**MATH 550**  Final Exam, Part A  Name: 
Spring, 2006

Note! For full credit you must show sufficient work to support your answer. In particular you must show the major steps of any integration. There are 120 points. Good luck!

**Change of Variables Theorem.** In two variables, \( \iint_D f(x, y) \, dx \, dy = \iint_{D^*} f(T(u, v)) |J(x, y)| \, du \, dv \) and in three variables \( \iiint_D f(x, y, z) \, dx \, dy \, dz = \iiint_{D^*} f(T(u, v, w)) |J(x, y, z)| \, du \, dv \, dw \), where \( D \) and \( D^* \) are suitable regions and \( T \) is a suitable transformation such that \( T(D^*) = D \).

**Stokes's Theorem.** Let \( S \) be a bounded, piecewise regular, oriented surface in \( \mathbb{R}^3 \) and suppose that \( C = \partial S \) consists of finitely many piecewise \( C^1 \) simple closed curves, oriented consistently with the orientation of \( S \). Suppose that \( F \) is a \( C^1 \) vector field with continuous partial derivatives defined on a domain that includes \( S \). Then \( \iint_S \nabla \times F \cdot dS = \int_C F \cdot n \, ds \).

**Divergence or Gauss' Theorem.** If \( W \) is a bounded symmetric elementary domain in \( \mathbb{R}^3 \), whose boundary \( S = \partial W \) consists of finitely many piecewise regular closed oriented surfaces, oriented so that the normal vectors point out of \( W \), and \( F \) is a \( C^1 \) vector field defined on \( W \), then \( \iiint_W F \cdot dV = \iint_S F \cdot n \, ds = \iint_S \nabla \cdot F \, dV \).

These theorems, and also Green's Theorem in the plane, which they generalize, can be applied to regions that have suitable decompositions.

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1. (6 points) Let \( A \) be the region in the \( xy \)-plane bounded below by \( y = x^2 \) for \( -2 \leq x \leq 2 \), and above by \( y = x^2 + 3 \) for \( -1 \leq x \leq 1 \) and \( y = 4 \) for \( -2 \leq x \leq -1, 1 \leq x \leq 2 \). Explain why Green's Theorem cannot be used directly for this domain, and show how Green's Theorem can be applied indirectly.

2. (10 points) a. In Stokes' Theorem what does it mean to say that the surfaces in question are regular?

\[ 4 \vec{T} \times \vec{T} \neq 0 \text{ everywhere} \]

\[ \text{Or there is a tangent plane everywhere} \]

\[ \text{Or there is an } \vec{n} \text{ everywhere} \]

b. What does it mean to say that the boundary curve(s) is (are) oriented compatibly with the surface(s)?

As you walk along the boundary, the positive (\( \vec{n} \)-oriented) side is on your left. Or curve orientation with right-hand rule gives \( \vec{n} \).
3. (12 points) Compute \( \iint_R (x + y)^2 e^{x-y} \, dA \) where \( R \) is the region bounded by the lines \( y = 1 - x, \, y = 4 - x, \, y = x + 1, \, y = x - 1 \).

Given the region \( R \), we can solve for \( x, y \):

\[
\begin{align*}
\frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2 \\
|\frac{\partial(x, y)}{\partial(u, v)}| &= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 2 \\
\iint_R (x + y)^2 e^{x-y} \, dA &= \iint_{R^*} \frac{1}{2} u^2 e^{u/2} \, du \, dv.
\end{align*}
\]

The Jacobian simplifies to:

\[
\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2
\]

The integral becomes:

\[
\iint_{R^*} \frac{1}{2} u^2 e^{u/2} \, du \, dv = \int_{-1}^{4} \int_{-1}^{1} \frac{1}{2} u^2 e^{u/2} \, du \, dv.
\]

After evaluating the integral and applying the limits, we get:

\[
\int_{-1}^{1} u^2 e^{u/2} \, du = \left[ \frac{2}{3} u^3 e^{u/2} \right]_{-1}^{1}.
\]

Thus, the solution is:

\[
\frac{2}{3} (e^1 - e^{-1})
\]

or

\[
\frac{2}{3} (e - e^{-1})
\]
4. (22 points) Compute \( \iiint_W \frac{dV}{(x^2 + y^2 + z^2)^2} \) where \( W \) is the region between the spheres \( x^2 + y^2 + z^2 = a^2 \), \( x^2 + y^2 + z^2 = 16 \), and \( 0 < a < 4 \). Use your result to evaluate the same integral for the entire region inside the sphere \( x^2 + y^2 + z^2 \leq 16 \).

\[
\iiint_2 \int_0^{2\pi} \int_0^\pi \frac{r^2 \sin \psi}{r^2} \, d\psi \, d\theta \, dr
\]

\[
= 2 \int_0^4 2\pi \int_0^{\frac{r}{a}} \frac{1}{r^2} \, dr
\]

\[
= 4\pi \left[ -\frac{1}{r} \right]_a^4 = 4\pi \left( -\frac{1}{4} + \frac{1}{a} \right) = \frac{4\pi}{a} - \pi
\]

\[
= \lim_{a \to 0} \left( \frac{4\pi}{a} - \pi \right) = \infty
\]

\( \frac{2}{3} \)
5. (15 points) Compute the area of the portion of the plane $2x + 4y + 6z = 12$ that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). Suggestion: orient the surface so that $\hat{n}$ points away from the origin.

Note $\hat{n} = \frac{1}{\sqrt{14 + 16 + 36}} \left< 2, 4, 6 \right> = \frac{1}{15} \left< 2, 4, 6 \right>$

everywhere on $S$

By the area cosine principle

Area = $\iint_{S} dA = \iint_{S} \hat{n} \cdot \hat{n} dA$

$= \sqrt{14 + 16 + 36} \cdot \frac{1}{2} (6) 8$

$= 3 \sqrt{46}$

6. (22 points) Compute $\int_{C} G \cdot ds$ for $G = (3xy^2, -3x^2y)$, where $C$ is given by the segment from $(1,0)$ to $(2,0)$, the circular arc of $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$, the segment from $(0,2)$ to $(0,1)$, and finally the circular arc of $x^2 + y^2 = 1$ from $(0,1)$ back to $(1,0)$.

Use Green's Theorem

$\oint_{C} F \cdot ds = \iint_{A} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$

$= \iint_{A} (6xy - 6xy) dA = 0$

Use polar coordinates

$= \int_{0}^{\pi/2} \int_{0}^{1} \frac{1}{r^2} r^2 d\theta dr$

$= \int_{0}^{\pi/2} \frac{\theta^2}{2} \bigg|_{0}^{\pi/2} d\theta$

$= \frac{\pi^3}{32} - \frac{\pi^3}{32} = -\frac{\pi^3}{16}$

$= -\frac{\pi^3}{16}$
7. (12 points) Let \( x = u^2 + \sin(v) \), \( y = \ln(1 + w^2) + uv \), and \( z = 1/(u^2 + w^2) \). Compute \( \frac{\partial (x, y, z)}{\partial (u, v, w)} \). If \( (u, v, w) = T(u, v, w) \) is given by the formulas above, what is the volume change factor in the vicinity of the point \( P \) \((u, v, w) = (0, \pi, -1)\)? Is this transformation orientation reversing or preserving near the point \( P \)?

\[
\begin{vmatrix}
2u & \cos v & 0 \\
0 & 0 & \frac{2u u}{1 + w^2} \\
-2u & 0 & -\frac{2u}{(u^2 + w^2)^2}
\end{vmatrix} = \left( -\cos v \right) \left( -\frac{2u^2}{(u^2 + w^2)^2} \right) + \frac{2u}{(u^2 + w^2)^2}
\]

At \( P \): \(-\frac{2}{1} = -2\)

volume change factor \( +2 \)

orientation reversing

8. (6 points) For the vector field \( \mathbf{F} \) shown below, show a path \( L \) from \( P \) to \( Q \) so that \( \int_L \mathbf{F} \cdot \mathbf{ds} > 0 \), and a path \( M \) from \( P \) to \( Q \) so that \( \int_M \mathbf{F} \cdot \mathbf{ds} = 0 \). What does this tell you about the vector field \( \mathbf{F} \)?

Independence of path fails. (\( \mathbf{F} \) is not conservative)
9. (15 points) Compute \( \int_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS \) if \( \mathbf{F} = (3x^2, y^2, x^2 + y^2) \) and \( S \) is the portion of the surface \( z = 10 - x^2 - y^2 \) that lies above the plane \( z = 1 \). Assume \( \mathbf{n} \) points away from the origin.

\[ S = \partial D \text{ oriented } \text{CCW} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]

\[ \text{Stokes} \]

Let \( D = \) interior of \( C \) in \( z=1 \) plane oriented by \( \mathbf{k} \).

\[ \int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} \]

\[ \nabla \times \mathbf{F} = (1, 6z - 1, 4) \]

\[ \int_D (1, 5, 4) \cdot \mathbf{k} \, dA = 4 \text{Area}(D) \]

\[ \int_D (1, 5, 4) \cdot \mathbf{k} \, dA = 4 \text{Area}(D) = 4 \cdot 9\pi = 36\pi \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3 + 12\cos \theta - \frac{9}{16} \sin^2 \theta) \cdot (3\cos \theta + 3\sin \theta) d\theta \]

\[ C: \quad 0 \leq \theta \leq 2\pi \]

\[ C: \quad \begin{align*} x &= 3\cos \theta \\ y &= 3\sin \theta \\ 0 \leq \theta \leq 2\pi \end{align*} \]

\[ C(t) = (3\cos t, 3\sin t, 0) \]

\[ ds = \sqrt{9\cos^2 t + 9\sin^2 t + 0} \, dt = 3 \, dt \]

Alternate approach:

\[ \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} \, dS \]

\[ \text{For integrals over a surface:} \]

\[ \text{If actual integral is done, it probably should be in polar coordinates.} \]

\[ \int_S \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3 + 12\cos \theta - \frac{9}{16} \sin^2 \theta) \cdot (3\cos \theta + 3\sin \theta) d\theta \]

\[ 3\cos \theta + 3\sin \theta \cdot (-3\sin \theta) \]

\[ 3\cos \theta, 0 \]
\[
\frac{1}{3} \left[ 0 + \int_{0}^{2\pi} 36 \left( \frac{1 + \cos(\theta)}{2} \right) d\theta \right] + 0
\]

\[
= 18 \left( 2\pi \right) = 36\pi \quad \text{(from } \sin(2\theta) \text{)}
\]

**Alternate #45**

\[\vec{A} = (-6, 0, 2)\]
\[\vec{B} = (-6, 3, 0)\]

\[
\text{Area} = \frac{1}{2} \| \vec{A} \times \vec{B} \| = \frac{1}{2} \| (-6, -12, -18) \|
\]

\[
= 3 \| (1, 2, 3) \|
= 3 \sqrt{1+4+9} = 3\sqrt{14}
\]
Attempt #6

\[ \int_{1}^{2} (0,0) \cdot (1,0) \, dt = 0 \]

\[
x = t \\
y = 0
\]

\[
x = 2 \cos t \\
y = 2 \sin t
\]

\[ 0 \leq t \leq 2\pi \]

\[
(-2 \sin t, 2 \cos t) \, dt
\]

\[ \int_{0}^{\pi} \left( -48 \cos t \sin^3 t - 48 \cos^3 t \sin t \right) \, dt \]

\[ = -48 \frac{\sin^4 t}{4} + 48 \frac{\cos^4 t}{4} \bigg|_{0}^{\pi} = -12 - 12 \]

\[ = -24 \]

\[
\int_{0}^{2} (0,0) \cdot (0,1) \, dt = 0
\]
\[
\int_0^{\pi/2} (3\cos^2 t \sin t, -3\cos^2 t \sin^3 t) \, dt
\]

\[
= \int_0^{\pi/2} (-3\cos^2 t \sin^3 t - 3\cos^3 t \sin t) \, dt
\]

\[
= \left[ -3 \left( \frac{\sin^4 t}{4} \right) + 3 \left( \frac{\cos^4 t}{4} \right) \right]_0^{\pi/2}
\]

\[
\frac{3}{4} - \left( -\frac{3}{4} \right) = \frac{6}{4}
\]

Total = \(-24 + \frac{6}{4}\) = \(-24 + \frac{3}{2}\)

\[
= -48 + \frac{3}{2}
\]

\[
= -\frac{45}{2}
\]

\[
= \frac{-45}{2}
\]
MATH 550  Final Exam, Part B  Name: \[ M^2 \]

Spring, 2006

Note! For full credit you must show sufficient work to support your answer. In particular you must show the major steps of any integration. There are 60 points. Unless otherwise stated you may assume that curves, surfaces, and regions of \( \mathbb{R}^3 \) are suitable for application of the major theorems.

1. (6 points) Let \( \mathbf{F} \) be defined on a domain \( D \) in either \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Give three different, but equivalent, statements that \( \mathbf{F} \) be a conservative vector field, without mentioning curl.

2. (2 points) Independence of path holds (\( \int_C \mathbf{F} \cdot d\mathbf{s} = f(Q) - f(P) \) for any path from \( P \) to \( Q \)).

3. (5 points) Suppose that \( \mathbf{F} \) is everywhere tangent to the closed regular surface \( S = \partial W \) for a suitable domain \( W \) in \( \mathbb{R}^3 \). Demonstrate that \( \iiint_W \text{div}(\mathbf{F}) \, dV = 0 \).

4. (8 points) Computing only derivatives (no integrals), explain why \( \mathbf{G} = (3y, \frac{3y}{1-z^2}, x^3) \) must be conservative in the slab \(-1 < y < 1, -\infty < x < \infty, -\infty < z < \infty \).

The slab is simply connected (even convex) domain, or has no exceptional points.

Together these imply \( \mathbf{G} \) is conservative.

[Diagram of a region between walls]
4. (12 points) Compute the flux \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) of \( \mathbf{F} = (xy^2, x^2y, z) \) through the closed cylinder \( S \) (oriented by outward normals) given by \( x^2 + y^2 = 4, \ z = -2, \ z = 3 \).

2. Use Divergence Theorem:
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV \]

\( W \) = interior of cylinder

use cylindrical coords:
\[ \int_0^{2\pi} \int_0^3 \left( \frac{\pi}{4} + \frac{\pi}{2} \right) \, d\theta \, dz \]
\[ = 6(2\pi)(5) = 60\pi \]

If they don't use the D.T. or they don't use polar, mark it for me to grade.

5. (18 points) In each of the following parts determine if \( \mathbf{F} \) is conservative or not, and give your reason. The compute \( \int_C \mathbf{F} \cdot d\mathbf{s} \) by whatever method seems easiest and most appropriate.

a. \( \mathbf{F} = (z, 0, -z) \); \( C \) is the line segment from \((0, -1, 3)\) to \((1, 1, 4)\).

\[ \mathbf{F} \cdot \mathbf{C} = (0, 2, 0) \neq 0 \]

1. \( \mathbf{F} \) is NOT conservative.

2. \( \mathbf{r}(t) = (t, -1 + 2t, 3 + t) \), \( 0 \leq t \leq 1 \)

3. \( \mathbf{r}'(t) = (1, 2, 1) \)

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (3 + t, 0, -t) \cdot (1, 2, 1) \, dt \]
\[ = \int_0^1 (3 + t - t) \, dt = 3 \]
b. \( \mathbf{F} = (2xy, x^2 + 4y^2/(y^4 + z^4), 4z^3/(y^4 + z^4)) \), \( C \) is given by \( x = \theta \), \( y = 4\cos\theta \), \( z = 3\sin\theta \), \( 0 \leq \theta \leq \pi/2 \).

\[ \text{Compute potential function } f: \quad \mathbf{F} \text{ is conservative} \]

\[ \begin{align*}
  f &= \int 2xy \, dx = x^2y + A(y, z) \\
  f &= \int \left(x^2 + \frac{4y^2}{y^4+z^4}\right) \, dy = x^2y + (\ln(y^4+z^4)) + B(x, z) \\
  f &= \int \frac{4z^3}{y^4+z^4} \, dz = \ln(y^4+z^4) + C(x, y)
\end{align*} \]

\[ \text{Compatible: } f_x(x, y, z) = x^2y + \ln(y^4+z^4) \]

\[ \begin{align*}
  P \times \Theta &= 0: \quad (0, 4, 0) \\
  Q \times \Theta &= \frac{\pi}{2}: \quad \left(\frac{\pi}{2}, 0, 3\right)
\end{align*} \]

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = f(Q) - f(P) = \ln(81) - \ln(256) \]

6. (5 points) We found for a vector field \( \mathbf{F} \) at a point \( P \), and an axis given by a unit vector \( \hat{n} \), that \( (\text{curl } \mathbf{F})(P) \cdot \hat{n} \) is equal to the “circulation” of \( \mathbf{F} \) about \( P \) on a surface with normal vector \( \hat{n} \). How is circulation defined?

\[ \lim_{A \to 0} \frac{1}{1} \int_{A} \mathbf{F} \cdot d\mathbf{s} \]

7. (6 points) Using “flowboxes”, determine if each of the following vector fields has positive, negative, or zero divergence in general.

\[ \begin{align*}
  \text{Shaded} &= \text{start, open} = \text{finish} \\
  \text{Same base, same height, top moves faster}
\end{align*} \]
\[ \int \int_{S_2} 32 \cos^2 \Theta \sin^2 \Theta \, d\Theta \, d\varphi \]
\[ = 5.32 \int_{0}^{2\pi} \frac{1}{2} (1 + \cos 2\Theta) (1 - \Theta \cos 2\Theta) \, d\Theta \]
\[ = 5.8 (2\pi) + 0 - 0 - 5.8 \int_{0}^{2\pi} \cos^2 2\Theta \, d\Theta \]
\[ = 80\pi - 5.4 (2\pi) = 40\pi \]