It is not known whether (2) and (3) are equivalent to (0) and (1) — in other words, whether Classic Problem X and Related Problem A are the same problem — but we do have:

Theorem 8.8. *If there is a a T_6 locally compact, locally connected nonmetrizable space, then there is either one that is a continuum, or every such space is locally metrizable, and hence each of its components is arcwise connected. These alternatives are mutually exclusive.*

Proof. Mutual exclusivity is clear: every locally metrizable compact space is second countable, hence metrizable.

If the first alternative is false, let X be as in the hypothesis, and for each point p of X let U_p be a connected open neighborhood of p with compact closure. Then U_p is (hereditarily) Lindelöf; and it is metrizable, since otherwise its one-point compactification would be a T_6 nonmetrizable locally connected continuum.

The components of X are arcwise connected because of the general fact that every Hausdorf space that is connected, locally connected, and locally completely metrizable is arcwise connected. This is an easy consequence of the case where X itself is completely metrizable [11, 6.3.11], because in the general case, any two points are contained in a connected open metrizable subspace via an elementary chaining argument [51, 26.15]. □

The second alternative in Theorem 8.8 would give a negative answer to Related Problem G, if it is consistent, because every locally metrizable space is rim-metrizable. This, together with Theorem 8.3, based on the first alternative, would then show that the existence of a space as in Related Problem G is ZFC-independent.

The two alternatives in Theorem 8.8 also help to compare the following two problems.

Problem 8.9. *If there is a T_6 locally compact, locally connected nonmetrizable space, is there one that is rim-metrizable?*

Problem 8.10. *If there is a T_6 locally connected nonmetrizable continuum, is there one that is rim-metrizable?*

If there is no T_6 locally connected nonmetrizable continuum, the answer to Problem 8.10 is vacuously affirmative. But then the answer to Problem 8.9 is also affirmative, either vacuously or by Theorem 8.8. In any case, a Yes answer to 8.10 implies one to 8.9

As a partial converse, if there is a rim-metrizable T_6 locally compact, locally connected space that is not *locally* metrizable, then there is one which is a continuum. Indeed, let p be a point without a metrizable neighborhood, and let U be an connected open neighborhood of p with compact closure. As in the proof of Theorem 8.4, the one-point compactification of U is a T_6 locally connected, rim-metrizable, nonmetrizable continuum.

9. Appendix 3

One of the most popular "small Dowker" space problems is whether there is a first countable Dowker space. There are many examples under various set-theoretic hypotheses [47]. Any space satisfying Classic Problem XI and its Related Problems A, B, and C would be such a space, but none of the known consistent examples has a point-countable base.

Classic Problem XI has the distinction of having had Mary Ellen Rudin and Zoltán Balogh — arguably the two greatest discoverers of intricate examples in the history of general and set-theoretic topology[4] — both announce consistent examples for it (normal, non-paracompact spaces with σ-disjoint bases) and then withdraw the claims.

I no longer remember when I proposed Classic Problem XI to Mary Ellen. It could have been as early as the 1974 Spring Topology Conference in Charlotte or as late as the 1976 AMS Annual meeting in San Antonio — possibly even later. I do remember that she announced the existence of an example under $\diamondsuit^*$ no later than the 1978 Spring Topology Conference. After having withdrawn it, she announced the existence of one under $\diamondsuit^+$ at the 1978 International Congress of Mathematicians (ICM) in Helsinki, the same conference at which she privately showed the consistency of all perfectly normal (T_6) Euclidean manifolds being metrizable.

Some details about this ICM announcement/withdrawal and Zoli's similar $\diamondsuit^+$ foray can be found in [20]. Zoli's dramatic withdrawal deserves additional mention. I had been working my way through his preprint at the 1993 Summer Topology Conference at Slippery Rock, PA. Shortly before Zoli was about to give his invited 1-hour talk on the claimed example, I discovered a snag in the proof. I showed it to him less than ten minutes before his talk.

Zoli did some very quick thinking. When the time came for him to begin his talk, he announced that he had no example after all. A lesser researcher than Zoli might have filled the allotted hour talking about some other excellent research, but Zoli simply told us that we all had an unexpected hour of free time, and left the podium.

10. Appendix 4

There is no overlap between the Dowker spaces that are discussed in Section 5 and Section 6. Every first countable symmetrizable space is semimetrizable, and thus is subparacompact, hence countably metacompact (more strongly, every closed subset is a G_δ).

The technique mentioned in the discussion of Classic Problem XII and its related problems begins with any space (X, τ) that is not countably metacompact and adds a non-G_δ point $\infty \notin X$ by using a descending sequence of closed sets $F_n \downarrow \emptyset$ such that if U_n is an open set containing F_n, then $\bigcap_n U_n \neq \emptyset$.

The topology on the extension $X \cup \{\infty\}$ is:

$$\tau' = \tau \cup \{W : W \cap X \in \tau \text{ and } \exists n(F_n \cup \{\infty\} \subset W)\}.$$

[4]The esteem in which we held these two great mathematicians is in no way diminished by the fact that we usually referred to them in conversation as simply "Mary Ellen" and "Zoli." The latter is the standard Hungarian (Magyar) diminutive of "Zoltán".

If X is symmetrizable by d, one can extend d to a symmetric on $X \cup \{\infty\}$ by:

$$d(x, \infty) = 2^{-n} \text{ if } x \in F_n \setminus F_{n+1},\ d(\infty, \infty) = 0,\ \text{and } d(x, \infty) = 2 \text{ otherwise.}$$

This extension is easily seen to conform to Definition 6.1, with $F_n \cup \{\infty\} = B_{n-1}(\infty, d)$, and ∞ is clearly not a G_δ point.

If X is normal, so is $X \cup \{\infty\}$, because no really new pairs of disjoint closed sets are created, modulo various X-closed $F \subset X$ being replaced by $F \cup \{\infty\}$ in the pairs where F has x in its closure. The same pairs of disjoint open sets work, *mutatis mutandis.*

On the other hand, regularity can easily be lost. It is not hard to see that:

If X is regular, then $X \cup \{\infty\}$ is regular if, and only if, cofinally many F_n have the property that if F is closed in X and disjoint from F_n, then there are disjoint open subsets of X containing F and F_n respectively.

This is a fairly demanding condition, and is not satisfied in the majority of published examples of regular, non-normal spaces that are not countably metacompact. Most of them feature a family of F_n for which there is a discrete collection of closed sets $\{D_n : n \in \omega\}$ such that $F_n = \bigcup\{D_m : m \geq n\}$. Regularity is inevitably lost in such spaces when ∞ is added. Indeed, suppose there were disjoint open sets U_n and V_n containing D_n and F_{n+1} respectively. Let $W_n = \bigcap_{i<n} V_i$; then $D_n \subset W_n$ for all n. Let $G_n = \bigcup\{W_i : i \geq n\}$; then $F_n \subset G_n$, and $\bigcap\{G_n : n \in \omega\} = \emptyset$, because the W_n are disjoint. But this is a contradiction.

This is exactly what happens in the central example of [9], which is still essentially the only known example of a regular symmetrizable space which is not countably metacompact. In the notation of [9], $F_n = \{\alpha \in F : k(\alpha) \geq n\}$, and $D_n = \{\alpha \in F : k(\alpha) = n\}$ for $n > 0$. $\{D_n : n \in \omega\}$ is a discrete collection of closed sets, since F is a closed discrete subspace of X. So $X \cup \{\infty\}$ is not regular. It is, however, Hausdorff, as is every $X \cup \{\infty\}$ if X is regular and the F_n arise from a discrete sequence of D_n as above.

Our understanding of how symmetrizability depends on covering or separation properties leaves much to be desired. Arhangel'skiĭ posed several questions about symmetrizable spaces in a 1966 survey paper [2] for which we still lack answers.

Problem 2.4 in [2]: Does every symmetrizable space have a σ-discrete network? [No for regular spaces, because of the central example in [9], but open for normal spaces.]

Problem 2.5 in [2]: Is every symmetrizable subspace of a T_6 compact space metrizable? [It is semimetrizable due to first countability of T_6 compacta.]

Problem in [2], not numbered: Is every collectionwise normal symmetrizable space paracompact?

Problem 4.4 in [2], attributed to Ceder: Is every paracompact symmetrizable space stratifiable?

Problem 4.5 in [2]: Can every paracompact symmetrizable space be condensed onto a metric space?

The central example of [9] is the brainchild of Gary Gruenhage. It is a remarkably original example: I, for one, am still unable to guess how Gary might have gotten the idea for it. I am similarly in awe of Mary Ellen Rudin's screenable Dowker space from $\diamondsuit^{++}$ and Zoltán Balogh's utterly different ZFC example of a screenable Dowker space.

Gary originally used CH for his example, but it was a routine matter for me to eliminate CH by using the discrete space of cardinality $\mathfrak{c}$ where Gary used one of cardinality $\aleph_1$ and by making a few minor adjustments; and to identify the descending sequence F_n inside F. Gary had originally used the closed set F to answer the problem by E. Michael that corresponds to Related Problem A. Then the addition of ∞ to Gary's example produced a Hausdorff example for Michael's question that corresponds to Classic Problem XII and Related Problem C.

Acknowledgements

I take this opportunity to thank the organizers of the conference in honor of Gary Gruenhage for their invitation, which spurred me on to first talk about these classic problems and then to write this paper.

I thank Alan Dow for keeping the Saturday Carolina Seminar up to date on progress on Classic Problem I.

I thank Grzegorz Plebanek and my colleagues Giorgis Androulakis, Steve Dilworth, and Maria Girardi for their help with Banach spaces.

I thank Lyubomyr Zdomskyy and Lynne Yengulalp for their help on Related Problem D of Classic Problem IX. I had been unaware of either the set-theoretic or topological version until Lynne told me about the Bešlagić - van Douwen paper [6] at the 2017 Auburn conference.

Special thanks are due to Andrzej Szymański for his many helpful comments on Classic Problem IX and related problems and research down through the years. I take this opportunity to congratulate him on his retirement[5] and new status as Professor Emeritus at Slippery Rock State University.

Special thanks are also due to Jacek Nikiel for inviting me to give a colloquium at Opole University, and for much information about the generalized Hahn-Mazurkiewicz theorem in which he had played a leading role for many decades. That role included his boldly posing what came to be known as "Nikiel's Conjecture," which Mary Ellen Rudin finally settled [45], and with this solution there followed the long-sought generalization.

Special thanks are also due to Sheldon Davis, currently chairman of the Department of Mathematics at the University of Texas at Tyler, Texas, for his advice for Section 6 at the 2017 Auburn Conference, and especially for his inestimable help with research related to Classic Problem XII in [9]. That joint paper with Sheldon and Gary was one of the most satisfying joint papers I have ever worked on. Sheldon's administrative talents were already in evidence back then as he kept us abreast of where we were and where we still had to go as our research progressed. If it hadn't been for him, I might not have had a joint paper with Gary Gruenhage in any of my three years at Auburn University.

[5]as of January 5, 2018

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