# Uniform box powers and products

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### 1. Introduction.

The following concept was introduced by Scott Williams at the 2001 Prague conference:

**Definition 1.1.** *Let $\mathcal{D}$ be a uniformity (a system of entourages) on the space $X$, and let $\kappa$ be a cardinal number. For each $D \in \mathcal{D}$ let*

$$\overline{D} = \{\langle x, y\rangle \in X^\kappa \times X^\kappa : \langle x_\alpha, y_\alpha)\rangle \in D \text{ for all } \alpha \in \kappa\}.$$

*The uniformity on $X^\kappa$ whose base is the collection of all $\overline{D}$ is called* **the uniform box product**.

A well known example is the usual uniformity on $\mathbb{R}$, whose base consists of the relations

$$D_\epsilon = \{\langle x, y\rangle : |x - y| < \epsilon\} \qquad (\epsilon > 0)$$

And $\overline{\mathcal{D}}$ is an extension of $\ell_\infty(\kappa)$ to all of $\mathbb{R}^\kappa$; $\ell_\infty(\kappa)$ is the component of $\overrightarrow{0}$ in $\mathbb{R}^\kappa$ with the uniform box product.

As usual, we use $D(x)$ to mean $\{y : \langle x, y\rangle \in D\}$. Given any uniform space $\langle X, \mathcal{D}\rangle$, the associated topological space has the sets $\{D(x) : D \in \mathcal{D}\}$ as a base for the neighborhoods of $X$. If $\mathcal{D}$ has a base of equivalence relations, then these relations partition the space into clopen sets.

Note that the uniform box product as described in Definition 1.1 might better be called a "uniform box power," whereas the metric uniformity whose base is as above but with $\epsilon \leq 1$ enables one to extend the concept to any product of metric spaces. There are a number of ways to extend Definition 1.1 to encompass this, and in this paper two ways are introduced. The first is a very general method that allows one to take the box product of any set of uniform spaces.

**Definition 1.2.** *Let $\kappa$ be a cardinal number and for each $\alpha < \kappa$ let $(X_\alpha, \tau_\alpha)$ be a space with a base $\mathcal{B}_\alpha$ for a uniformity $\mathcal{D}_\alpha$ on $X_\alpha$. Let $\psi_\alpha : \mathcal{B}_0 \to \mathcal{B}_\alpha$ be an order-preserving surjection for each $\alpha \in \kappa$, with $\psi_0$ the identity on $\mathcal{B}_0$.* **The uniform box product over** $\Gamma \subset \kappa$ *is the product, $X = \Pi_{\alpha\in\Gamma} X_\alpha$ together with the uniformity $\overline{\mathcal{D}}$ whose base is $\{\overline{B} : B \in \mathcal{B}_0\}$, where*

$$\overline{B} = \Pi_{\alpha\in\Gamma}\psi_\alpha(B) = \{\langle x, y\rangle \in X^2 : \langle x_\alpha, y_\alpha\rangle \in \psi_\alpha(B) \text{ for all } \alpha \in \kappa\}$$

*The uniform space $(X_0, \mathcal{D}_0)$ is called the* **pivot** *of the uniform box product. If* $0 \in \Gamma$ *then $X_0$ is called the* **internal pivot**, *otherwise it is called the* **external pivot**.

There is no requirement that $\psi_\alpha$ be associated with any meaningful function from the underlying set of $X_0$ to $X_\alpha$. For instance, the unique uniformity on the compact metrizable space $\omega + 1$ can serve as a pivot for uniform spaces with countable bases, and thus for all uniform box products of metrizable uniform spaces.

The broadened definition even subsumes the usual box product, which is notoriously badly behaved — even the box product of metric spaces often fails to be normal [W].

**Example 1.3.** Let $\{(X_\alpha, \mathcal{D}_\alpha) : 0 < \alpha < \kappa\}$ be a family of uniform spaces. Let $X_0, \mathcal{D}_0$ be the uniform direct sum of these spaces. That is, $X_0$ is the disjoint union of the $X_\alpha$, $\bigcup\{X_\alpha \times \{\alpha\} : 0 < \alpha < \kappa\}$, and $\mathcal{D}_0$ is the uniformity whose base is the collection of all sets of the form

$$\bigcup\{D_{\beta_\alpha,\alpha} : 0 < \alpha < \kappa\} \text{ where } D_{\beta_\alpha,\alpha} = \{\langle\, \langle x, \alpha\rangle\, \langle y, \alpha\rangle\, \rangle : \langle x, y\rangle \in D_{\beta_\alpha} \text{ and } D_{\beta_\alpha} \in \mathcal{D}_\alpha\}.$$

The (usual) box product of the associated topological spaces is the underlying space of the uniform box product whose external pivot is $(X_0, \mathcal{D}_0)$ and for which $\psi_\alpha(D) = D_{\beta_\alpha,\alpha}$ for each $\mathcal{D}$ of the above form. Recall that a base for the box product topology consists of all sets of the form $\Pi\{B_\alpha : B_\alpha \in calB_\alpha\}$, where $\mathcal{B}_\alpha$ is a base for the topology on $X_\alpha$.

The second method is more restrictive. It simply consists of the special case where all the $\psi_\alpha$ are bijections. This obviously restricts the kinds of families whose uniform box products we can take. Still, it is general enough to let $\omega + 1$ serve as a pivot for all uniform spaces with countable bases, except for those that give the discrete uniformity. We will see some other interesting examples in the next section.

The following notation will be used in this paper.

**Notation 1.4.** *Let $S$ and $T$ be sets and let $g : S \to T$. If $F \subset S$, the image of $F$ under $g$ will be denoted $g^{\to}F$. If $\mathfrak{F}$ is a filter on $S$, then the filter on $T$ generated by $\{g^{\to}F : F \in \mathfrak{F}\}$ will be denoted $g[\mathfrak{F}]$.*

## 2. Some basics

We recall some basic definitions in the theory of uniform spaces, then move on to some basic results about uniform box products.

**Definition 2.1.** *A* **uniformity entourages separated equivalence uniformity**

All through this article, uniform spaces are assumed to be separated.

**Definition 2.2.** *Let $(X, \mathcal{D})$ be a uniform space. A* **Cauchy filter on** $X$

**Theorem 2.3.** *Every uniform box product of $\kappa$ nontrivial uniform spaces contains a discrete collection of $2^\kappa$ open sets.*

A well-known pair of results carry over from the theory of ordinary products of uniform spaces.

**Theorem 2.4.** *Let $\mathfrak{F}$ be a Cauchy filter on the uniform box product $(X, \overline{D})$ of uniform spaces $(X_\alpha, \mathcal{D}_\alpha)$. Then $\pi_\alpha[\mathfrak{F}]$ is a Cauchy filter on $X_\alpha$ for each factor space $X_\alpha$.*

**Theorem 2.5.** *The product of any family of complete uniform spaces is complete in the uniform box product.*

*Proof.* Let $\mathfrak{F}$ be a Cauchy filter on $X = \Pi_{\alpha\in\Gamma} X_\alpha$, and let $X_\alpha$ be complete in $\mathcal{D}_\alpha$. For each $\alpha \in \Gamma$, let $x_\alpha = \lim \pi_\alpha[\mathfrak{F}]$. The resulting $x \in X$ is the pointwise limit of $\mathfrak{F}$, and it remains to show that it is also the limit in the uniform box uniformity.

Let $\overline{D}$ be a symmetrical entourage in $\overline{\mathcal{D}}$. Let $D_\alpha(x_\alpha) = \pi_\alpha^{\rightarrow}\overline{D}(x)$ for all $\alpha$. Using the Cauchy property of $\mathfrak{F}$, let $F \in \mathfrak{F}$ be such that $F \times F \subset \overline{D}$.

*Claim.* For each $\alpha \in \Gamma$, $\pi_\alpha^{\rightarrow} F \subset (D_\alpha \circ D_\alpha)(x_\alpha)$.

Equivalently, $F \subset (\overline{D} \times \overline{D})(x)$. Once the Claim is proved, the theorem follows from the fact that entourages of the form $\overline{D} \circ \overline{D}$ form a base for the uniformity $\overline{D}$.

*Proof of Claim.* Because $\pi_\alpha[\mathfrak{F}]$ converges to $x_\alpha$, we have $x_\alpha \in c\ell_{X_\alpha}(\pi_\alpha^{\rightarrow} F)$. By the symmetry of $\overline{D}$, this in turn is a subset of $\{\bigcup\{D_\alpha(y) : y \in \pi_\alpha^{\rightarrow} F\}$. This last set is a subset of $\bigcup\{D_\alpha(y) : y \in D_\alpha(x_\alpha)\} = D_\alpha \circ D_\alpha(x_\alpha)$. $\square$

Note that the foregoing proof made no use of the fact that $D_\alpha(x_\alpha) = \pi_\alpha^{\rightarrow}\overline{D}(x) = (\psi_\alpha(D_0))(x)$ nor even of $\psi$ at all.

**Corollary 2.6.** *Let $\mathfrak{m}$ denote the least measurable cardinal. The uniform box product of any family of $< \mathfrak{m}$ complete uniform spaces, none of which contains a closed discrete subset of cardinality $\mathfrak{m}$, is realcompact.*

In the introduction, the flexibility of the uniform box product concept was illustrated by metric spaces even when the maps $\psi_\alpha$ are isomorphisms. Here is another illustration.

**Definition 2.7.** *An* **AD family of subsets of** $\omega$ *is a family $\mathcal{A}$ of infinite subsets of $\omega$ such that $A_0 \cap A_1$ is finite for all distinct $A_0, A_1 \in \mathcal{A}$. The* **$\Psi$-like space** *$\Psi(\mathcal{A})$ is the locally compact space whose set of nonisolated points is $\mathcal{A}$, such that a neighborhood of $A \in \mathcal{A}$ is a set that contains $(A \setminus F) \cup \{A\}$ for some finite set $F$.*

The notation $\Psi^*(\mathcal{A})$ will be used for the one-point compactification of $\Psi(\mathcal{A})$. A nice base for the usual uniformity on $X = \Psi^*(\mathcal{A})$ is provided by a base for the entourages associated with the partitions $\mathcal{P}_F$ of $X$ clopen sets, where $\mathcal{F}$ is a finite subset of $\mathcal{A}$, defined as follows. Let $D(\mathcal{F})$ be the "disjointifier" of $\mathcal{F}$, defined by $D(\mathcal{F}) = \bigcup\{A_1 \cap A_2 : A_1, A_2 \in \mathcal{F}\}$. Then

$$\mathcal{P}_{\mathcal{F}} = [D(\mathcal{F})]^1 \cup \{A \setminus D(\mathcal{F}) : A \in \mathcal{F}\} \cup \{X \setminus \bigcup \mathcal{F}\}.$$

Here, as usual, the notation $[S]^\lambda$ [*resp.* $[S]^{<\lambda}$ stands for the collection of $\lambda$-element [*resp.* $< \lambda$-element subsets of $S$.

The natural map that takes each $\mathcal{F}$ to the entourage associated with $\mathcal{P}_{\mathcal{F}}$ is an order-isomorphism from $([\mathcal{A}]^{<\omega}, \subset)$ to a base for the unique uniformity on $X$, with the order of reverse inclusion. This sets up one-point compactifications of discrete spaces as external pivots for the uniform box product of spaces of the form $\Psi^*(\mathcal{A})$ of equal or lesser cardinality. If the cardinalities are the same, the $\psi_\alpha$ can all be isomorphisms.

Here is a broader definition of AD family, which still makes possible many effective bijections $\psi_\alpha$.

**Definition 2.8.** *An* **AD family** *of sets is a family $\mathcal{A}$ of infinite subsets of a set $S$, such that $|A_1 \cap A_2| < min\{|A| : A \in \mathcal{A}\}$ for all $A_1, A_2 \in \mathcal{A}$.*