Antidiamond and anti-PFA axioms and topological applications
[Incomplete preliminary draft]
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Abstract: In the last ten years, a number of axioms have been identified which might be called "antidiamond" axioms and "anti-PFA" axioms. The antidiamond ones featured in this talk are all compatible with $2^{\aleph_0} < 2^{\aleph_1}$ but have a strong PFA-like flavor besides being implied by the PFA. The anti-PFA axioms are all compatible with $MA(\omega_1)$ and, with one possible exception, follow from V=L and have a diamond-like flavor. Some are purely set-theoretic, others are unavoidably topological. Among the applications of the antidiamond axioms are the consistency of every normal, locally compact, linearly Lindelöf space being Lindelöf, and several metrization theorems for manifolds. Contrasted with the latter is the construction of a nonmetrizable hereditarily collectionwise normal 2-manifold from a club-guessing axiom compatible with $MA(\omega_1)$. This in turn is in contrast to the PFA implying that every hereditarily collectionwise normal 2-manifold of dimension > 1 is metrizable.

Set theoretic consistency results have been with us for a long time in the metrization theory of manifolds, beginning with Mary Ellen Rudin's theorem that the Continuum Hypothesis (CH) implies the existence of a perfectly normal, nonmetrizable manifold, [RZ] and her theorem that $MA + \neg CH$ implies all perfectly normal manifolds are metrizable.[R2] She also used $\diamondsuit$ to construct a perfectly normal, countably compact nonmetrizable manifold, while I recently used the Proper Forcing Axiom (PFA) to show the consistency of the statement that all normal, hereditarily strongly cwH manifolds of dimension > 1 are metrizable [N3].

In this talk we give some results using "antidiamond" axioms with a PFA-like flavor and "antiPFA axioms" with a diamond-like flavor to give theorems and examples that run against the current of the second pair of results just mentioned. The "antidiamond" axioms are featured in the first section.

Section 1. Some antidiamond axioms with applications to manifolds

We begin with some purely topological axioms. I have not been able to find any natural combinatorial axioms from which the first can be derived:

Axiom 1. *The FCCC Dichotomy:* Every first countable, countably compact Hausdorff space is either compact or contains a copy of ω_1.

The FCCC Dichotomy is a pure dichotomy: if W is a copy of ω_1 in a compact Hausdorff space, then the closure of W is a copy of $\omega_1 + 1$, which is not first countable. The FCCC Dichotomy was first shown consistent by Zoltan Balogh in 1987. A proof that it follows from the PFA appears in [D]. Its consistency with

CH was shown in 1999 and published in [EN1]. The consistency of its negation was shown by Ostaszewski in 1973 using CH $+\clubsuit$; see [R]. In 1979 I showed that the weakening of $\clubsuit$ to Axiom 3 [see below], which is even compatible with MA $+\neg$ CH, is enough to produce a counterexample. This was written up in [F]. In this paper we will show that the strictly weaker Axiom 5 is enough for this (Example 1, Section 3.) In Section 5 we will construct an entirely different counterexample using Axiom 3, a 2-manifold.

Axiom 1 has been used to settle the ZFC-independence of a statement motivated by a classic theorem:

Theorem. [Sneider, 1945] *A compact space is metrizable if, and only if, it is Hausdorff and has a G_δ-diagonal; that is, the diagonal $\{(x,x) : x \in X\}$ is a countable intersection of open sets.*

This theorem was extended to all regular countably compact spaces by J. Chaber in 1975. One might naturally expect these two theorems to either stand or fall together if "G_δ-diagonal" is weakened to "small diagonal":

Definition. A space has a *small diagonal* if, whenever A is an uncountable subset of $X \times X$ that is disjoint from the diagonal Δ, there is a neighborhood U of Δ such that $U \setminus A$ is uncountable.

But in fact, this is not the case. On the one hand, we know that CH implies every compact Hausdorff space with a small diagonal is metrizable. We do not know whether ZFC implies this as well; but, be that as it may, the corresponding statement about regular countably compact spaces is independent not only of ZFC, but also of CH. On the one hand, Gary Gruenhage has shown [G]:

Theorem. [CH + FCCC Dichotomy] *Every countably compact regular space with a small diagonal is metrizable.*

On the other hand, Oleg Pavlov [DP] has constructed a counterexample assuming $\diamondsuit^+$. His example is a perfect preimage of ω_1. [A continuous function is called **perfect** if it is closed and the preimage of each point is compact.]

Here is a pair of topological axioms which can be derived from a pair of combinatorial axioms given in Section 3.

Axiom 2 [$\mathbf{2^+}$]. *The [Strong] LCπ Dichotomy:* If X is locally compact and Hausdorff, and $\pi : X \to \omega_1$ is continuous and onto, then at least one of the following is true:

(1) There is a subspace W of X such that $\pi \restriction W$ is a perfect map onto a club.

(2)(2^+) There is a closed discrete subspace D of X such that $\pi(D)$ is stationary [resp. a club].

I am being somewhat loose in calling this a dichotomy, since the alternatives are not mutually exclusive.

The LCπ Dichotomy was first shown consistent by Zoltán Balogh around 1982, when he showed that it followed from MA+not-CH (in fact he got every locally countable subspace to be the countable union of closed discrete subspaces of X) unless X contained a perfect preimage of ω_1. Its consistency with CH was shown in 1997 and will appear in [EN2]. Ostaszewski's space shows that its negation is consistent with CH.

The strong LCπ dichotomy cannot be obtained by ccc forcing, and does not follow from MA+not-CH, but it does follow from the PFA. It is consistent with $2^{\aleph_0} < 2^{\aleph_1}$, but it is unknown whether it is consistent with CH.

Theorem 1.1. [$2^{\aleph_0} < 2^{\aleph_1}$ + FCCC Dichotomy + Strong LCπ Dichotomy] *Every T_5, countably compact manifold of dimension > 1 is metrizable.*

Suppose M is a counterexample.

Step 1: By FCCC Dichotomy, there is a copy of ω_1 in any countably compact, nonmetrizable subspace of M.

Step 2: Use $2^{\aleph_0} < 2^{\aleph_1}$ to show M is ω-bounded, *i.e.,* every countable subset has compact closure.

Step 3: Write M as $\bigcup\{M_\alpha : \alpha < \omega_1\}$ where each M_α is a metrizable submanifold and $\overline{M_\alpha} \subset M_\beta$ whenever $\alpha < \beta$, and $M_\beta = \bigcup\{M_\alpha : \alpha < \beta\}$ when β is a limit ordinal.

Step 4: Let $X = \bigcup\{\overline{M_\alpha} \setminus M_\alpha : \alpha < \omega_1\}$ and let $\pi : X \to \omega_1$ be defined by $\pi(x) = \alpha$ for all $x \in \overline{M_\alpha} \setminus M_\alpha$. Then π is continuous onto.

Step 5: Remove a copy W_0 of ω_1 from M. This gives either an uncountable closed discrete subspace D by (1) of Strong LCπ Dichotomy, or another copy W_1 of ω_1 in two steps using (2) and then FCCC Dichotomy.

Step 6a: If (1), get a ladder system space on a club and use $2^{\aleph_0} < 2^{\aleph_1}$, which implies [DS] that there is a piecewise monochromatic, non-uniformizable 2-coloring of any ladder system on a club. These colorings are associated with a pair of closed discrete subspaces of $M \setminus W_0$ which cannot be put into disjoint open sets.

Step 6b: If (2), use Urysohn's lemma and line up copies of ω_1 to get a non-normal subspace as in [N3]. The subspace can be an "NN-plank" as defined in [N2].

For 6b we can also rely on the highly useful lemma in [N1] about how a perfect preimage of ω_1 containing infintely many disjoint copies of ω_1 can be neither T_5 nor hereditarily strongly cwH.

Theorem 1.2. [FCCC Dichotomy + LCπ Dichotomy] *Every hereditarily strongly cwH, countably compact manifold of dimension > 1 is metrizable.*

This time, Step 2 can be done in ZFC, Step 5 needs only a stationary image for D, and Step 6a uses the Pressing-Down Lemma. For Step 6b we repeat the construction of a copy of ω_1 infinitely many times, and use the lemma from [N1].

The Rudin-Zenor perfectly normal manifold constructed using CH is an S-space [RZ], so "countably compact" cannot be replaced with "ω_1-compact" in either Theorem 1 or Theorem 2: in both cases the axioms used are compatible with CH. ($2^{\aleph_0} < 2^{\aleph_1}$ even follows from it.) The best I have been able to do with ω_1-compact manifolds is:

Theorem 1.3. *[PFA] Every ω_1-compact hereditarily strongly cwH manifold of dimension > 1 is metrizable.*

Section 2. Anti-PFA axioms that negate Axiom 1

All the "Anti-PFA" axioms with which we deal are preserved by ccc forcing, and hence compatible with $MA + \neg CH$. But some have better preservation properties than others. The following axiom is not preserved by countably closed forcing: it is destroyed by adding $\aleph_2$-many Cohen subsets of ω_1.

Axiom 3. KH^+: There is a base $\mathcal{B}$ for the club filter on ω_1 such that $\mathcal{B} \restriction \alpha$ $(= \{B \cap \alpha : B \in \mathcal{B}\})$ is countable for all $\alpha \in \omega_1$.

On the other hand, the following axiom is not even destroyed by any ω-proper forcing [I]:

Axiom 4. *There is a club-guessing ladder system on ω_1.* That is, there is a ladder system $\{L_\alpha : \alpha \in \Lambda\}$ such that for every club $C \subset \omega_1$, there is α such that $L_\alpha \subset C$.

As in [EN2], I will often refer to Axiom 4 as $\clubsuit_C$. because $\clubsuit$ is the axiom which strengthens "every club" to "every uncountable". Similarly, I let $\clubsuit_S$ strengthen it to the intermediate "every stationary."

In Section 5, we will show that Axiom 4 is enough to show the existence of a hereditarily collectionwise normal, countably compact 2-manifold. This manifold shows that the weakening of FCCC Dichotomy by substituting "perfect preimage of ω_1" for "copy of ω_1" does not work in Theorem 1 or Theorem 2. The long line shows dim > 1 is needed.

Section 6 gives more information about Axioms 3 (KH^+) and 4 ($\clubsuit_C$), including a short proof that the former implies the latter. In fact, as will be seen there, KH^+ has many consequences much stronger than $\clubsuit_C$. Here we look at some other axioms that progressively weaken $\clubsuit_C$, and show that the second weakest is already enough to negate FCCC Dichotomy.

Axiom 5. ["Kunen's Axiom"] There is a sequence of ladders $\{L_\alpha : \alpha \in \Lambda\}$ such that for every club C, there is a limit point α of C such that for almost every ξ in L_α, there is an element of C between ξ and the next element of L_α.

If we apply $\clubsuit_C$ (or Kunen's Axiom) to the derived set C' of C, then there are infinitely many elements of C between ξ and the next element of C. Hence $\clubsuit_C$ implies Kunen's Axiom even if we interpret "between" strictly.

Conjecture. *Kunen's Axiom does not imply* $\clubsuit_C$.

Kunen's Axiom can be obtained by the following forcing, beginning with any ground model; but if we begin with a model the PFA, then I conjecture that the following forcing is not enough to get $\clubsuit_C$.

Example 1. Let $\mathcal{U}$ be a Q-point ultrafilter; that is, given any finite-to-one function $f : \omega \to \omega$, there is $A \in \mathcal{U}$ such that $f \restriction A$ is one-to-one. Let P be "the second half of Mathias forcing" applied to $\mathcal{U}$. That is:

$P = \{([p], \mathcal{A}_p) | [p] \in [\omega]^{<\omega}, \text{ and } \mathcal{A}_p \in [\mathcal{U}]^{<\omega}\}$

$p < q \iff [p] \subset [q], \mathcal{A}_p \subset \mathcal{A}_q \text{ and } [q] \setminus [p] \subset \bigcap \mathcal{A}_p.$

Claim. Forcing with P gives a model of Kunen's Axiom.

The following axiom appears in the joint paper of Gruenhage and Moore in *Open Problems in Topology II*:

Axiom 6. $\mho$*:* There is a family of continuous functions $\{f_\alpha : \alpha \in \Lambda\}$, $f_\alpha \to \omega$ such that, for each club $C \subset \omega_1$, there is α such that $f_\alpha \restriction C \cap \alpha$ is surjective.

This axiom is implied by Kunen's Axiom: with L_α listed in order as $\xi_n (n \in \omega)$ let $x_n (n \in \omega)$ list ω^2, and let f_α take the interval $(\xi_n, \xi_{n+1}$ to $\pi_1(x_n)$. Besides the forcing in Example 1, $\mho$ can be produced by adding either a Cohen or random real to any ground model. It is well known that if the ground model satisfies PFA, then the forcing extension does not satisfy $\clubsuit_C$, and I conjecture that it also does not satisfy Kunen's Axiom.

It is easy to see that $\mho$ implies the following axiom: just reduce f_α modulo 2 to get g_α in:

Axiom 7. There is a family of continuous functions $\{g_\alpha : \alpha \in \Lambda\}$, $g_\alpha \to \{0, 1\}$ such that, for each club $C \subset \omega_1$, there is α such that $g_\alpha \restriction C \cap \alpha$ is not eventually constant.

Theorem 2.1. *The following axioms are equivalent.*

1. *Axiom 1*
2. *There is a banded perfect 2-1 preimage of* ω_1 *without a copy of* ω_1*.*
3. *There is a ladder system such that for each club* C *there is a ladder* L_α *for which there are infinitely many pairs of successive members* c, c' *of* C *such that* L_α *meets the interval* $[c, c')$ *in an odd number of elements.*

"Banded" means each point x has a clopen nbhd which does not contain the other member of the fiber containing x, but which contains every fiber $\pi^{\leftarrow}\{\xi\}$ that it meets whenever $\xi < \pi(x)$.

If we strengthen ”an odd number” to ”exactly one” in (3), the resulting axiom follows from $\clubsuit_C$; I do not know whether it follows from Kunen's Axiom or $\mho$, nor whether the reverse implications hold.

The following example shows how (1) implies (2):

Example 2. Let $X = \omega_1 \times \{0,1\}$, let $\pi : X \to \{0,1\}$ be the projection map, and define basic nbhds of each point of X as follows. For each α we define a pair of complementary subsets $B(\alpha,0)$ and $B(\alpha,1)$ of $[0,\alpha]$, declaring them to be (cl)open, letting a (clopen) base at $\langle\alpha,i\rangle$ be the collection of all sets of the form

$\{\langle\alpha,i\rangle\} \cup ([(\beta,\alpha) \times \{0,1\}] \cap B(\alpha,i))$.

For nonlimit α, we let $B(\alpha,1) = \{\langle\alpha,1\rangle\}$, so that $B(\alpha,0)$ be the complement of $B(\alpha,1)$ in $[0,\alpha] \times \{0,1\}$. For limit α we define:

$$B(\alpha,i) = \{\langle\alpha,i\rangle\} \cup (g_\alpha^{-1}\{i\} \times \{0,1\}) \text{ for } i \in \{0,1\}.$$

where g_α is as in Axiom 7.

With the resulting topology, X is countably compact (in fact, it is ω**-bounded:** every countable subset has compact closure), noncompact, and locally countable, hence first countable. Clearly, X is banded, and it does not contain a copy of ω_1. Indeed, every uncountable closed subset meets the fibers over a club C, because π is a closed map; then we can apply Axiom 7 to find a stationary set of α such that $C \cap \pi^{-1}[0,\alpha)$ meets both $B(\alpha,0)$ and $B(\alpha,1)$ over a cofinal subset of α. But no copy of ω_1 can behave like this.

Corollary. *Axiom 7 negates the FCCC Dichotomy (Axiom 1).*

Problem 1. *Is Axiom 7 strictly weaker than Axiom 6?*

Section 3. More antidiamond axioms and a topological application

In this section we give some purely combinatorial axioms that imply Axioms 2 and 2^+ and:

Axiom 8. *The LC Trichotomy:* If X is locally compact and Hausdorff, then at least one of the following is true:

(1) X is the countable union of ω-bounded subspaces.
(2) X has an uncountable closed discrete subspace.
(3) X has a countable subset with non-Lindelöf closure.

Obviously, (1) is incompatible with either (2) or (3), but there are easy examples of X satisfying both (2) and (3); so I am speaking loosely here again. The power of this axiom is evident from Theorem 3.1 below, which gives a consistent

Yes answer, modulo large cardinals, to the following problem of Arhangel'skiĭ and Buzyakova [AB]:

Problem 2: Is every normal, locally compact, linearly Lindelöf space Lindelöf?

If "locally compact" is omitted, we have one of the most basic unsolved problems in general topology, for which we do not even have any consistent answers.

Definition. A space X is *linearly Lindelöf* if every open cover that is totally ordered by $\subset$ has a countable subcover. A space X is *pseudocompact* if every continuous real-valued function on X is bounded. A space X is *countably metacompact* if every countable open cover has a point-finite open refinement.

Here are some basic facts about these concepts; the first two have very easy proofs.

• Every countably compact T_1 space is countably metacompact, and so is every T_1 space that is the countable union of closed, countably compact subspaces.

• Every countably compact space is pseudocompact; more generally, so is every space with a dense countably compact (or pseudocompact) subspace.

• Every normal, pseudocompact space is countably compact [E, p. 206].

• Every countably metacompact, linearly Lindelöf space is Lindelöf [H].

The following appears in [EN2].

Theorem 3.1. [Axiom $8 + \mathfrak{c} < \aleph_\omega$] *Every locally compact, normal, linearly Lindelöf space is Lindelöf.*

Proof. Alternative (2) in Axiom 6 cannot hold in a linearly Lindelöf space: if D is an uncountable closed discrete subspace of a space X, let $\{d_\xi : \xi < \omega_1\}$ list $\aleph_1$ points of D, let $E = X \setminus \{d_\xi : \xi < \omega_1\}$ and let $U_\alpha = E \cup \{d_\xi : \xi < \alpha\}$ for all $\alpha < \omega_1$. Then $\{U_\alpha : \alpha < \omega_1\}$ is an open cover with no countable subcover.

Alternative (3) cannot hold in a regular linearly Lindelöf space if $\mathfrak{c} < \aleph_\omega$. For then, the closure of any countable subset has a base of cardinality less than $\aleph_\omega$, and in a linearly Lindelöf space, every open cover of cardinality $< \aleph_\omega$ has a countable subcover [H]. Clearly, every closed subspace of a linearly Lindelöf space is linearly Lindelöf, so the closure is Lindelöf.

So now we turn to alternative (1). Let X be a linearly Lindelöf space which is the union of countably many ω-bounded subspaces X_n. The closure $\overline{X_n}$ is pseudocompact since X_n is a dense countably compact subspace. If X is normal, so is each $\overline{X_n}$. Hence $X = \bigcup_n \overline{X_n}$ is the countable union of closed countably compact subspaces, and is thus countably metacompact. It now follows that X is Lindelöf. $\square$

Now we give some purely combinatorial axioms from which all our topological axioms except Axiom 1 can be derived. They use the following concepts:

Definition. An ideal $\mathcal{J}$ of subsets of a set X is *countable-covering* if for each $Q \in [X]^\omega$, the ideal $\mathcal{J} \restriction Q$ is countably generated.

In other words, for each countable subset Q of X, there is a countable subcollection $\{J_n^Q : n \in \omega\}$ of $\mathcal{J}$ such that every member J of $\mathcal{J}$ that is a subset of Q satisfies $J \subset J_n^Q$ for some n.

Definition. Given an ideal $\mathcal{J}$ of subsets of a set S, a subset A of S is *orthogonal to* $\mathcal{J}$ if $A \cap J$ is finite for each $J \in \mathcal{J}$. The ω*-orthocomplement of* $\mathcal{J}$ is the ideal $\{I : |I| \leq \omega,\ I \text{ is orthogonal to } \mathcal{J}\}$ and will be denoted $\mathcal{J}^\perp$.

Axiom 9. For every countable-covering ideal $\mathcal{J}$ on a set X, either

(i) X is the union of countably many sets $\{B_n : n \in \omega\}$ such that every countable subset of each B_n is in $\mathcal{J}$, or

(ii) there is an uncountable subset A of X such that every countable subset of A is in $\mathcal{J}^\perp$.

Theorem 3.2. *Axiom 9 implies Axiom 8.*

Axiom 9 follows from what is called $(*)$ in [T], which in turn is a consequence of the PFA and is also compatible with CH. It has considerable large-cardinal strength, whereas the following variations on it are ZFC-equiconsistent. When $\{i, j\} \subset \{1, 2\}$ then C_{ij} is also compatible with CH.

Axioms. CC_{11} is the axiom that for each countable-covering ideal $\mathcal{J}$ on a stationary subset S of ω_1, either:

(i) there is an uncountable $A \subset S$ such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there is an uncountable $B \subset S$ such that $[B]^\omega \subset \mathcal{J}^\perp$.

Axioms CC_{12}, CC_{21} and CC_{22} are defined by replacing "uncountable" with "stationary" in (ii), (i), and both, respectively. By strengthening "stationary" to "club" and replacing S by ω_1, we get a pair of axioms that are compatible with $2^{\aleph_0} < 2^{\aleph_1}$:

Axioms. CC_{13} [*resp.* CC_{23}] is the axiom that for each countable-covering ideal $\mathcal{J}$ on ω_1, either:

(i) there is an uncountable [resp. stationary] A such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there is a club $B \subset \omega_1$ such that $[B]^\omega \subset \mathcal{J}^\perp$.

Problem 3. *Is CC_{13} or CC_{23} compatible with CH?*

Theorem 3.3. *(a) $CC_{11} \implies \neg\clubsuit$.*

(b) $CC_{12} \implies \neg\clubsuit_S$.

(c) $CC_{13} \implies \neg\clubsuit_C$.

Proof. Let $\mathcal{J}$ be the ideal generated by the sets S_α. This is countable-covering: if $\sup Q = \alpha$, then $\mathcal{J} \restriction Q$ is generated by $\{S_\xi \cap Q : \xi \leq \alpha\} \cup [Q]^{<\omega}$. Now, it is impossible for there to be a set of order type $\geq \omega^2$ in $\mathcal{J}$, so alternative (i) fails in CC_{1n}. But alternative (ii) gives an uncountable [*resp.* stationary] [*resp.* club] subset of ω_1 which meets each ladder S_α in a finite set, very strongly negating the respective variants of $\clubsuit$. $\square$

The following is shown in [EN2].

Theorem. *Axiom* CC_{12} [*resp.* CC_{13}] *implies Axiom 2* [*resp.* 2^+].

Section 4. Principal T-bundles: some ZFC results

The 2-manifold which will be constructed using Axiom 4 ($\clubsuit_C$) is formed by adding a single point to a principal T-bundle over the long ray L^+, where T is the so-called "torus group," the group of complex numbers of absolute value 1. For L^+ we take $\{\alpha + r : \alpha \in \omega_1, r \in (0,1]\}$ with the lexicographic "connect-the-dots" order: $\alpha + r \leq \beta + s \iff \alpha < \beta$ or $(\alpha = \beta \wedge r < s)$.

To emphasize the geometry of the situation, we use the alternative description of T as the group of angles θ $(0 \leq \theta < 2\pi)$ with the obvious addition modulo 2π. A principal T-bundle X over the long ray is locally like $\mathbb{R} \times T$; in fact, there is a projection $\pi : X \to L^+$ and homeomorphisms $f_\alpha : \mathbb{R} \times T \to \pi^{-1}(0, \alpha + 1)$ such that π-fibers are preserved, i.e., $f(\pi_1^{-1}\{r\} = \pi^{-1}\{p\}$ for a unique $p \in L^+$. [Here $\pi_1 : \mathbb{R} \times T \to \mathbb{R}$ is the projection.] We call a subset of X **unbounded** if its image under π is unbounded.

Notation. We let $X_p = \pi^{\leftarrow}\{p\}$ for each point p of L^+. [X_p is referred to as *the fiber over* p.] We let $x + \theta$ represent the unique point y in $X_{\pi(x)}$ that is θ radians from x in the positive direction. This notation also gives the action of T on X.

Principal T-bundles have the property that convergent sequences preserve angular separation. In fact, if $x_n \to x$ and $y_n = x_n + \theta_n$, then $\theta_n \to \theta$ iff $\langle y_n \rangle$ converges to $x + \theta$. Because of this, it is convenient to treat each fiber X_p as an isometric copy of T, with $d(x, y)$ equal to the angular separation between x and y. Moreover, the homeomorphisms f_α can be chosen so that their restrictions to the various rings $\{r\} \times T$ are isomorphisms to the images X_p.

Combining these features gives us a convenient way of defining a local base at any point x. Fix a choice of $f_\alpha : \mathbb{R} \times T \to \pi^{\leftarrow}(0, \alpha + 1)$ for each $\alpha \in \omega_1$, and for $p = \alpha + r$ $(\alpha \in \omega_1, r \in (0,1))$ let $f_p = f_\alpha$. For each point x there is an arc A_x that contains x and meets each fiber X_p in $\pi^{\leftarrow}(0, \pi(x) + 1)$ exactly once. Once whe choose A_x for some $x \in X_p$, we can let

$$A_{x+\theta} = \{y \in X : \exists z \in A_x \ (y = z + \theta)\}$$

for all $\theta \in [0, 2\pi)$. So we can define a local base at x by taking the members B of a local base at $\pi(x)$ and taking the intersections of the sets $\pi^{\leftarrow} B$ with the strips of width $1/n$ $(n \in \mathbb{Z}^+)$ centered on A_x.

For the rest of this section, X will be principal T-bundle over L^+, beginning with the following easy lemma.

Lemma 1. *If F is a closed unbounded subset of X, then $\pi^{\rightarrow}(F) \cap \omega_1$ is a club, and $F \cap X_p$ is compact for all $p \in L^+$.* $\square$

Definition. Let $p \in L^+$ and $F \subset X$. A *gap in $F \cap X_p$* is a pair $x, x+\theta$ ($0 < \theta \leq 2\pi$) of (not necessarily distinct) elements of $F \cap X_p$ such that $x + \phi \notin F \cap X_p$ whenever $0 < \phi < \theta$. The number θ is called the *width* of the gap $\{x, x+\theta\}$.

In this definition, $x + 2\pi = x$: a gap of width 2π is the case $F \cap X_p = \{x\}$.

Lemma 2. *Let F be a closed unbounded subset of X. For each $\alpha \in \pi^{\rightarrow} F \cap \omega_1$ let*

$$h(\alpha) = max(\{\theta : \text{ there is a gap of width } \theta \text{ in } F \cap X_\alpha\}).$$

Then h attains its minimum r_0 on a club subset C of ω_1.

[As usual, $max(\emptyset) = 0$]

Lemma 3. *Let F and C be as in Lemma 2. If $r_0 > 0$ let $g(\alpha)$ be the number of gaps of maximum width in $F \cap X_\alpha$, for each $\alpha \in C$. Then $g(\alpha)$ is constant on a club subset C of ω_1.*

Theorem 4.1. *If F is a closed unbounded subset of X, then there is a club $C \subset \omega_1$ such that either:*

(1) *$X_\alpha \subset F$ for all $\alpha \in C$ or*
(2) *There exists $n \in \mathbb{Z}^+$ and a closed unbounded $F_0 \subset F$ such that $|F_0 \cap X_\alpha| = n$ for all α in C.*

Definition. The *order of X* is ∞ if (1) of Theorem 4 holds for all closed unbounded subsets F of X, otherwise it is the least n for which there is a closed unbounded F meeting a club-indexed set of X_α in exactly n points.

Obviously, the trivial bundle $L^+ \times T$ is of order 1. More generally:

Theorem 4.2. *The order of X is 1 iff X contains a copy of ω_1.*

Corollary. *The FCCC Dichotomy (Axiom 1) implies every X is of order 1.*

Theorem 4.3. *If X is of order ∞, then X is totally normal and hereditarily cwn. Otherwise X is neither T_5 nor hereditarily strongly cwH.*

Section 5. Club-guessing gives a hereditarily cwn nonmetrizable manifold

In this section it will be shown how the club-guessing axiom $\clubsuit_C$ can be used to construct a principal T-bundle X of infinite order. The construction can be modified to produce X of any desired order $n \in \mathbb{Z}^+$, again from $\clubsuit_C$.

Here is the key concept guiding the construction.

Definition. Let $\alpha > \beta \in \omega_1$, $p \in L^+$, $\theta \in (0, 2\pi)$. We say *α jolts β at p by θ* if for one (hence all) $x \in X_\alpha$ the following holds: if y is the unique point of X_β in A_x, then the unique point of X_p in A_y is θ radians above the unique point of X_p in A_x. If S is a subset of $(0, \beta]$ we say *α jolts β on S by θ* if there exists $p \in S$ such that α jolts β at p by θ.

Let Λ_2 stand for the derived set of Λ, *i.e.,* the set of limits of countable limit ordinals. Let $\{L_\alpha : \alpha \in \Lambda\}$ be a club-guessing ladder system, with the extra properties (adopted to make the description move smoothly) that $L_{\omega^2} = \{\omega \cdot (n+1) : n \in \omega\}$ and that L_α consists of successor ordinals whenever $\alpha \in \Lambda \setminus \Lambda_2$, *i.e.,* whenever α is of the form $\beta + \omega$. Our goal is to construct X in such a way that the following induction hypothesis holds:

(1) If $\alpha \in \Lambda_2$ and $\langle \beta_n \rangle_{n \in \omega}$ is a sequence of limit ordinals converging to α, then for each n there is a choice of $\xi_n \in L_{\beta_n}$ such that ξ_n converges to α and α jolts β_n by 1 at ξ_n.

Our underlying set will be simply $L^+ \times T$, but the topology will be very different from the product topology, except on $(0, \omega^2) \times T$ where the two agree, and $A_{\omega \cdot n}$ is $(0,\, \omega \cdot n + 1) \times \{0\})$. But ω^2 jolts $\omega \cdot (n+1)$ by 1 at the first member of $L_{\omega \cdot (n+1)}$ that is greater than $\omega \cdot n$.

The simplest way to do this, and the most adaptable to orders other than ∞ is the following: A_{ω^2} agrees with $(0, \omega^2) \times \{0\}$ *except* between one unit before and one unit after the points at which these jolts take place; in these intervals of length 2, A_x continuously falls away in the negative direction from $A_{\omega \cdot (n+1)}$ until it is 1 radian below it at the specified jolting point. Then it returns to join the graph of $A_{\omega \cdot (n+1)}$ one unit later.

If we let $\sigma_\beta(n)$ stands for the $n + 1$st point in the ladder L_β, then clearly the following is satisfied for $\beta = \omega^2$:

(2) If $\gamma \in \Lambda,\ \beta \in \Lambda_2,\ \sigma_\beta(k) < \gamma \le \sigma_\beta(k+1)$,
then β jolts γ on $(\sigma_\beta(k), \gamma)$.

After this, we assume that when we reach each later $\alpha \in \Lambda_2$, then each earlier β satisfied (2), and we will define A_α so that (2) continues to hold for α in place of β. Then it is easy to see that (1) is satisfied.

Theorem 5.1. *If* (1) *holds then X is of order ∞.*

Section 6. More about Axiom 3 ($\clubsuit_C$) and Axiom 4 (KH^+)

The construction in the preceding section is a simplification of my original construction, which used KH^+ and a more complicated technique. KH^+ is still looking for an essential application, so to speak, but it is so elegant that it seems only a matter of time before a construction is discovered in which it, and it alone, is needed. Up until now it has mainly been used as an ingredient in applications of $\diamondsuit^+$ where $\diamondsuit^*$ is inadequate for the arguments used. In fact:

Theorem 6.1. $\diamondsuit^+ \iff \diamondsuit^* + KH^+$.

Proof. Let $\langle \mathcal{S}_\alpha : \alpha \in \omega_1 \rangle$ witness $\diamondsuit^+$. In other words, $\mathcal{S}_\alpha$ is a countable collection of subsets of α such that for each subset A of ω_1, there is a club C_A such that for all $\alpha \in C_A$, $A \cap \alpha$ and $C_A \cap \alpha$ are both in $\mathcal{S}_\alpha$. [$\diamondsuit^*$ leaves out the condition that $C_A \cap A \in \mathcal{S}_\alpha$]

If F is a club we let $K_F = F \cap C_F$. Then $\mathcal{B} = \{K_F : F \text{ is a club}\}$ is a base for the club filter on ω_1, such that $\mathcal{B} \restriction a$ is countable for each α. In fact, if we let $\mathcal{T}_\alpha = \{A \cap B : A, B \in \mathcal{S}_\alpha\}$, let $\mathcal{V}_\alpha = \mathcal{T}_\alpha$ when α is a limit ordinal, and let

$$\mathcal{V}_{\alpha+1} = \mathcal{T}_{\alpha+1} \cup \{A \cup \{\alpha\} : A \in \mathcal{T}_\alpha\} \quad \text{and} \quad \mathcal{W}_\alpha = \bigcup \{\mathcal{V}_\beta : \beta \leq \alpha\}$$

then $\mathcal{W}_\alpha$ is obviously countable, and $\mathcal{B} \restriction \alpha \subset \mathcal{W}_\alpha$ for all α. Indeed, if $\alpha \in K_F$, then clearly $K_F \cap \alpha \in \mathcal{T}_\alpha$, otherwise $K_F \cap \alpha \in \mathcal{V}_{\beta+1}$ where $\beta = max(K_F \cap \alpha)$ and $\beta + 1 \leq \alpha$. [The sup is a max since $K_F \cap \alpha$ is closed in α.]

Conversely, if $\{\mathcal{S}_\alpha : \alpha \in \omega_1\}$ witnesses $\diamondsuit^*$ and $\mathcal{B}$ witnesses KH^+, then $\{\mathcal{S}_\alpha \cup \mathcal{B} \restriction \alpha : \alpha \in \omega_1\}$ witnesses $\diamondsuit^+$. Indeed, for each $A \in \omega_1$ let C_A be as in $\diamondsuit^*$ and let $K_A \in \mathcal{B}$ be a subset of C_A; then $K \cap \alpha \in \mathcal{B} \restriction \alpha$ for all α, and $A \cap \alpha \in \mathcal{S}_\alpha$ for all $\alpha \in K_A$; so K_A can be put for C_A in $\diamondsuit^+$. $\square$

Similarly, we have $\diamondsuit^{++} \implies KH^{++}$, which is KH^+ with the added condition that there is a stationary set E of limit ordinals α such that $\mathcal{U}_\alpha$ is a filterbase, where $\mathcal{U}_\alpha = \{A \in \mathcal{B} \restriction \alpha : A \text{ is unbounded in } \alpha\}$.

To get this, let $\langle \mathcal{S}_\alpha : \alpha \in \omega_1 \rangle$ above also witness $\diamondsuit^{++}$ and let $\mathcal{B}$ be as before. Let S be a stationary set of limit ordinals α such that the unbounded members of $\mathcal{S}_\alpha$ form a filterbase. Then $S \subset E$, because $(K_F \cap \alpha) \in \mathcal{T}_\alpha$ is unbounded in α whenever $\alpha \in S$ and F and C_F are unbounded in α.

Both KH^+ and KH^{++} are preserved by ccc forcing, because every club subset of ω_1 in the forcing extension contains one in the ground model. Hence both axioms are compatible with $MA + \mathfrak{c} = \aleph_\alpha$ for any regular $\aleph_\alpha$.

It is easy to show that KH^{++} implies the club-guessing axiom $\clubsuit_C$; but KH^+ is already adequate for this:

Theorem 6.2. $KH^+ \implies \clubsuit_C$.

Proof. First, note that $\clubsuit_C$ obviously implies the following axiom:

Axiom 4$'$. There is a family $\mathcal{L} = \{L^n_\alpha : \alpha \in \Lambda, n \in \omega\}$ such that each L^n_α is a ladder at α and, for every club $C \subset \omega_1$, there are α and n such that $L^n_\alpha \subset C$.

Conversely, if $\mathcal{L}$ witnesses Axiom 4′, one of the families $\mathcal{L}_n = \{L_\alpha^n : \alpha \in \Lambda, n \in \omega\}$ must witness $\clubsuit_C$. Were this not so, we could pick clubs C_n so that no $L_\alpha^n \subset C_n$ and then $C = \bigcap_{n=0}^\infty C_n$ would be a counterexample to $\mathcal{L}$ witnessing Axiom 4′.

To finish the proof, let $\mathcal{B}$ be a base for the club filter as in KH^+ and, for each limit α, let $\{C_\alpha^n : n \in \omega\}$ list the members of $\mathcal{B} \restriction \alpha$ that are cofinal in α, and let L_α^n be a cofinal subset of C_α^n of order type ω. Now if C is a club, let $B \in \mathcal{B}$ be a subset of C. If α is in the derived set B' of B, then $B \cap \alpha = C_\alpha^n$ for some n, and $L_\alpha^n \subset B \subset C$. $\square$

It is clear from the preceding proof that KH^+ implies the following stengthening of Axiom 4′:

$\clubsuit_C^*$: There is a family $\{L_\alpha^n : \alpha \in \omega_1, n \in \omega\}$ such that each L_α^n is a cofinal subset of α of order type ω, and such that for each club $C \subset \omega_1$ there is a club $K(C)$ such that $\exists n \in \omega(L_\alpha^n \subset C)\}$ for all $\alpha \in K(C)$.

In fact, the proof of Theorem 6.2 can easily be modified to show something even stronger:

Axiom 3$^+$. There is a family $\{C_\alpha^n : \alpha \in \omega_1, n \in \omega\}$ such that each C_α^n is a club subset of α of the same order type as α, and such that for each club $C \subset \omega_1$ there is a club $K(C)$ such that $\exists n \in \omega(C_\alpha^n \subset C)\}$ for all $\alpha \in K(C)$.

This axiom is destroyed by adding ω_2 Cohen subsets of ω_1, which is α-proper for all $\alpha < \omega_1$. In contrast, Axiom 4 ($\clubsuit_C$) cannot be destroyed by any ω-proper forcing [I]. This makes it compatible with not only the LCπ dichotomy (Axiom 2) and Axiom 9, but also with the weakening of the FCCC dichotomy (Axiom 1) which substitutes "perfect preimage of ω_1" for "copy of ω_1". The combination of these three axioms can be produced using forcing that is α-proper for all countable α and also does not add reals. This substitution is enough to take care of steps 1 through 4 of the sequence of proofs for Theorem 1 and Theorem 2, as well as Theorem 4.

At the opposite extreme, the following axiom, though even weaker than Axiom 7, is known to be negated by the PFA. I do not know whether it is also negated by FCCC Dichotomy (Axiom 1).

Axiom 10. There is a family of functions $\{g_\alpha : \alpha \in \Lambda\}$, $g_\alpha \to \{0,1\}$ such that $g_\alpha^{-1}\{1\}$ is open for all α and, for each club $C \subset \omega_1$, there is α such that $g_\alpha \restriction C \cap \alpha$ is not eventually constant.

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