1. (26 points) Short answers; no proof required.
   a. If a square matrix $M$ can be partitioned into submatrices $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ in which $A$ and $D$ are square, what is the determinant of $M$ in terms of the determinants of the submatrices?
   b. Assume you have a $3 \times 3$ matrix. Give the elementary matrix corresponding to the row operation $cR_2$; do the same for the row operation $cR_1 + R_3$ (this changes row 3 but leaves row 1 unchanged).
   c. Let $\beta = \{e_1, e_2\}$ be the standard ordered basis for $\mathbb{R}^2$ and $\gamma$ be the ordered basis given by $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Give the matrix that gives $\gamma$ in terms of $\beta$; and give the matrix that gives $\beta$ in terms of $\gamma$.
   d. Define similar matrices. How is this important for defining the determinant of a linear operator on a finite dimensional space?
   e. A system $AX = 0$ of $m$ linear equations in $n$ unknowns $x_1, \ldots, x_n$ always has the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. What condition on $A$ will guarantee that there exist non-trivial solutions?

2. (14 points) Let $T$ be a linear operator on a finite dimensional vector space $V$ whose characteristic polynomial splits (that is, factors into a product of linear factors, possibly repeated). Let $W$ be a nontrivial $T$-invariant subspace. Prove that $W$ contains at least one eigenvector for $T$.

3. (16 points) Let $A$ be an upper triangular matrix $n \times n$ matrix, and suppose $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_t$ with corresponding algebraic multiplicities $k_1, \ldots, k_t$.
   a. Compute the characteristic polynomial $\chi_A$ from the definition, and show that it splits.
   b. How are the diagonal entries of $A$, namely the $a_{ii}$’s, related to the eigenvalues of $A$?
   c. Prove that the trace of $A$, defined as $\sum_{i=1}^{n} a_{ii}$, is equal to $\sum_{i=1}^{t} k_i \lambda_i$, and that $\sum_{i=1}^{t} k_i = n$. How is $\text{tr}(A)$ related to one of the coefficients of $\chi_A$?

4. (16 points) A linear operator $T$ on $V = \mathbb{R}^4$ has characteristic polynomial $x^4 - 1$. Determine whether $T$ is diagonalizable or not. Determine whether $T$ is invertible or not, and if so, give $T^{-1}$ in terms of non-negative powers of $T$. How would your answers to these questions change if $V = \mathbb{C}^4$ (as a vector space over $\mathbb{C}$) instead?
5. (14 points) Suppose $V$ is an $n$-dimensional vector space and the non-zero linear operator $T$ satisfies $T^\ell = 0$ for some $\ell > 0$. What conclusions can you draw about the minimal polynomial $p_T(x)$, the characteristic polynomial $\chi_T(x)$, and the operator $T^n$? Explain.

If $W$ is a subspace of $V$, recall that $W^\circ = \{ f \in V^* \mid f(w) = 0 \text{ for all } w \in W \}$. In this problem $T^*$ represents the dual map on $V^*$ if $T$ is a linear operator on $V$, not the adjoint.

6. (14 points) Let $V$ be a finite dimensional vector space and $T$ be a linear operator on $V$. Suppose $W$ is a $T$-invariant subspace of $V$. Prove that $W^\circ$ is $T^*$-invariant.